A Solution to Einstein's Field Equations in which the Λ Discrepancy is Resolved

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The acceptance of the multiverse by prominent cosmologists opens the door to exploring alternative solutions to Einstein's field equations. This brief paper explores the mathematics of an alternative solution in which it is postulated that $\mathrm{d}\tau$ physically behaves differently than it does in the FLRW cosmology (*i.e.*, $\mathrm{d}\tau = a(t)\mathrm{d}t$ as opposed to the FLRW's $\mathrm{d}\tau = \mathrm{d}t$). The equations that are analogous to the Friedmann equations contain an additional a^2 term, and the equation that is analogous to the Friedmann acceleration equation has a change in sign. The age-based $a(t) = t/t_0$ solution to these equations results in an Einstein variable gravity model in which the cosmological constant (Λ) discrepancy is resolved. The analogs to the Friedmann equations, when evaluated between the Planck era and t_0, effectively reduce to the same expression as the calculation of the vacuum Zero-point energy, *i.e.* $\Omega_{\Lambda_Z} = 10^{120}$.

Keywords: Einstein's field equations, Λ discrepancy

I. INTRODUCTION

Dirac hypothesized that certain large numbers that occur in nature (*i.e.*, 10^{40}, 10^{80}) are unlikely coincidences but rather imply a cosmology with the unusual feature that certain physical constants vary with the age of the universe.[1, 2] Dirac did not live to witness the discovery that the Universe's expansion is accelerating requiring the reintroduction of Λ into the Friedmann Lemaitre Robertson-Walker (FLRW) cosmology. The prime candidate for Λ is the Zero-point energy (ZPE) of the quantum vacuum which is 120 orders of magnitude greater than that predicted by FRW (*i.e.*, $\Omega_\Lambda \approx 1$). Hobson et al. has referred to this discrepancy as, "The worst theoretical prediction in the history of physics."[3] If Dirac had lived, he might have concluded that Λ is also age dependent as the 10^{120} number is next in progression to the large numbers he frequently cited.

The 10^{120} discrepancy in Λ also contributes significantly to the realization that the Universe is extremely unique, which then leads to the multiverse. Susskind et al. has suggested that a minimum of 10^{500} types of universes make up the multiverse, whereas Linde and Vanchurin have estimated that number to be closer to 10^{700}.[4] Tegmark extended the possibilities beyond FRW universes by employing a taxonomic classification system with varying levels of constraints.[5] This suggests that a possible approach for reconciling the 10^{120} number would be to explore alternative Einstein-based cosmologies based on other (evolving) geometries. There should be no restriction to this approach as the R-W does not rise to the same level of exclusivity as does, for instance, the Schwarzschild (*i.e.*, There is no uniqueness theorem similar to Birkhoff's Theorem for the R-W.). Consequently, it would be incorrect to state that any other coordinate(s) under consideration must be the R-W coordinate(s) which have undergone a coordinate transformation. Indeed, the principle feature of the R-W is its adherence to the observed spatial homogeneity and isotropy of the universe.[6] Accordingly, any alternative geometry considered must remain similar to the R-W with respect to spatial coordinates.

An Einstein-based model that incorporates a variation of Mach's principle is a possibility worth exploring. According to the Mach-Einstein principle, inertial forces are themselves determined by the gravitational field of the whole universe. This implies that the total energy condition (inertial (m) + gravitational (U_g)) of a particle at rest with respect to the universe is zero.[7][8] As Nottale states this requires $GM_Um/R_0 \approx mc^2$, *i.e.*, it is expressed by the Schwarzschild-like condition $GM_U/c^2R_0 \approx 1$.[9] Interestingly, the above simple relationship does indeed effectively render one of Dirac's coincidental numbers, *i.e.*:

$$\frac{GM_Um}{R_0} \approx mc^2 \rightarrow M_U \approx \frac{c^3}{G}t_0.$$

Setting $t_0 = 13.7$by or 4.32×10^{17}s and given that a proton's mass equals 1.67×10^{-27}kg then gives:

