Lerch’s Φ and the Polylogarithm at the Positive Integers

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Abstract

This is a final treatise on generalized harmonic progressions, with an overview of all the formulae we created for them and the closed-forms of their associated partial Fourier sums. We derive an asymptotic expression for \( \sum_{j=1}^{n} \frac{1}{j+b} \) (where \( 2b \) is a non-integer complex number) as a way to obtain formulae for the full Fourier series (if \( b \) is such that \( |b| < 1 \), we get a surprising pattern), \( \sum_{j=1}^{n} \frac{1}{j+b} \sim -1/(2b) + \pi(cot 2\pi b + \csc 2\pi b)/2 + H(n) + \sum_{k=1}^{\infty} \zeta(2k+1)b^{2k} \). Finally, we use the found Fourier series formulae to obtain the values of the Lerch transcendent function, \( \Phi(e^{m}, k, b) \), and by extension the polylogarithm, \( Li_k(e^{m}) \), at the positive integers \( k \).

Summary

1 Introduction

2 The partial Fourier sums
   2.1 \( C_{2k}^{m}(a, b, n) \) and \( S_{2k+1}^{m}(a, b, n) \) .................................................................................. 3
   2.1.1 Limits of \( C_{2k}^{m}(n) \) and \( S_{2k+1}^{m}(n) \) .................................................................................. 3
   2.2 \( C_{2k+1}^{m}(a, b, n) \) and \( S_{2k}^{m}(a, b, n) \) .................................................................................. 4
   2.2.1 Limits of \( C_{2k+1}^{m}(n) \) and \( S_{2k}^{m}(n) \) .................................................................................. 4

3 Overview of the various \( HP_k(n) \) formulae
   3.1 Integer \( a \) and \( b \) ............................................................................................................................... 5
       3.1.1 Based on sine ................................................................................................................................. 5
       3.1.2 Based on exponential ...................................................................................................................... 5
   3.2 Complex \( b \) with \( b \notin \mathbb{Z} \) .................................................................................................................... 6
   3.3 Complex \( b \) with \( 2b \notin \mathbb{Z} \) .................................................................................................................... 6
       3.3.1 Based on cosine .............................................................................................................................. 6
       3.3.2 Based on sine ............................................................................................................................... 7

4 Limits of the integrals ................................................................................................................................. 7

5 \( HP(n) \) asymptotic behavior ..................................................................................................................... 8

6 The full Fourier series .................................................................................................................................. 10
1 Introduction

In [3] we derived expressions for the partial Fourier sums, \( C_m^k(a, b, n) \) and \( S_m^k(a, b, n) \), which we reproduce again in the next section, with a short description.

Our objective in this paper is to obtain the limit of those expressions as \( n \) gets large, and then combine them to obtain the Lerch transcendent function, \( \Phi \), at the positive integers. They are a function of the harmonic progressions of order \( k^3, 4 \):

\[
HP_k(n) = \sum_{j=1}^{n} \frac{1}{(an + b)^k},
\]

of which the harmonic numbers of order \( k \), \( H_k(n) \), are particular cases (when \( b = 0 \)). Without loss of generality, let’s set \( a = 1 \) to simplify the calculations:

\[
C_k^m(b, n) = \sum_{j=1}^{n} \frac{1}{(j + b)^k} \cos \frac{2\pi(j + b)}{m} \quad \text{and} \quad S_k^m(b, n) = \sum_{j=1}^{n} \frac{1}{(j + b)^k} \sin \frac{2\pi(j + b)}{m}
\]

In the process, we need to obtain the limit of \( HP(n) - H(n) \), with \( 2b \) a non-integer complex number. Since this limit can also be attained by means of the digamma function, \( \psi(n) \), this is just a new, more interesting way of deriving that limit.

In section (4), we review the limits of the integrals that appear in the expressions of \( HP_k(n) \) as \( n \) tends to infinity, which are central to this solution.

The process of obtaining the limits of \( C_{2k}^m(b, n) \) and \( S_{2k+1}^m(b, n) \) is much simpler than that of \( C_{2k+1}^m(b, n) \) and \( S_{2k}^m(b, n) \), since the latter involve the limit of \( HP(n) \), which is not finite.

2 The partial Fourier sums

The subsequent expressions are the partial sums of the Fourier series associated with the generalized harmonic progressions from [3], and hold for all complex \( m, a \) and \( b \) and for all integer \( n \geq 1 \).
By definition $HP_0(n) = 0$ for all positive integer $n$, so they actually have no effect in the sums. If $b = 0$, we can discard any term that has a null denominator and the equation still holds (technically, we take the limit as $b$ tends to 0, as we’ll see in section (7.1.1)).

