# A straightforward and Lagrangian proof of the Einsteinian equivalence between the mass and the inernal energy (i.e. rest energy) V2

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I propose a Lagrangian proof of Einstein's well-known law that the mass system is its internal energy. The interest of this proof is to show how the distinction between internal degrees of freedom and the center of mass appears in the Lagrangian formalism. Considering that the Lagrangian depends on a particular set of variables for the internal degree of freedom, I show in a standard Lagrangian way how one can naturally find the desired law. This proof does not use the tensors of energy-momentum and can be easily used by students familiar with Lagrangian mechanics and the basis of special relativity. I apply the method for the particles and for the field, using the scalar field for simplification but it is easy to generalize for other fields (containing only the first derivative in Lagrangian). I give the example for the gravitation field. The method permits us to observe a strong relation between the Einstein's $E=mc^2$ law and his other famous law of the time dilatation. I carefully analyze the meaning of the particular choice of the variable and showing a sort of a modified speed addition formula without contradicting, of course, the one of Einstein (& Poincaré). I also try to untangle (for myself at least) the relation between the mass and the origin of the energy scale. Finally I analyze the reason why in Newtonian mechanic we don't have a such law. In future complement I will apply this way of thinking in the toy model of the electron (useful for an explicit classical renormalization of the mass) and the effective description of a complex system in term of a particle in order to better understand the passage from this 2 forms of description often used but never really explained.

Keywords: Special Relativity, $E=mc^2$, center of mass, Internal Energy, Lagrangian, Lorentz Transformation

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Sommaire

## 1. The Einstein law

### 1.1. The law

According the expression of the law of physics via the principle of least action [1] and the relativistic invariance: the mass $m_a$ of a material point "a" is simply the multiplicative coefficient appearing in the Lagrangian of this material point, interacting or not with an external field.

$$S[\boldsymbol{r_a}(t)] = -\int_{s_{a,1}}^{s_{a,2}} m_a.c.ds_a + \cdots = -\int_{t_1}^{t_2} \frac{m_a.c^2}{\gamma(\boldsymbol{v_a})} dt + \cdots$$

In 1905, Einstein tells us that whatever the system: a set of material points (dynamically characterised with a Lagrangian $L\left(\{\boldsymbol{r_a}\},\left\{\frac{dr_a}{dt}\right\}\right)$) or a field (dynamically characterised with the Lagrangian $\Lambda\left(\varphi,\frac{\partial\varphi}{\partial r},\frac{\partial\varphi}{\partial t}\right)$) we should have:

$$S[\boldsymbol{R_c}(t),\dots] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma(\boldsymbol{V_c})} dt + \cdots$$

- With $E^* = \sum_a \frac{d\boldsymbol{r_a^*}}{dt^*}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}} - L^*\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}\right)$ for a material system;
- Or $E^* \equiv \iiint \left(\frac{\partial\varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right) dV^*$ for a scalar field (for example).

Where the quantities with a star * are relative to the reference frame associated to the mass center K*. So $E^*$ is the internal energy.

Thus, every system has a centre of mass which has a Lagrangian, analogous to a material point with a mass $M = \frac{E^*}{c^2}$**.** This is the famous law of Einstein.

### 1.2. The current proof

This law is well established since its first publication in 1905 and was re-demonstrated more clearly after by other (Einstein himself, Von Laue ...). The simpler way (that the author know and read in [1]), is to demonstrate that the momentum is a 4 vector.

Indeed, tanks to the stress energy tensor $T^{ik}$ of the system, we can always associate to it a 4-vector

$P^i(K^*) \equiv \frac{1}{c}\int\iiint_{space-tim\ ,K} T^{ik} dS_k$ , where we choose the hyper-surface of integration as the hyperplane of the reference frame K* ($t^* = cte$).

In any frame ([3]), $P^i(K^*)$ can be written equivalently

$$P^i(K^*) = \frac{1}{c}\int\iiint_{space-tim} T^{ik}\delta\left(n_{lm}x^l\eta^m(K^*)\right).\eta_k(K^*)\, d^4x$$

where $\eta_k(K^*)$ is an orthogonal vector of the hyperplane $t^* = cte$ of K* such that $\eta^*{}_k(K^*) = (1,0,0,0)$ in $K^*$.