$$M_U = \frac{(3 \times 10^8 \mathrm{m} \cdot \mathrm{s}^{-1})^3 (4.32 \times 10^{17}\mathrm{s})}{(6.67 \times 10^{-11}\mathrm{kg}^{-1} \cdot \mathrm{m}^3 \cdot \mathrm{s}^{-2})(1.67 \times 10^{-27}\mathrm{kg})} \approx$$

$$10^{80}\text{protons}.$$

Such a Machian Universe could be described by envisioning spacetime as a viscous fluid in which constituent particles move through naturally in the direction of time's arrow (*i.e.* $a(t)$). The Machian total energy condition also holds true if m and U_g both

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evolved based on the same parametric function, *i.e.* $m(f(a(t))) + U_g(f(a(t))) = 0$. In this instance, the viscosity of the spacetime fluid would lessen as time progressed, and every position in time would be tantamount to its own unique inertial frame referenced with respect to each other by time. Unnatural movements (accelerations/decelerations) would interrupt the natural flow and cause displacement into frames that correspond to the increase/decrease in constituent particle mass/energy. Substituting a potential that increases with time in place of the fluid description would then satisfy the requirements of Mach's principle.

Normally, the Friedmann equations are solved by specifying ρ in terms of a (*e.g.*, $\rho_m \propto a^{-3}$ or $\rho_\nu \propto a^{-4}$) which then gives $a = f(t)$. However, within a multiverse of almost unlimited possibilities, other relationships could be postulated if even on a purely mathematical basis. One such possibility would be to simply postulate $a \propto t$, which then requires the Hubble parameter $H = \dot{a}/a \rightarrow H = 1/t$. The $a \propto t$ postulate is thus readily falsifiable by recent measurements of H_0, specifically: $74.2^{+2.7}_{-3.0}$[11]; 76.8 ± 2.6[12]; 73.5 ± 1.4[13]; 69.8 ± 1.9[14–16]; $73.3^{+1.7}_{-1.8}$[17]; $70.3^{+5.3}_{-5.0}$[18]; $68.0^{+4.2}_{-4.1}$[19]; 74.03 ± 1.42[20]; $67.78^{+0.91}_{-0.87}$[21]; giving a mean $\approx 72\text{km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$. However, falsification does not necessarily occur because the Hubble time (t_H) of this mean value is:

$$\bar{t}_H = \frac{1}{\bar{H}_0} = \left(\frac{3.086\times10^{19}\text{km}\cdot\text{Mpc}^{-1}}{72\text{km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}}\right) = 4.3\times10^{17}\text{s},$$

which is very close to t_0 (*i.e.* 4.32×10^{17}s). (FRW cosmology attributes this coincidence to the Universe's expansion being close to the inflexion point where $\ddot{a}$ turns positive.) Applying the boundary condition $a_0 = 1$ to the postulated $a \propto t$ relationship then requires:

$$a(t) = \frac{t}{t_0} \rightarrow \left(H = \frac{1}{t}\right). \tag{1}$$

The acceptance of the multiverse by prominent cosmologists and the R-W not being unique in the same manner as is the Schwarzschild opens the door to alternative cosmologies. Based on the above discussion, there are three possible criteria that might feasibly be considered in formulating one possible alternative: 1) The spatial portion of the metric must remain unchanged; 2) The Machian total energy condition $m(f(a(t)))+U_g(f(a(t))) = 0$ must hold true; and 3) An age-based solution featuring (1) could be trialed.

II. AN ALTERNATIVE SOLUTION TO EINSTEIN'S FIELD EQUATIONS

A feature of evolving geometries such as the R-W is that it is the universal rest frame that evolves. Consequently, the R-W metric requires $d\tau = dt$. However, in a multiverse of possibilities a universe might be hypothesized in which $d\tau = a(t)dt$. This equation may resemble the Conformal-Time R-W, but in this instance no coordinate transformation has been employed. Rather, Einstein's field equations will be directly solved utilizing the following geometry as input:

$$ds^2 = a^2(t)(-dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)), \tag{2}$$

(in which $c = 1$). In addition, the Machian total energy condition $m(f(a(t))) + U_g(f(a(t))) = 0$ requires $T_{\mu\nu} = [-g_{00}\rho, g_{ij}p]$.