2.1 $C_{2k}^m(a, b, n)$ and $S_{2k+1}^m(a, b, n)$

For all integer $k \geq 1$:

$$
\sum_{j=1}^{n} \frac{1}{(aj + b)^{2k}} \cos \frac{2\pi(aj + b)}{m} = -\frac{1}{2b^{2k}} \left( \cos \frac{2\pi b}{m} - \sum_{j=0}^{k} \frac{(-1)^j}{(2j)!} \left( \frac{2\pi b}{m} \right)^{2j} \right)
$$

$$
+ \frac{1}{2(an + b)^{2k}} \left( \cos \frac{2\pi(an + b)}{m} - \sum_{j=0}^{k} \frac{(-1)^j}{(2j)!} \left( \frac{2\pi(an + b)}{m} \right)^{2j} \right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(2k - 2j)!} \left( \frac{2\pi}{m} \right)^{2k-2j} HP_{2j}(n)
$$

$$
+ \frac{(-1)^k}{2(2k - 1)!} \left( \frac{2\pi}{m} \right)^{2k} \int_{0}^{1} (1 - u)^{2k-1} \left( \sin \frac{2\pi(an + b)u}{m} - \sin \frac{2\pi bu}{m} \right) \cot \frac{\pi au}{m} du
$$

For all integer $k \geq 0$:

$$
\sum_{j=1}^{n} \frac{1}{(aj + b)^{2k+1}} \sin \frac{2\pi(aj + b)}{m} = -\frac{1}{2b^{2k+1}} \left( \sin \frac{2\pi b}{m} - \sum_{j=0}^{k} \frac{(-1)^j}{(2j + 1)!} \left( \frac{2\pi b}{m} \right)^{2j+1} \right)
$$

$$
+ \frac{1}{2(an + b)^{2k+1}} \left( \sin \frac{2\pi(an + b)}{m} - \sum_{j=0}^{k} \frac{(-1)^j}{(2j + 1)!} \left( \frac{2\pi(an + b)}{m} \right)^{2j+1} \right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(2k + 1 - 2j)!} \left( \frac{2\pi}{m} \right)^{2k+1-2j} HP_{2j}(n)
$$

$$
+ \frac{(-1)^k}{2(2k)!} \left( \frac{2\pi}{m} \right)^{2k+1} \int_{0}^{1} (1 - u)^{2k} \left( \sin \frac{2\pi(an + b)u}{m} - \sin \frac{2\pi bu}{m} \right) \cot \frac{\pi au}{m} du
$$

2.1.1 Limits of $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$

For comparison purposes, let’s review some limits that we derived previously for the particular cases $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$ (that is, $a = 1$ and $b = 0$). We expect the limits of the more general expressions to coincide with them.

At infinity these particular cases become Fourier series (denoted here by $C_{2k}^m$ and $S_{2k+1}^m$), which have limits given by:

$$
C_{2k}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k}} \cos \frac{2\pi j}{m} = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(2k - 2j)!} \left( \frac{2\pi}{m} \right)^{2k-2j} \zeta(2j) + \frac{(-1)^k}{4(2k - 1)!} \left( \frac{2\pi}{m} \right)^{2k} \quad (\forall \text{ integer } k \geq 1)
$$

$$
S_{2k+1}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k+1}} \sin \frac{2\pi j}{m} = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(2k + 1 - 2j)!} \left( \frac{2\pi}{m} \right)^{2k+1-2j} \zeta(2j) + \frac{(-1)^k}{4(2k)!} \left( \frac{2\pi}{m} \right)^{2k+1} \quad (\forall \text{ integer } k \geq 0)
$$

These limits only hold for real $|m| \geq 1$ ($k = 0$ and $|m| = 1$ are exceptions and also trivial cases). So for $S_{1}^1 = 0$ the formula breaks down (see section (4) to know why). Both these results are known in the literature, they’re rewrites of equations that feature in [1] (page 805).
2.2 \( C_{2k+1}^m(a, b, n) \) and \( S_{2k}^m(a, b, n) \)

For all integer \( k \geq 0 \):

\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} \cos \frac{2\pi (aj+b)}{m} = -\frac{1}{2(2k+1)} \left( \cos \frac{2\pi b}{m} - \sum_{j=0}^{k} \frac{(-1)^j}{(2j)!} \left( \frac{2\pi b}{m} \right)^{2j+1} \right) + \frac{1}{2(2k+1)} \left( \cos \frac{2\pi (an+b)}{m} - \sum_{j=0}^{k} \frac{(-1)^j}{(2j)!} \left( \frac{2\pi (an+b)}{m} \right)^{2j+1} \right) + \frac{1}{2(2k+1)} \left( \frac{2\pi}{m} \right)^{2k+1} \int_{0}^{1} (1-u)^{2k} \left( \cos \frac{2\pi (an+b)}{m} - \cos \frac{2\pi bu}{m} \right) \cot \frac{\pi au}{m} \, du
\]

For all integer \( k \geq 1 \):

\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} \sin \frac{2\pi (aj+b)}{m} = -\frac{1}{2(2k)} \left( \sin \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j)!} \left( \frac{2\pi b}{m} \right)^{2j+1} \right) + \frac{1}{2(2k)} \left( \sin \frac{2\pi (an+b)}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j)!} \left( \frac{2\pi (an+b)}{m} \right)^{2j+1} \right) - \frac{1}{2(2k)} \left( \frac{2\pi}{m} \right)^{2k-1} \int_{0}^{1} (1-u)^{2k-1} \left( \cos \frac{2\pi (an+b)}{m} - \cos \frac{2\pi bu}{m} \right) \cot \frac{\pi au}{m} \, du
\]

