Comments on: A new additive decomposition of velocity gradient, by B. Sun [Phys. Fluids 31, 061702 (2019)]

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Abstract

Comments on "A new additive decomposition of velocity gradient [Phys. Fluids 31, 061702 (2019)]" is presented.

1 Introduction

The Cauchy-Stokes decomposition of the velocity gradient tensor into a symmetric strain rate tensor $\mathbf{D}$ and an anti-symmetric spin tensor $\mathbf{W}$ is well-known (Kundu and Cohen 2002).

$$\nabla\mathbf{u} = \mathbf{D} + \mathbf{W}$$

The spin tensor $\mathbf{W}$ is the tensor representation of the vorticity $2\boldsymbol{\omega}$, where $\mathbf{W}_{ij} = -\varepsilon_{ijk}\omega_k$ (ε_{ijk} is the permutation tensor). In the quest of finding characteristics that define a vortex, the vorticity field has been found lacking due to various reasons (Epps 2017). An interesting, alternative proposition of a novel decomposition of the velocity gradient tensor is presented by Sun (2019) based on the Lie algebra of the special orthonormal Lie group $SO(3)$. This decomposes the velocity gradient tensor into a component which is a rotation tensor instead of the usual spin tensor. Sun (2019) had noted that a deeper significance of this decomposition is not yet clear and further investigations are necessary in that direction. The comments here are intended to interpret and rectify some of the aspects of Sun (2019).

Sun (2019) decomposes the velocity gradient tensor as,

$$\nabla\mathbf{u} = \mathbf{K} + \mathbf{Q} \tag{1}$$

where $\mathbf{Q} \in SO(3)$ is a rotation tensor and $\mathbf{K}$ is the residual. One of questions raised by Sun (2019) is under what condition(s) $\mathbf{K}$ be symmetric? From the arguments of Sun (2019), $\mathbf{K}$ can be symmetric only for vortical flows with vanishing vorticity ($\omega \to 0$). It will, however, be shown here that it is impossible for $\mathbf{K}$ to be symmetric in a flow with vorticity. The difference in inferences occur due to disregard of a fundamental property of Lie algebras and Lie groups by Sun (2019).

Anti-symmetric tensors like $\mathbf{W}$ belong to the Lie algebra ($so(3)$) of the Lie group $SO(3)$. There exists an exponential map from $so(3) \to SO(3)$. Exploiting this, Sun (2019) expresses a rotation matrix $\mathbf{Q} \in SO(3)$ as,

$$\mathbf{Q} = e^{\mathbf{W}} \tag{2}$$

First, Sun (2019) does not address the issue of dimensional inconsistency in eq.(2). Physical dimension of $\mathbf{W}$ is sec^{-1} - there are obvious problems to exponentiate a dimensional quantity. It is unclear if the all the physical quantities are non-dimensional. Second, presuming that $\nabla\mathbf{u}, \mathbf{D}$ and $\mathbf{W}$ are non-dimensional right from the outset, there is a more basic problem with eq.(2). $\mathbf{Q}$ in eq.(2) does not represent a *one-parameter subgroup* of $SO(3)$ in the neighborhood of $\mathbf{I}$ (Hall 2015; Fegan 1991), where $\mathbf{I}$ is identity element of $SO(3)$. This is an essential requirement for an isomorphism from $so(3)$ to $SO(3)$ in the neighborhood of identity. The second objection is fundamental because fixing it will fix the first objection seamlessly, and not vice-versa. In other words, even if all the quantities were dimensionless, eq.(2) still do not represent a one-parameter subgroup of $SO(3)$. The basis of this rectification is discussed in the next section and implication of this decomposition is presented.

2 Discussion and Conclusion

A Lie group, such as the $SO(3)$, has the structure of a differentiable manifold in the vector space of real matrices. On any integral curve induced by the tangent tensor field like $\mathbf{W}$ on $SO(3)$, the following hold (Hall 2015) in the neighborhood of $\mathbf{I}$,

$$\frac{d\sigma(\tau)}{d\tau} = \mathbf{W} \tag{3}$$

where τ is the parameter in the map $\sigma : I_R \to SO(3)$, with $\tau \in I_R = [a,b] \in \mathbb{R}$ and I_R contains 0 ($a,b \in \mathbb{R}$). τ might be interpreted as the time increment/decrement, $t - t_0$, where $\sigma(0) = \mathbf{I}$. Along the integral curve $\sigma(\tau) \in SO(3)$, eq.(3) demands,

$$\sigma(\tau) = \mathbf{Q}(\tau) = e^{\mathbf{W}\tau} \tag{4}$$

Equation(4) would describe a family of rotations parameterised by τ: a one parameter subgroup of $SO(3)$. Equation(4) can also be derived by a much simpler consideration of the rotation matrix $\mathbf{Q}(\tau)$. Time derivative of $\mathbf{Q}\mathbf{Q}^T$ $(= \mathbf{I})$ is