For reference the Einstein field equations are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Given the following metric:

$$g_{\mu\nu} = \begin{pmatrix} -a(t)^2 & & & \\ & a(t)^2 & & \\ & & a(t)^2r^2 & \\ & & & a(t)^2r^2sin^2\theta \end{pmatrix}, \tag{2}$$

and the corresponding rest frame e-m tensor:

$$T_{\mu\nu} = \begin{pmatrix} -g_{00}\rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix}, \tag{3}$$

the analogs to the Friedmann equations can be determined in the following manner.

First, calculate the lefthand (geometric) side of the Einstein equations. The Christoffel symbols (along with their symmetric counterparts) are:

$$\Gamma^0_{00} = \tfrac{\dot{a}}{a} \qquad \Gamma^1_{01} = \tfrac{\dot{a}}{a} \qquad \Gamma^2_{02} = \tfrac{\dot{a}}{a}$$

$$\Gamma^0_{11} = \tfrac{\dot{a}}{a} \qquad \Gamma^1_{22} = -r \qquad \Gamma^2_{12} = \tfrac{1}{r}$$

$$\Gamma^0_{22} = \tfrac{\dot{a}}{a}r^2$$

$$\Gamma^0_{33} = \tfrac{\dot{a}}{a}r^2sin^2\theta \quad \Gamma^1_{33} = -rsin^2\theta \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{03} = \tfrac{\dot{a}}{a} \qquad \Gamma^3_{13} = \tfrac{1}{r} \qquad \Gamma^3_{23} = cot\theta.$$

The pertinent Ricci tensors are:

$$R_{00} = -3\left[\tfrac{\ddot{a}}{a} - \left(\tfrac{\dot{a}}{a}\right)^2\right].$$

$$R_{11} = \left[\tfrac{\ddot{a}}{a} + \left(\tfrac{\dot{a}}{a}\right)^2\right].$$

$$R_{22} = \left[\tfrac{\ddot{a}}{a} + \left(\tfrac{\dot{a}}{a}\right)^2\right]r^2.$$

$$R_{33} = \left[\tfrac{\ddot{a}}{a} + \left(\tfrac{\dot{a}}{a}\right)^2\right]r^2sin^2\theta.$$

The Ricci scalar is:

$$R = 6\frac{\ddot{a}}{a^3}.$$

The lefthand side of the $_{00}$ Einstein equation is:

$$R_{00} - \frac{1}{2}Rg_{00} = -3\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] - \frac{1}{2}\left(6\frac{\ddot{a}}{a^3}\right)(-a^2) =$$

$$3\left(\frac{\dot{a}}{a}\right)^2.$$

The $_{00}$ Einstein equation is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2. \quad (4)$$

The lefthand side of the $_{11}$ Einstein equation is:

$$R_{11} - \frac{1}{2}Rg_{11} = \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] - \frac{1}{2}\left(6\frac{\ddot{a}}{a^3}\right)(a^2) =$$

$$-2\frac{\ddot{a}}{a} + \frac{8\pi G}{3}\rho a^2.$$

The $_{11}$ Einstein equation is:

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\rho - 3\frac{p}{c^2}\right)a^2. \quad (5)$$

Differentiating (4) and eliminating $\ddot{a}$ by inserting the result into (5) gives the conformal-time version of the fluid equation:

$$(\dot{a})^2 = \frac{8\pi G}{3}\rho a^4 \to 2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^4 + 4\rho a^3\dot{a})$$

$$\to 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\dot{a}\left(\rho - 3\frac{p}{c^2}\right)a^3 = \frac{8\pi G}{3}(\dot{\rho}a^4 + 4\rho a^3\dot{a})$$

$$\to \rho\dot{a} - 3\frac{p}{c^2}\dot{a} = \dot{\rho}a + 4\rho\dot{a}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0, \quad (6)$$

whose basic form is unchanged. (*Math in this section has been independently verified.*)