2.2.1 Limits of \( C_{2k+1}^m(n) \) and \( S_{2k}^m(n) \)

The limits of \( C_{2k+1}^m(n) \) and \( S_{2k}^m(n) \) for real \(|m| \geq 1\) are given by:

\[
C_{2k+1}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k+1}} \cos \frac{2\pi j}{m} = \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k-2j)!} \left( \frac{2\pi}{m} \right)^{2k-2j} \zeta(2j+1) + \frac{(-1)^{k}}{(2k)!} \left( \frac{2\pi}{m} \right)^{2k} \log |m| - \frac{(-1)^{k}}{2(2k)!} \left( \frac{2\pi}{m} \right)^{2k+1} \int_{0}^{1} (1-u)^{2k} \cot \frac{\pi u}{m} - m(1-u) \cot \pi u \, du \quad (\forall \text{ integer } k \geq 0)
\]

\[
S_{2k}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k}} \sin \frac{2\pi j}{m} = -\sum_{j=1}^{k-1} \frac{(-1)^{k-j}}{(2k-1-2j)!} \left( \frac{2\pi}{m} \right)^{2k-1-2j} \zeta(2j+1) + \frac{(-1)^{k}}{(2k-1)!} \left( \frac{2\pi}{m} \right)^{2k-1} \log |m| + \frac{(-1)^{k}}{2(2k-1)!} \left( \frac{2\pi}{m} \right)^{2k} \int_{0}^{1} (1-u)^{2k-1} \cot \frac{\pi u}{m} - m(1-u) \cot \pi u \, du \quad (\forall \text{ integer } k \geq 1)
\]

The exception is \( C_{1}^{1} = \infty \), since integral \( \int_{0}^{1} \cot \pi u - (1-u) \cot \pi u \, du \) diverges, which means that \( H(n) \) diverges. These results are probably original.

3 Overview of the various \( HP_{k}(n) \) formulae

Here we revisit the various results obtained in previous papers with different approaches and introduce new ones.
3.1 Integer $a$ and $b$

When $a$ and $b$ are integers, we have two possible approaches, one of them new.

3.1.1 Based on sine

As seen in [3], using equation $\sin 2\pi (aj + b) = 0$ we can obtain the following relations, which hold for all integers $a \neq 0$, $b$ and $k \geq 0$. For the even and odd cases, respectively, we have:

$$\sum_{j=1}^{n} \frac{1}{(aj + b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(an + b)^{2k}}$$

$$\frac{(-1)^k (2\pi)^{2k}}{2} \int_0^{1} \sum_{j=0}^{k} \frac{B_{2j} (2 - 2^{2j}) (1 - u)^{2k-2j}}{(2j)! (2k-2j)!} \frac{(\sin 2\pi (an + b)u - \sin 2\pi bu)}{\cot \pi au} du,$$

and

$$\sum_{j=1}^{n} \frac{1}{(aj + b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an + b)^{2k+1}}$$

$$\frac{(-1)^k (2\pi)^{2k+1}}{2} \int_0^{1} \sum_{j=0}^{k} \frac{B_{2j} (2 - 2^{2j}) (1 - u)^{2k+1-2j}}{(2j)! (2k+1-2j)!} \frac{(\cos 2\pi (an + b)u - \cos 2\pi bu)}{\cot \pi au} du,$$

The polynomials in $u$ within the integrand are generated by the functions:

$$f(x) = \frac{x \cos x (1-u)}{\sin x} \Rightarrow f^{(2k)}(0) = (-1)^k \sum_{j=0}^{k} \frac{B_{2j} (2 - 2^{2j}) (1 - u)^{2k-2j}}{(2j)! (2k-2j)!},$$

and

$$f(x) = \frac{x \sin x (1-u)}{\sin x} \Rightarrow f^{(2k+1)}(0) = (-1)^k \sum_{j=0}^{k} \frac{B_{2j} (2 - 2^{2j}) (1 - u)^{2k+1-2j}}{(2j)! (2k+1-2j)!}.$$

As mentioned before, if we discard any term of the equation that has a zero denominator, the equation still holds.

3.1.2 Based on exponential

This approach is new and consists in using equation $e^{2\pi i (aj + b)} = 1$. It’s even better than the sine-based approach, since it yields a single formula for both odd and even powers. If $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{(aj + b)^k} = -\frac{1}{2b^k} + \frac{1}{2(an + b)^k}$$

$$+ \frac{i (2\pi)^k}{2} \int_0^{1} \sum_{j=0}^{k} \frac{B_{j} (1 - u)^{k-j}}{j! (k-j)!} \frac{(e^{2\pi i (an+b)u} - e^{2\pi bu})}{\cot \pi au} du,$$
where the generating function of the polynomial in $u$ within the integral is:

$$f(x) = \frac{x e^{x(1-u)}}{e^x - 1} \Rightarrow \frac{f^{(k)}(0)}{k!} = \sum_{j=0}^{k} \frac{B_j (1 - u)^{k-j}}{j!(k-j)!}$$

### 3.2 Complex $b$ with $b \notin \mathbb{Z}$

First, let’s introduce the Lerch transcendent function, $\Phi$, and the Hurwitz zeta function, both of which are generalizations of the Riemann zeta function:

$$\Phi(z, s, \alpha) = \sum_{j=0}^{\infty} \frac{z^j}{(j + \alpha)^s}, \text{ and } \zeta(s, \alpha) = \sum_{j=0}^{\infty} \frac{1}{(j + \alpha)^s}$$

Note that $\zeta(s) = \Phi(1, s, 1)$. By the end of this paper we will find out the closed-form of $\Phi(e^m, k, b)$.