Thus, the Lorentz transformations tells us:

$$P^i(K^*) = \frac{1}{c}\int\iiint_{space-time} L^i_r L^k_s T^{*rs}\delta(t^*).L^m_k.\eta^*{}_m(K^*)\,d^4x^* \;= L^i{}_r\frac{1}{c}\iiint_{x^{*\alpha}\epsilon V^*} T^{*r0}(0,x^{*\alpha})dV^*$$

So $P^i(K^*) = L^i{}_r P^{*r}(K^*)$ where $P^{*r}(K^*) = \frac{1}{c}\iiint_{Space} T^{*r0}(0,x^{*\alpha})dV^*$

But $E^* \equiv \iiint_{Space} T^{*00}(0,x^{*\alpha})dV^*$ and $P^{*\alpha}(K^*) \equiv 0$ by definition of K*

So we have $P^i(K^*) = \left(\gamma\frac{E^*}{c},\gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{K*/K}}\right)$, hence $\boldsymbol{P} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{K*/K}}$ **=>** $M = \frac{E^*}{c^2}$

That is to say, the 3-momentum of any system is the same as a material point:

- with a mass $M = \frac{E^*}{c^2}$;
- and a speed $\boldsymbol{v} = \boldsymbol{V}_{\mathbf{K*/K}}$.

2 remarks:

- $P^i(K^*)$ is here relative to the particular time $t^* = 0$ and is not a priori constant;
- $P^i(K^*)$ is not the only one 4-momentum since we can define a different one for each frame of reference, $P^i(K), P^i(K'), P^i(K^*)$ ..., all are associated to different hyperplane of simultaneity linked to each possible (an infinity) frame of reference K,K',K*...(see [3]).

It exists a particular case where there is only one 4-momentum $P^i$: $P^i(K) = P^i(K') = P^i(K^*)$...In [1] we know that (in a general field theory):

- if the system is locally conserved : the stress-energy tensor has a null divergence $\partial_k T^{ik} = 0$;
- and if there is "nothing (other than gravitation field)" in infinity (in the sense of convergence to infinity).
- $P^i(K) \equiv \frac{1}{c}\int\iiint_{space-time,K} T^{ik}dS_k$ is conserved and doesn't depend on the hyperplane of integration (thanks to the conservation law).

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In a less general theorem (but more old) from Von Laue (cf. [4]) we can also say that if $\partial_0 T^{ik} = 0$ (and nothing to infiny):

$$P^i = \frac{1}{c}\iiint_{Space} T^{i0}dV \text{ is a 4-momentum} \Leftrightarrow \frac{1}{c}\iiint_{Space} T^{\alpha\beta}dV{=}0$$

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### 1.3. Why (I am) searching another proof ?

The proof above does not use the Lagrangian directly but indirectly via the stress energy tensor. However, the base of all dynamics in physics laws is (until now) always to start from the Lagrangian of a system with the appropriate variables (including degrees of freedom). We should be able to select the center of mass and the complementary degrees of freedom (which we called logically the internal degrees of freedom since they are seen in the "hidden" K*). Unfortunately (for myself at least...), I never found any proof using this point of view. With the current approach (even if it is sufficient for physics) it is not clear, for me, how the centre of mass appears in the Lagrangian, in parallel with the internal degrees of freedom. Indeed the Lagrangian is reconstructed only a posterior, after to demonstrate that $\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{C}}$ (using the stress-energy tensor) (see [1]). So we don't clearly see the passage:

- From an initial Lagrangian $S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$ or $S[\{\varphi(x,t)\}] = \frac{1}{c}\int \iiint \Lambda\left(\varphi, \frac{\partial \varphi}{\partial \boldsymbol{r}}, \frac{\partial \varphi}{\partial t}\right) d\Omega$
- To a Lagrangian of an apparent material point $S[\boldsymbol{R_c}(t), \dots] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma(\boldsymbol{V_c})} dt + \cdots$

In this article, I propose, using directly the Lagrangian formalism, to give the proof, for a material system (to present the method), for a field (scalar in order to simplified) and finally a system where the two interact.