III. A SIMPLE SOLUTION TO THE ANALOG FRIEDMANN EQUATIONS

(1) trivially solves (4) and gives:

$$\rho \propto a^{-4}. \quad (7)$$

Similarly, the expression in parenthesis in (5) equals the trace of the stress-energy tensor. Consequently, the (1) solution results in (5) resembling an equation of state appropriate to an energy density proportional to a^{-4}. Not surprisingly, combining the reduced (5) (*i.e.* $p/\rho c^2 = 1/3$) with (6) also gives (7):

$$\rho' + 3\frac{a'}{a}\left(\rho + \frac{1}{3}\rho\right) = 0 \to \left[\frac{\rho'}{\rho} = -4\frac{a'}{a}\right] \to \rho \propto a^{-4}. \quad (7)$$

Hence, the (1) solution effectively consolidates the principal equations of a Conformal-Time cosmology into (7).

It should be reiterated that (7) was attained by specifying $a \propto t$, *i.e.*, (1), not by specifying $\rho_m \propto a^{-n}$. Consequently, in this alternativie $\rho \propto a^{-4}$ for energy in general (which then requires):

$$\rho_m \propto \frac{1}{a^4(t)} \to m \propto \frac{1}{a(t)} \to U_g \propto -\frac{1}{a(t)}. \quad (8)$$

With regard to the third term in (8), if this were otherwise the total Machian energy conservation condition $m(f(a(t))) + U_g(f(a(t))) = 0$ throughout the expansion would not hold true.

IV. A POSSIBLE RESOLUTION TO THE Λ DISCREPANCY

Since a sign change occurs in the analog to the Friedmann 2 equation (5), it would be reasonable to conclude that the cosmological constant (Λ) discrepancy is, in some manner, rectified by these equations. Differentiating (8) with respect to a leads to an expression for an evolving gravitational field:

$$F(a) = \frac{\mathrm{d}U_g}{\mathrm{d}a} \propto \frac{1}{a^2}, \quad (9a)$$

and since a and t are interchangeable, (9a) can be expressed as a ratio between the Planck era (t_P) and t_0:

$$\frac{\left.\frac{\mathrm{d}U_g}{\mathrm{d}t}\right|_{t_P}}{\left.\frac{\mathrm{d}U_g}{\mathrm{d}t}\right|_{t_0}} = \left(\frac{t_0}{t_P}\right)^2 = 10^{120}, \quad (9b)$$

resulting in gravity diminishing by a factor of 10^{-120} in the interval t_P to t_0.

(9b) provides a possible explanation for the elimination of the Λ discrepancy, particularly if a relationship can be established between it and the equation for Ω_{Λ_Z}. The prime candidate for Λ is the ZPE of the quantum vacuum. E_{vac} is generally obtained by calculating the number of allowable waveforms in a maximum-sized (*i.e.*, $a = a_0$) cosmological box. The following equation for ρ_{vac} is taken from Cheng:[22]

$$\rho_{vac} = \frac{E_{vac}}{Vc^2} = \left(\frac{1}{16\pi^2}\frac{E_P^4}{\hbar^3 c^5}\right). \tag{10}$$

(4) and (1) together require ρ_c at a_0 to be:

$$\rho_{c_0} = \frac{3{H_0}^2}{8\pi G{a_0}^2} = \frac{3}{8\pi G{a_0}^2{t_0}^2}. \tag{11}$$

Ω_{Λ_Z} can be obtained using (10) and (11):

$$\Omega_{\Lambda_Z} = \frac{\rho_{vac}}{\rho_{c_0}} = \left(\frac{1}{16\pi^2}\frac{E_P^4}{\hbar^3 c^5}\right)\left(\frac{8\pi G{a_0}^2{t_0}^2}{3}\right). \tag{12a}$$

Converting E_P by substituting $(\hbar c^5/G)^{1/2}$, then reducing to t_P gives:

$$\Omega_{\Lambda_Z} = \frac{1}{6\pi}\left(\frac{t_0}{t_P}\right)^2 \approx \left(\frac{t_0}{t_P}\right)^2 = 10^{120}, \tag{12b}$$

which is the same expression as (9b) (which originates from the Friedmann analogs).