Per paper [4], if we use equation $e^{2\pi i(j+b)} = e^{2\pi b}$, we are able to derive the following identity, if $b \notin \mathbb{Z}$:

$$\sum_{j=1}^{n} \frac{1}{(j + b)^k} = -\frac{1}{2b^k} + \frac{1}{2(n+b)^k} + (2\pi i)^k \int_{0}^{1} \sum_{j=1}^{k} \frac{e^{-2\pi ib} \Phi(e^{-2\pi ib}, -j + 1, 0) (1-u)^{k-j}}{(j-1)!(k-j)!} e^{\pi i u (n+2b)} \sin \pi nu \cot \pi u du,$$

with associated generating function:

$$f(x) = -\frac{ixe^{ix(1-u)}}{e^{ix} - e^{2\pi ib}} \Rightarrow \frac{f^{(k)}(0)}{k!} = i^{k} \sum_{j=1}^{k} \frac{e^{-2\pi ib} \Phi(e^{-2\pi ib}, -j + 1, 0) (1-u)^{k-j}}{(j-1)!(k-j)!}$$

### 3.3 Complex $b$ with $2b \notin \mathbb{Z}$

If, however, we use an approach based on $\cos 2\pi(j + b) = \cos 2\pi b$ and $\sin 2\pi(j + b) = \sin 2\pi b$ as starting points, we can obtain new equations.

This solution will be detailed in a subsequent paper, and only two of the possibilities are presented next. The formulae derived with this approach usually break for integer and half-integer $b$.

#### 3.3.1 Based on cosine

Let $f(x)$ and $g(x)$ be the below functions:

$$f(x) = \frac{x \cos x(1-u)}{\cos x - \cos 2\pi b} \text{ and } g(x) = \sin x \frac{x \cos x(1-u)}{\cos x - \cos 2\pi b}$$
Then, for complex \( b \):

\[
\sum_{j=1}^{n} \frac{1}{(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(n+b)^{2k+1}} - \frac{(2\pi)^{2k+1}}{2} \int_{0}^{1} \frac{f^{(2k+1)}(0)}{(2k+1)!} \left( \cos 2\pi(n+b)u - \cos 2\pi bu \right) \cot \pi u \, du,
\]

\[
\sum_{j=1}^{n} \frac{1}{(j+b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(n+b)^{2k}} - \frac{(2\pi)^{2k}}{2 \sin 2\pi b} \int_{0}^{1} \frac{(-1)^k (1-u)^{2k-1}}{(2k-1)!} + \frac{g^{(2k)}(0)}{(2k)!} \left( \cos 2\pi(n+b)u - \cos 2\pi bu \right) \cot \pi u \, du,
\]

3.3.2 Based on sine

Now let \( f(x) \) and \( g(x) \) be the below functions:

\[
f(x) = \frac{x \sin x(1-u)}{\cos x - \cos 2\pi b} \quad \text{and} \quad g(x) = \frac{x \sin x(1-u)}{\cos x - \cos 2\pi b}
\]

Then, for complex \( b \):

\[
\sum_{j=1}^{n} \frac{1}{(j+b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(n+b)^{2k}} + \frac{(2\pi)^{2k}}{2} \int_{0}^{1} \frac{f^{(2k)}(0)}{(2k)!} \left( \sin 2\pi(n+b)u - \sin 2\pi bu \right) \ cot \pi u \, du,
\]

\[
\sum_{j=1}^{n} \frac{1}{(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(n+b)^{2k+1}} + \frac{(2\pi)^{2k+1}}{2 \sin 2\pi b} \int_{0}^{1} \frac{(-1)^k (1-u)^{2k}}{(2k+1)!} + \frac{g^{(2k+1)}(0)}{(2k+1)!} \left( \sin 2\pi(n+b)u - \sin 2\pi bu \right) \cot \pi u \, du
\]

4 Limits of the integrals

In [2] we introduced the following theorems, whose validity we now fully extend. For all real \( k \geq 0 \) and real \( m \):

**Theorem 1** \( \lim_{n \to \infty} \int_{0}^{1} u^k \sin \frac{2\pi n(1-u)}{m} \cot \frac{\pi(1-u)}{m} \, du = \begin{cases} 1, & \text{if } k = 0 \text{ and } |m| = 1 \\ |\frac{m}{2}|, & \text{if } |m| \geq 1 \end{cases} \)

Another result we need is in the following theorem, which holds for all real \( k \geq 0 \) and real \( |m| \geq 1 \) (except \( k = 0 \) and \( |m| = 1 \), for which the integral doesn’t converge):

**Theorem 2** \( \lim_{n \to \infty} \int_{0}^{1} u^k \cos \frac{2\pi n(1-u)}{m} \cot \frac{\pi(1-u)}{m} - m u \cos 2\pi n(1-u) \cot \pi(1-u) \, du = \frac{m \log |m|}{\pi} \)
We don’t provide a proof for these results due to the scope, but they should be simple.