## 2. Material system free

### 2.1. The proof for a material system

We begin with the action principle for a set of particles:

$$S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$$

In this expression, we are using coordinates in a Galilean reference frame K.
The degrees of freedom are the vectors $\{\boldsymbol{r_a}\}$, and we integrate the expression between the plan $H_1$ ($t_1 = cte$), and $H_2$ ( $t_2 = cte'$) in this frame.
We want now separate:

- the internal degree of freedom $\{\boldsymbol{r}_a^*\}$ , defined in the frame K*of the center of mass ;
- from the external degree of freedom $\boldsymbol{R_c}$ defined in the Galilean frame K.

So the degrees of freedom $\{\boldsymbol{r_a}\}$, are equivalent to the degree o freedom $\{\boldsymbol{r}_a^*, \boldsymbol{R_c}\}$.

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**Note 1: a plane t=cte is seen differently for different internal particle in the frame of the center of mass K***

Thanks to the relativist invariance we know that each terms of the action associated to a particle is invariant ($L.dt = \sum_a -m_a.cds_a$). However in the frame K*, the border plan $H_1$ and $H_2$ are associated to different time for each particle (in Einstein relativity the simultaneity is relative to a frame).

More explicitly, the Lorentz transformation said that a coordinate $t'$ seen in the frame K is expressed like

$$t' - t = \gamma(t)\left((t'^* - t^*_{c(t)}) + \frac{\boldsymbol{\beta}(t)}{c}\boldsymbol{r}^*\right)$$

With:

- $\gamma(t) = \gamma(\boldsymbol{V}_{\mathrm{C}}(t))$ ,
- $\boldsymbol{V}_{\mathrm{C}} \equiv \boldsymbol{V}_{\mathrm{K*/K}}(t)$,
- and $t^*_c$, the time measured by a clock in C: $t^*_{c(t)} = \int_0^t \frac{dt'}{\gamma(t')}$

, in the frame $\mathrm{K}^*(t)$ at the instant t (t'≠t, a priori, since t' is a generic coordinate of K but t defines the time of K for which the center of mass has the speed $\boldsymbol{V}_{\mathrm{C}}(t)$ ).

So a plane $t' = cte$ in K is seen like a plane $\gamma(t)\left((t'^* - t^*_{c(t)}) + \frac{\boldsymbol{\beta}}{c}\boldsymbol{r}^*\right) + t = cte$ in the frame $\mathrm{K}^*(t)$ around t.

Thus a particle at the position $\boldsymbol{r}_a^*$, sees the plane $t' = cte$ at the instant $t'^* = \frac{t'-t}{\gamma(t)} - \frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}\boldsymbol{r}_a^* + t^*_{c(t)}$

This is the proof of the assertion in the title.

**Note 2: measurement of a clock fixed on** $\mathrm{K}^*(t)$

Around t (t given and constant), a clock in $\boldsymbol{r}^*$ of $\mathrm{K}^*(t)$**, and always in** $\boldsymbol{r}^*$ **,** measures the duration time

$t'^* - t^*_{c(t)} = \frac{t'-t}{\gamma(t)} - \frac{\boldsymbol{V}_{\mathrm{C}}(t)}{c^2}\boldsymbol{r}^*$ between the event $\left(ct^*_{c(t)}, 0^*\right)_{\mathrm{K}*(t)}$ associated to C in $\mathrm{K}^*(t)$ and a certain event $(ct'^*, \boldsymbol{r}^*)_{\mathrm{K}*(t)}$ localised, by definition, in a **different position** than C: that is to say $\boldsymbol{r}^*$.

If we demand to this clock to measures now the duration between 2 events localised in its own position, the duration is now $\Delta\left(t'^* - t^*_{c(t)}\right) = \Delta\left(\frac{t'-t}{\gamma(t)} - \frac{\boldsymbol{V}_{\mathrm{C}}(t)}{c^2}\boldsymbol{r}^*\right) <=> (\Delta t'^* - 0) = \left(\frac{\Delta t'-0}{\gamma(t)} - 0\right)$ since $\gamma(t), \boldsymbol{V}_{\mathrm{C}}(t)$, t are constant since we work always in the **same reference frame** $\mathrm{K}^*(t)$. More over $\boldsymbol{r}^* = \boldsymbol{cte}$ by definition of the 2 events considered.