V. CONCLUSION

Inquiry into the possible reconciliation of the 10^{120} number is not limited exclusively to quantum vacuum theory; alternative Einstein-based, non-FRW universes might also be explored. This course of inquiry is legitimate because (unlike the Schwarzschild) there is no uniqueness theorem prohibiting such opportunities. A metric in which the time coordinate of the spacetime line element behaves as $a(t)\mathrm{d}t$ is one such possibility. The trivial solution $a = t/t_0$ corresponds to the $t_H \approx t_0$ coincidence. The result is a simplified Einstein evolving gravity model in which the Machian total energy condition $m(a(t)) + U_g(a(t)) = 0$ is realized, which then leads to the recovery of the 10^{120} number. Perhaps even more revealing is that (9b) and (12b), which originate separately from the two pillars of modern physics, essentially reduce to the same expression.

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Appendix A (8) AND GALAXY ROTATIONAL CURVES

Observational data has revealed that galaxy rotational velocities (v) do not experience the predicted Keplerian fall-off but rather become constant with increasing r. The prevailing theory is that halos of Dark Matter surround galaxies providing the additional mass necessary to support these increased velocities. However, other theories such as Milgrom's MOdified Newtonian Dynamics (MOND) modify gravity in order to achieve similar results. (MOND has been shown to be highly accurate in its ability to predict galaxy rotations.)[23]

The expansion (a) is thought to occur between galaxies but not within galaxies (which are gravitationally bound). However, there are no papers that clearly demark just where and how the transition from full expansion to zero expansion ($a \to 0$) occurs. Additionally, Milgrom noted that while Newton's laws have been extensively tested in high-acceleration environments (in the Solar System and on Earth), they have not been verified for objects with extremely low accelerations, such as stars in the outer parts of galaxies.[24] Thus, if the expansion is allowed to seep into the low acceleration regions of galaxies then (8) can be utilized, ($m \propto -U_g \propto a(t)^{-1}$). Consequently, when the acceleration is high (*i.e.*, when the center of mass frame and the orbital frame both reside in the $a = 0$ region) the m that occurs on both sides of:

$$m\frac{v^2}{r} = \frac{GMm}{r^2}$$

is the same, and the expected Newtonian result occurs. However, when comparing the center of mass frame ($a = 0$) where the acceleration (a) is very high to a frame at the galactic extremity ($a > 0$) where the acceleration is very very low and where (8) might apply, the solution for galaxy rotational curves becomes:

$$m_\mathrm{l} = \left[\frac{GMm_\mathrm{h}}{v^2}\right]\frac{1}{r} \to m_\mathrm{l} \propto \frac{1}{a(t)},\ (i.e.\ (8)) \tag{A.1}$$

in which v is constant and in which M and m_h are also constant ($a = 0$ frame). Thus, if the expansion were allowed to occur in the low acceleration region of galaxies, then (8) predicts that galaxy rotational curves will flatten. Substituting a/a_0 for $m_\mathrm{l}/m_\mathrm{h}$ in (A.1) then gives Milgrom's equation $v = \sqrt[4]{GM\mathsf{a}_0}$. (For reference, Milgrom's threshold constant a_0 is equivalent to the acceleration of an object with a solar orbit approximately 40× that of Pluto.) This solution is applicable to galaxies which by nature have strong gravitational sources (black holes) that serve as references where (in accordance with (1) and (8) $t \to 0$ as $U_g \to -\infty$. It also explains why MOND does not work for galactic clusters in which no such singular center of mass reference is available.

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