Though these limits shouldn’t converge for non-real complex $m$, when they are linearly combined like $l_1 + il_2$ their infinities cancel out giving a finite value. This property is what allows our final formula from section (7.2) to converge always, even when the parameters are not real.

5 $HP(n)$ asymptotic behavior

Here we figure out the relationship between $HP(n)$ and $H(n)$.

For this exercise, we make use of the formula from section (3.3.2), which for $k = 0$ is:

$$\sum_{j=1}^{n} \frac{1}{j + b} = -\frac{1}{2b} + \frac{1}{2(n + b)} + \frac{\pi}{\sin 2\pi b} \int_{0}^{1} (\sin 2\pi (n + b) u - \sin 2\pi bu) \cot \pi u \, du$$

We can “expand” the sine (that is, use the identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$), getting:

$$\sum_{j=1}^{n} \frac{1}{j + b} = -\frac{1}{2b} + \frac{1}{2(n + b)} + \frac{\pi}{\sin 2\pi b} \int_{0}^{1} (\cos 2\pi bu \sin 2\pi nu + \sin 2\pi bu \cos 2\pi nu - \sin 2\pi bu) \cot \pi u \, du$$

We take the first part of the integrand, change the variables, expand $\cos 2\pi b (1 - u)$ (with identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$), and by means of the theorem 1 we are able to conclude that:

$$\lim_{n \to \infty} \frac{\pi}{\sin 2\pi b} \int_{0}^{1} \cos 2\pi b(1 - u) \sin 2\pi n(1 - u) \cot \pi (1 - u) \, du = \frac{\pi}{2} (\cot 2\pi b + \csc 2\pi b)$$

Now we need to work out the second part of the integrand. We make a change of variables, expand $\sin 2\pi b(1 - u)$, and when using theorem 2 we need to avoid the case $k = 0$ and $m = 1$ (since that integral doesn’t converge), which leads us to:

$$\int_{0}^{1} (\sin 2\pi b(1 - u) \cos 2\pi n(1 - u) - \sin 2\pi b(1 - u)) \cot \pi (1 - u) \, du =$$

$$\sin 2\pi b \int_{0}^{1} (\cos 2\pi bu - 1 - u(\cos 2\pi b - 1)) \cos 2\pi n(1 - u) \cot \pi (1 - u) \, du$$

$$- \cos 2\pi b \int_{0}^{1} (\sin 2\pi bu - u \sin 2\pi b) \cos 2\pi n(1 - u) \cot \pi (1 - u) \, du$$

$$+ \int_{0}^{1} (-\sin 2\pi b(1 - u) + (\sin 2\pi b)(1 - u) \cos 2\pi n(1 - u)) \cot \pi (1 - u) \, du$$

8
The two first integrals on the right-hand side cancel out, per theorem 2, when \( n \) goes to infinity, leaving only the third integral to be figured.

But if we look back at the expression for \( H(n) \) from [2], we notice it matches part of the last integral:

\[
\sum_{j=1}^{n} \frac{1}{j} = \frac{1}{2n} + \pi \int_{0}^{1} u \left(1 - \cos 2\pi n(1 - u)\right) \cot \pi (1 - u) \, du,
\]

which means the last integral can be further split:

\[
\frac{\pi}{\sin 2\pi b} \int_{0}^{1} \left(- \sin 2\pi b(1 - u) + (\sin 2\pi b)(1 - u) \cos 2\pi n(1 - u)\right) \cot \pi (1 - u) \, du =
\]

\[
\frac{\pi}{\sin 2\pi b} \int_{0}^{1} \left(- \sin 2\pi b(1 - u) - u \sin 2\pi b + \sin 2\pi b \cos 2\pi n(1 - u)\right) \cot \pi (1 - u) \, du
\]

\[
+ \pi \int_{0}^{1} u \left(1 - \cos 2\pi n(1 - u)\right) \cot \pi (1 - u) \, du
\]

At this point, there’s only the limit of the first integral on the right-hand side left to figure out, but fortunately that integral is constant for all integer \( n \). Therefore, we conclude that for sufficiently large \( n \):

\[
\sum_{j=1}^{n} \frac{1}{j + b} \sim -\frac{1}{2b} + \frac{\pi}{2} \left(\cot 2\pi b + \csc 2\pi b\right) - \pi \int_{0}^{1} \left(\frac{\sin 2\pi bu}{\sin 2\pi b} - u\right) \cot \pi u \, du + H(n)
\]

(2)

Coincidentally, the above integral is identical to the generating function of the zeta function at the odd integers, that we’ve seen in [2]:

\[
\sum_{k=1}^{\infty} \zeta(2k+1) x^{2k+1} = -\pi x \int_{0}^{1} \left(\frac{\sin 2\pi xu}{\sin 2\pi x} - u\right) \cot \pi u \, du
\]

