A NEW ASYMPTOTIC FORMULA FOR THE PARTIAL SUM OF THE LOGARITHMIC FUNCTION

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Abstract. In this short paper we estimate the size of the partial sum of the logarithmic function \( \sum_{n \leq x} \log n \) using a different method. We also use it to give an upper and lower bound for the factorial of any positive integer.

INTRODUCTION

Stirling’s formula gives \( n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \) as \( n \to \infty \). Expressed as an inequality
\[
\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \leq n! \leq \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} e^{-n} \sqrt{\frac{1}{1 - \frac{11}{11520}}}
\]

The Ramanujan estimate [1] also gives
\[
\sqrt{\pi} \left( \frac{n}{e} \right)^n \left( 8n^3 + 4n^2 + n + \frac{1}{30} \right)^{\frac{1}{2}} \left( 1 - \frac{11}{11520(n+a)^4} \right) < n!
\]

where \( a = \frac{39}{54} \) and
\[
b = \left( \frac{11}{11520} \left( 1 - \left( \frac{30\sqrt{2}}{391\pi} \right)^2 \right) \right)^{\frac{1}{4}} - 1 = 0.35499112666\ldots
\]

1. REPRESENTATION FORMULA

Theorem 1.1. If \( a \) is an odd positive integer, then
\[
\frac{a-1}{2} = \sum_{\substack{1 \leq j \leq a \\ \ (2,j)=1}} \left| \frac{\log \left( \frac{a}{j} \right)}{\log 2} \right|
\]

Proof. Consider an odd positive integer \( a \). There are as many even integers as odd integers less than \( a \). That is, for \( 1 \leq m < a \), there are \( \frac{a-1}{2} \) such possibilities. On the other hand, notice that the set of all positive even integers is the union of the sequences \( \{m2^\alpha\} \), where \( m \) is distinct for each sequence and runs through the
odd integers and $\alpha$ runs through the positive integers ($\alpha = 1, 2, \cdots$). Consider the sequence

$$2, 2^2, \ldots, 2^m.$$ 

Clearly there are $\left\lfloor \log \frac{a}{\log 2} \right\rfloor$ such terms in this sequence. Again consider those of the form

$$3 \cdot 2, 3 \cdot 2^2, \ldots, 3 \cdot 2^m.$$ 

Clearly there are $\left\lfloor \log \frac{a}{3 \log 2} \right\rfloor$ such number of terms in this sequence. We terminate the process, by considering the sequence

$$2 \cdot j, j \cdot 2^2, \ldots, j \cdot 2^m,$$

where $(2, j) = 1$ and there are $\left\lfloor \log \frac{a}{j \log 2} \right\rfloor$ such number of terms in this sequence.

In comparison with the previous count, we obtain

$$a - \frac{1}{2} = \sum_{1 \leq j \leq a} \left\lfloor \frac{\log \left( \frac{a}{j} \right)}{\log 2} \right\rfloor + O(1),$$

(1.1)

which is what we set out to prove. □

**Remark 1.2.** Having establish this formula, we are ready to give a different estimate for the partial sum of the logarithmic function. Here is how it goes.

**Corollary 1.1.** For all positive integers $a$

$$\sum_{j \leq a} \log j = a \log a - a \log 2 + \log 2 - \sum_{j \leq a} \left\lfloor \frac{\log \left( \frac{a}{j} \right)}{\log 2} \right\rfloor + O(1).$$

**Remark 1.3.** In the forthcoming proof we consider $a$ to be an odd integer. And it is very easy to see that it will remain valid for all positive integers once the proof goes through for all odd integers.

**Proof.** Invoking theorem (?), we have that

$$\frac{a - 1}{2} = \sum_{j \leq a \atop (2, j) = 1} \log \left( \frac{a}{j} \right) - \sum_{j \leq a \atop (2, j) = 1} \left\lfloor \log \left( \frac{a}{j} \right) \right\rfloor.$$

(1.1)

Also we observe that

$$\frac{a - 1}{2} = \frac{1}{\log 2} \sum_{j \leq a \atop (2, j) \neq 1} \log \left( \frac{a}{j} \right) - \sum_{j \leq a \atop (2, j) \neq 1} \left\lfloor \log \left( \frac{a}{j} \right) \right\rfloor + O(1).$$

(1.2)
Combining equations (1) and (1.2), we have

\[
a - 1 = \frac{1}{\log 2} \sum_{j \leq a} \log \left( \frac{a}{j} \right) - \sum_{j \leq a} \left\{ \log \left( \frac{a}{j} \right) / \log 2 \right\} + O(1) 
\]

(1.3)

\[
= \frac{1}{\log 2} \left( \sum_{j \leq a} \log a - \sum_{j \leq a} \log j \right) - \sum_{j \leq a} \left\{ \log \left( \frac{a}{j} \right) / \log 2 \right\} + O(1). \]

It follows from (1.3) that

\[
\log 2^{a-1} + \log 2 \sum_{j \leq a} \left\{ \log \left( \frac{a}{j} \right) / \log 2 \right\} = \sum_{j \leq a} \log a - \sum_{j \leq a} \log j + O(1) \]

(1.4)

\[= a \log a - \sum_{j \leq a} \log j + O(1). \]

Arranging terms, we have

\[
\sum_{j \leq a} \log j = a \log a - a \log 2 + \log 2 + \log 2 \sum_{j \leq a} \left\{ \log \left( \frac{a}{j} \right) / \log 2 \right\} + O(1), \]

where \( \left\{ \log \left( \frac{a}{j} \right) / \log 2 \right\} = G(n) \) denotes the fractional part of \( \frac{\log \left( \frac{a}{j} \right)}{\log 2} \). \( \Box \)

Remark 1.4. Stirling’s formula gives an upper and lower bound for the factorial of any positive integer. We present here a slightly different version.

**Theorem 1.5.** For all \( n \geq 1 \) and for some constant \( C(n) > 1 \)

\[
\frac{n^n}{2^{n-1+G(n)}} \leq n! \leq C(n) \frac{n^n}{2^{n-1+G(n)}}, \]

where

\[
G(n) := \sum_{m \leq n} \left\{ \log \left( \frac{n}{m} \right) / \log 2 \right\}. \]

**Proof.** Recall from Corollary 1.1

\[
\sum_{m \leq n} \log m \geq n \log n - n \log 2 + \log 2 - \log 2 \sum_{m \leq n} \left\{ \log \left( \frac{n}{m} \right) / \log 2 \right\}. \]
Recall $\sum_{m \leq n} \log m = \log n!$. Using this identity, it follows that
\[ \log n! \geq n \log n - n \log 2 + \log 2 - \log 2G(n), \]
from which we see that
\begin{equation}
(1.5) \quad n! \geq \frac{n^n}{2^{n-1+G(n)}}.
\end{equation}
Also it is clear from Corollary 1.1
\begin{equation}
(1.6) \quad n! = \frac{n^n}{2^{n-1+G(n)}} e^{O_n(1)} \leq C(n) \frac{n^n}{2^{n-1+G(n)}}.
\end{equation}
From (1.5) and (1.6), we obtain the inequality
\[ \frac{n^n}{2^{n-1+G(n)}} \leq n! \leq C(n) \frac{n^n}{2^{n-1+G(n)}}, \]
where $G(n) := \sum_{m \leq n} \left\{ \frac{\log \left( \frac{n}{m} \right)}{\log 2} \right\}$.

References


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