$$\frac{d(\mathbf{Q}\mathbf{Q}^T)}{d\tau} = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \dot{\mathbf{I}} = \mathbf{0} \tag{5}$$

where $\dot{\mathbf{Q}}, \dot{\mathbf{I}}$ denotes time derivative of $\mathbf{Q}$ and $\mathbf{I}$ respectively. From eq.(5), it is obvious that $\dot{\mathbf{Q}}\mathbf{Q}^T$ is anti-symmetric. Thus for any $\mathbf{Q}$, there always exists a $\dot{\mathbf{Q}}$ such that,

$$\dot{\mathbf{Q}} = \mathbf{W}\mathbf{Q} \tag{6}$$

This tensorial differential equation is equivalent to eq.(3), and the following satifies eq.(6),

$$\mathbf{Q}(\tau) = e^{\mathbf{W}\tau}\mathbf{Q}_0 \tag{7}$$

Consider the integral curve through the identity with $\mathbf{Q}_0 = \mathbf{I}$ and eq.(7) reduces to eq.(4), reiterating the fact that $\mathbf{Q}(\tau)$ is a one-parameter sub-group of $SO(3)$ near $\mathbf{I}$. This is mathematically and dimensionally a more consistent exponential map from $so(3) \to SO(3)$ than eq.(2). If $\mathbf{W}$ is independent of time, there is no restriction on τ in eq.(4), and eq.(2) is recovered only for a special case of $\tau = 1$. But, in a generic fluid flow field, $\mathbf{W}$ must be a function of time for a material fluid parcel. Therefore, this limits the validity of eq.(4) to $|\tau| \to 0$.

Based on this modified exponential map in eq.(4), the Rodrigues' formula used by Sun (2019) (eq.16 of the paper) is valid only for $\tau = 1$. The complete Rodrigues' formula for $\mathbf{Q}$ is,

$$\mathbf{Q} = \mathbf{I} + \frac{\sin \omega\tau}{\omega}\mathbf{W} + \frac{1-\cos \omega\tau}{\omega^2}\mathbf{W}^2$$

And, if decomposition (1) for a non-dimensional $\nabla \mathbf{u}$ is demanded with the requirement that $\mathbf{K}$ be symmetric, then the following has to be true,

$$\left(1 - \frac{\sin \omega\tau}{\omega}\right)\mathbf{W} = \mathbf{0} \tag{8}$$

Equation 8 can be satisfied for any allowable τ ($|\tau| \to 0$), if and only if $\mathbf{W} = 0$ identically. Thus, $\mathbf{K}$ can never be symmetric in a vortical flow. This is in distinction to the possibility of a symmetric $\mathbf{K}$ from Sun's exposition, where symmetric $\mathbf{K}$ is allowable for vortical flows with $\omega \to 0$. It is clear that incorrect use of the transformation from the Lie algebra to Lie group, $so(3) \to SO(3)$ (eq.(2)) is the source of this discrepancy.

For a turbulent flow, as mentioned earlier, $\mathbf{W}$ would have erratic dependence on time, and the exponentiation map would be valid for infinitesimal time duration, i.e., $|\tau| \to 0$. In that limit, $\mathbf{Q} = \mathbf{I}$ and $\mathbf{K} = \mathbf{D} + \mathbf{W} - \mathbf{I}$, severely restricting the applicability of this new decomposition of the velocity gradient tensor.

References

[1] Brenden Epps. "Review of Vortex Identification Methods". In: *55th AIAA Aerospace Sciences Meeting*. 2017. DOI: 10.2514/6.2017-0989. eprint: https://arc.aiaa.org/doi/pdf/10.2514/6.2017-0989. URL: https://arc.aiaa.org/doi/abs/10.2514/6.2017-0989 (cit. on p. 1).

[2] Howard D Fegan. *Introduction to Compact Lie Groups*. WORLD SCIENTIFIC, 1991. DOI: 10.1142/1436. eprint: https://www.worldscientific.com/doi/pdf/10.1142/1436. URL: https://www.worldscientific.com/doi/abs/10.1142/1436 (cit. on p. 2).

[3] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations*. Second. Graduate Texts in Mathematics. Springer International Publishing, 2015. DOI: 10.1007/978-3-319-13467-3 (cit. on p. 2).

[4] P. K. Kundu and I. M. Cohen. *Fluid Mechanics*. Second. Academic Press, U.S.A., 2002 (cit. on p. 1).

[5] Bohua Sun. "A new additive decomposition of velocity gradient". In: *Physics of Fluids* 31.6 (2019), p. 061702. DOI: 10.1063/1.5100872. eprint: https://doi.org/10.1063/1.5100872. URL: https://doi.org/10.1063/1.5100872 (cit. on pp. 1–3).