That means that for sufficiently large \( n \) and \( 0 < |b| < 1 \) we can write the interesting approximation:

\[
\sum_{j=1}^{n} \frac{1}{j + b} \sim -\frac{1}{2b} + \frac{\pi}{2} \left(\cot 2\pi b + \csc 2\pi b\right) + \sum_{k=0}^{\infty} H_{2k+1}(n)b^{2k},
\]

or alternatively:

\[
\sum_{j=1}^{n} \frac{1}{j + b} \sim -\frac{1}{2b} + \frac{\pi}{2} \left(\cot 2\pi b + \csc 2\pi b\right) + H(n) + \sum_{k=1}^{\infty} \zeta(2k+1) b^{2k}
\]

\(^1\text{It stems from } \int_{0}^{1} (1 - \cos 2\pi n u) \cot \pi u \, du = 0 \text{ for all integer } n.\)
6  The full Fourier series

Although the expressions of $C^m_k(a, b, n)$ and $S^m_k(a, b, n)$ hold for all positive integers $k$ and $n$ and complex $m$, $a$ and $b$, the limits that we find next are constrained by the requirements of the theorems 1 and 2 from section (4).

In the next sections, we let $a = 1$ and omit this parameter in the notations (ditto for $HP_k(n)$). And since $k = 0$ and $|m| = 1$ leads to trivial cases, we’re not going to account for them in the following reasoning (so remember the final formulae may not be true for $k = 0$ and $|m| = 1$).

6.1 The limits of $C^m_{2k+1}(b, n)$ and $S^m_{2k+1}(b, n)$

The limit of $C^m_{2k+1}(b, n)$ is much harder to figure out than the limit of $S^m_{2k+1}(b, n)$, which should come as no surprise given the limits we’ve seen in sections (2.1.1) and (2.2.1) for the particular cases. So, without further ado, let’s see how to go about it:

$$\lim_{n \to \infty} C^m_{2k+1}(b, n) = -\frac{1}{2b^{2k+1}} \left( \cos \frac{2\pi b}{m} \sum_{j=0}^{k} \frac{(-1)^j (\frac{2\pi b}{m})^{2j}}{(2j)!} \right) + \lim_{n \to \infty} \frac{(-1)^k}{2(2k)!} \left( \frac{2\pi}{m} \sum_{j=1}^{n} \frac{1}{j + b} + \int_{0}^{1} (1 - u)^{2k} \left( \cos \frac{2\pi(n + b)u}{m} - \cos \frac{2\pi bu}{m} \right) \cot \frac{\pi u}{m} \, du \right)$$

Now, if we recall the approximation we found for $HP(n)$ in (2), $HP(n) \sim c + H(n)$ for large $n$ (where $c$ is the part that doesn’t depend on $n$), we only need to solve the limit:

$$\lim_{n \to \infty} \left( \frac{m}{\pi} (c + H(n)) + \int_{0}^{1} (1 - u)^{2k} \left( \cos \frac{2\pi(n + b)u}{m} - \cos \frac{2\pi bu}{m} \right) \cot \frac{\pi u}{m} \, du \right)$$

We can expand the cosine in the last integral:

$$\int_{0}^{1} (1 - u)^{2k} \left( \cos \frac{2\pi bu}{m} \left( -1 + \cos \frac{2\pi nu}{m} \right) - \sin \frac{2\pi bu}{m} \sin \frac{2\pi nu}{m} \right) \cot \frac{\pi u}{m} \, du$$

But due to theorem 1, the below limit is 0 (we just need to expand the first sine):

$$\lim_{n \to \infty} \int_{0}^{1} u^{2k} \sin \frac{2\pi b(1 - u)}{m} \sin \frac{2\pi nu(1 - u)}{m} \cot \frac{\pi (1 - u)}{m} \, du = 0$$

Now, by replacing $H(n)$ with its equation (1) and adding it up to what’s left in the integral:

$$\int_{0}^{1} (1 - u)^{2k} \cos \frac{2\pi bu}{m} \left( -1 + \cos \frac{2\pi nu}{m} \right) \cot \frac{\pi u}{m} + m(1 - u) (1 - \cos 2\pi nu) \cot \pi u \, du$$

Looking at theorem 2, we can recombine the terms conveniently into an integral that converges as $n$ goes to infinity:

$$\lim_{n \to \infty} \int_{0}^{1} (1 - u)^{2k} \cos \frac{2\pi bu}{m} \cos \frac{2\pi nu}{m} \cot \frac{\pi u}{m} - m(1 - u) \cos 2\pi nu \cot \pi u \, du = \frac{m \log |m|}{\pi},$$
which is justified by the following (only one piece shown):

\[
\cos \frac{2\pi b}{m} \int_0^1 u^{2k} \cos \frac{2\pi bu}{m} \cos \frac{2\pi n(1-u)}{m} \cot \frac{\pi(1-u)}{m} - m \cos \frac{2\pi b}{m} u \cos 2\pi n(1-u) \cot \frac{\pi(1-u)}{du} \\
\rightarrow \left( \cos \frac{2\pi b}{m} \right)^2 \frac{m \log |m|}{\pi},
\]

whereas the remaining integral converges on its own:

\[- \int_0^1 (1-u)^{2k} \cos \frac{2\pi bu}{m} \cot \frac{\pi u}{m} - m(1-u) \cot \pi u du\]

Hence, after we put everything together, the final conclusion is that for integer \(k \geq 0\) and real \(|m| \geq 1\):

\[
\sum_{j=1}^{\infty} \cos \frac{2\pi(j+b)}{m(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} \left( \sin \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \left( \frac{2\pi b}{m} \right)^{2j+1} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k+1-2j)!} \left( \frac{2\pi}{m} \right)^{2k+1-2j} \zeta(2j, b+1)
\]

\[
\ldots + \lim_{n \to \infty} \frac{(-1)^k}{2(2k)!} \left( \frac{2\pi}{m} \right)^{2k+1} \int_0^1 (1-u)^{2k} \left( \sin \frac{2\pi(n+b)u}{m} - \sin \frac{2\pi bu}{m} \right) \cot \frac{\pi u}{m} du
\]

As we can see, it takes a really convoluted function to generate this simple Fourier series.

In the case of \(S^m_{2k+1}(b, n)\), we have:

\[
\lim_{n \to \infty} S^m_{2k+1}(b, n) = -\frac{1}{2b^{2k+1}} \left( \sin \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \left( \frac{2\pi b}{m} \right)^{2j+1} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k+1-2j)!} \left( \frac{2\pi}{m} \right)^{2k+1-2j} \zeta(2j, b+1)
\]

This one is much simpler and we can easily deduce the limit of the integral by means of the theorem 1, without even having to expand the sine in the integrand. Thus, for integer \(k \geq 0\) and real \(|m| \geq 1\):

\[
\sum_{j=1}^{\infty} \sin \frac{2\pi(j+b)}{m(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} \left( \sin \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \left( \frac{2\pi b}{m} \right)^{2j+1} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k+1-2j)!} \left( \frac{2\pi}{m} \right)^{2k+1-2j} \zeta(2j, b+1)
\]

\[
\ldots + \frac{(-1)^k}{4(2k)!} \left( \frac{2\pi}{m} \right)^{2k+1} - \frac{(-1)^k}{2(2k)!} \left( \frac{2\pi}{m} \right)^{2k+1} \int_0^1 (1-u)^{2k} \sin \frac{2\pi bu}{m} \cot \frac{\pi u}{m} du
\]
6.2 The analogs

The next two formulae, $C_{2k}^m(b)$ and $S_{2k}^m(b)$, are analogous and don’t require further explanations.

\[
\sum_{j=1}^{\infty} \frac{\cos 2\pi j (j+b)}{(j+b)^2k} = -\frac{1}{2b^{2k}} \left( \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j)!} \left( \frac{2\pi b}{m} \right)^{2j} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k-2j)!} \frac{2\pi}{m} \left( \frac{1}{(2k-2j)!} \right)^{2k-j} \zeta(2j, b+1)
\]

\[
\sum_{j=1}^{\infty} \frac{\sin 2\pi j (j+b)}{(j+b)^2k} = -\frac{1}{2b^{2k}} \left( \sum_{j=0}^{k-2} \frac{(-1)^j}{(2j+1)!} \left( \frac{2\pi b}{m} \right)^{2j+1} \right) - \sum_{j=1}^{k-1} \frac{(-1)^{k-j}}{(2k-1-2j)!} \frac{2\pi}{m} \left( \frac{1}{(2k-1-2j)!} \right)^{2k-1} \zeta(2j+1, b+1)
\]

7 Lerch’s $\Phi$ at the positive integers

In this section we find out the values of the Lerch transcendent function, $\Phi(e^m, k, b)$, at the positive integers $k$.

7.1 Partial sum, $E_k^m(b, n)$

It’s straightforward to derive an expression for $\sum_{j=1}^{n} e^{mj}/(j+b)^k$ using the formulae from (2.1) and (2.2). If $i$ is the imaginary unit, we just make:

\[
\sum_{j=1}^{n} \frac{e^{2\pi i j (j+b)/m}}{(j+b)^k} = C_k^m(b, n) + i S_k^m(b, n) = E_k^{2\pi i/m}(b, n)
\]

Omitting the calculations and making a transformation $(m := 2\pi i/m)$ (to bring the variables to domain of the real numbers, which are easier to understand), we are able to come up with a single formula for both the odd and even powers:

\[
\sum_{j=1}^{n} \frac{e^{mj+b}}{(j+b)^k} = \frac{-e^{mb}}{2b^k} + \frac{e^{m(n+b)+1}}{2(n+b)^k} + \frac{1}{2b^k} \sum_{j=0}^{k} \frac{(mb)^j}{j!} - \frac{1}{2(n+b)^k} \sum_{j=0}^{k} \frac{(m(n+b))^j}{j!}
\]

\[
+ \sum_{j=1}^{k} \frac{m^{k-j}}{(k-j)!} HP_j(n) + \frac{m^k}{2(k-1)!} \int_{0}^{1} (1-u)^{k-1} \left( e^{m(n+b)u} - e^{mbu} \right) \coth \frac{mu}{2} \, du
\]

From this new equation, it’s easy to see that as $n$ goes to infinity, the sum on the left-hand side converges only if $\Re(m) < 0$. However, we can obtain an analytic continuation for this sum, by removing the second term on the right-hand side, which explodes out to infinity if $\Re(m) > 0$. Perhaps not surprisingly, this analytic continuation coincides with the Lerch $\Phi$ function.
7.1.1 A particular case, $E_k^m(0,n)$

When $b = 0$, we have one interesting particular case:

$$
\sum_{j=1}^{n} \frac{e^{mj}}{j^k} = \frac{e^{mn}}{2n^k} - \frac{1}{2n^k} \sum_{j=0}^{k} \frac{(mn)^j}{j!} + \sum_{j=1}^{k} \frac{m^{k-j}}{(k-j)!} H_j(n) + \frac{m^k}{2(k-1)!} \int_{0}^{1} (1-u)^{k-1} (e^{mnu} - 1) \coth \frac{mu}{2} du \quad (3)
$$

To obtain this expression, we take the limit as $b$ tends to 0:

$$
\lim_{b \to 0} - \frac{e^{mb}}{2b^k} + \frac{1}{2b^k} \sum_{j=0}^{k} \frac{(mb)^j}{j!} = 0
$$

7.2 Limit of $E_k^m(b,n)$

The limits we found in section (6) allow us to find the below infinite sum:

$$
\sum_{j=1}^{\infty} \frac{e^{i2\pi(j+b)/m}}{(j+b)^k} = \lim_{n \to \infty} C_k^m(b,n) + i S_k^m(b,n)
$$

After we do all the algebra, we find that for all integer $k \geq 1$:

$$
\sum_{j=1}^{\infty} \frac{e^{mj+b}}{(j+b)^k} = -\frac{1}{2b^k} \left( e^{mb} - \sum_{j=0}^{k-2} \frac{(mb)^j}{j!} \right) + \sum_{j=2}^{k} \frac{m^{k-j}}{(k-j)!} \zeta(j,b+1)
$$

$$
+ \frac{\pi m^k}{2(k-1)!} \left( \cot 2\pi b + \csc 2\pi b \right) - \frac{m^{k-1}}{(k-1)!} \log \left( \frac{-m}{2\pi} \right) - \frac{m^k}{2(k-1)!} \int_{0}^{1} (1-u)^{k-1} e^{mbu} \coth \frac{mu}{2} + \frac{2\pi}{m} \left( -1 + \sin 2\pi bu \right) \cot \pi ud\nu
$$

The infinite sum on the left-hand side converges whenever $\Re(m) < 0$, whereas the formula on the right-hand size, $E_k^m(b)$, converges always, except when $m = 0$ or $2b$ is an integer. This sum is related to the Lerch function by:

$$
e^{mb} \left( \Phi(e^m,k,b) - \frac{1}{b^k} \right) = \sum_{j=1}^{\infty} \frac{e^{mj+b}}{(j+b)^k} = E_k^m(b)
$$

Hence, for all integer $k \geq 1$, complex $m \neq 0$ and non-integer complex $2b$:

$$
\Phi(e^m,k,b) = \frac{1}{b^k} + e^{-mb} E_k^m(b)
$$

Note that when $b$ is a positive integer, $E_k^m(b)$ becomes an incomplete polylogarithm series (which we cover next). Therefore it’s very simple to derive its formula, we just need to subtract the missing part from the full polylogarithm (that is, subtract $E_k^m(0,b)$). A similar reasoning is used if $b$ is a negative integer, which leaves only the half-integers unaccounted for, though it’s probably easy to figure them out.
7.2.1 The polylogarithm, $\text{Li}_k(e^m)$

The limit of $E^m_k(0, n)$ when $n$ tends to infinity is the limit of the expression we just found when $b$ tends to 0, and it relies on the following two notable limits:

$$\lim_{b \to 0} -\frac{1}{2b^k} \left( e^{mb} - \sum_{j=0}^{k-2} \frac{(mb)^j}{j!} \right) + \frac{\pi m^k}{2(k-1)!} \left( \cot 2\pi b + \csc 2\pi b \right) = -\frac{m^k}{2k!}, \quad \text{and} \quad \lim_{b \to 0} \frac{\sin 2\pi bu}{\sin 2\pi b} = u$$

Therefore, for all integer $k \geq 1$ and all complex $m \neq 0$:

$$\sum_{j=1}^{\infty} \frac{e^{mj}}{j^k} = -\frac{m^{k-1}}{(k-1)!} \log \left( \frac{m}{2\pi} \right) + \sum_{j=0}^{k} \frac{m^{k-j}}{(k-j)!} \zeta(j)$$

$$-\frac{m^k}{2(k-1)!} \int_{0}^{1} (1-u)^{k-1} \coth \frac{mu}{2} - \frac{2\pi}{m} (1-u) \cot \pi u \, du$$

This infinite sum is known as the polylogarithm, $\text{Li}_k(e^m)$, and the formula on the right-hand side provides an analytic continuation for it when $\Re(m) > 0$.

Note how the first limit fit perfectly into the second sum (together with the other $\zeta(j)$ values, except for the singularity). And it’s easy to show that when $m$ goes to 0, the formula we found goes to $\zeta(k)$, if $k \geq 2$.

7.2.2 A very interesting consequence

The formula we found for $\text{Li}_k(e^m)$ in the previous section allows us to deduce the following power series for $e^m$:

$$\lim_{k \to \infty} \sum_{j=2}^{k} \frac{m^{k-j}}{(k-j)!} \zeta(j) = e^m$$

References