So we have $\Delta t'^{*} = \frac{\Delta t'}{\gamma(t)}$ and $dt'^{*} = \frac{dt'}{\gamma(t)}$ for 2 infinitesimal events.

When we observe 2 events associated to a particle, we study the duration time between 2 hyperplanes $t'^{*} = ct$ of K*$(t)$ where the 2 successive positions of the particle occurred. The duration is **always measured by a clock fixed** in K*$(t)$. So we can apply the relation above for the duration time associated to a particle:

$$\forall \text{ particle } a\text{: } dt_a^* = dt^* = \frac{dt}{\gamma(t)}$$

**Note 3 : On the Lorentz transformation**

A more general Lorentz transformation is:

$$\begin{pmatrix} t_a - t \\ \boldsymbol{r_a}(t_a) - \boldsymbol{R_c}(t) \end{pmatrix} = L(t).\begin{pmatrix} t_a^* - t_c^* \\ \boldsymbol{r_a^*} \end{pmatrix} <=> \begin{cases} t_a - t = \gamma(t)\left((t_a^* - t_c^*) + \frac{\boldsymbol{\beta}}{c}\boldsymbol{r_a^*}\right) \\ \boldsymbol{r_a}(t_a) - \boldsymbol{R_c}(t) = c(t_a^* - t_c^*)\gamma(t)\boldsymbol{\beta} + \boldsymbol{r_a^*} + (\gamma - 1)\frac{\boldsymbol{\beta}}{\beta^2}.(\boldsymbol{\beta r_a^*}) \end{cases}$$

For a movement of K* along x, we have the special Lorentz transformation principally used in this article:

$$\begin{cases} t_a - t = \gamma(t)\left((t_a^* - t_{c(t)}^*) + \dfrac{\beta(t)}{c}.x_a^*\right) \\ x_a - X_c = \gamma(t)\left(c(t_a^* - t_{c(t)}^*)\beta(t) + x_a^*\right) \end{cases}$$

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Now we express the action in the local frame K*(t):

$$S[\{\boldsymbol{r_a^*}(t^*), \boldsymbol{R_c}(t)\}] = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dt^*$$

Taking account $dt^* = \frac{dt}{\gamma(t)}$ and returnig to the Galilean frame K we have:

$$S = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{dt}{dt^*}\frac{d\boldsymbol{r_a^*}}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)\frac{dt^*}{dt}dt = \int_{t_1}^{t_2} \frac{L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r_a^*}}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}dt$$

So far, nothing new.

The important point to keep in mind is that we are not considering the variation of the internal degree of freedom $\boldsymbol{r_a^*}$ **:**

- relative to the internal time $t^*$of K*: $\frac{d\boldsymbol{r_a^*}}{dt^*}$ ;
- **but instead relative to time t of K:** $\frac{d\boldsymbol{r_a^*}}{dt}$.

That is to say, the Lagrangian considered is $L'\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) \equiv \frac{L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r_a^*}}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$, instead of using the most « natural » $L\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) \equiv \frac{L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$

So, we can now calculate the momentum of the center of mass, with $\boldsymbol{V}_\mathrm{C} \equiv \boldsymbol{V}_{\mathrm{K}*/\mathrm{K}}$:

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{c}} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{c}} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)}{\gamma(V_\mathrm{c})}$$

$$= L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\frac{\partial}{\partial \boldsymbol{V}_\mathrm{c}}\frac{1}{\gamma(\boldsymbol{V}_\mathrm{c})} + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{c})}\frac{\partial}{\partial \boldsymbol{V}_\mathrm{c}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)$$

- $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{c}}\frac{1}{\gamma(\boldsymbol{V}_\mathrm{c})} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{c}}\sqrt{1-\frac{V_\mathrm{c}^2}{c^2}} = \frac{-\frac{1}{2}2\frac{\boldsymbol{V}_\mathrm{c}}{c^2}}{\sqrt{1-\frac{V_\mathrm{c}^2}{c^2}}} = -\gamma(\boldsymbol{V}_\mathrm{c})\frac{\boldsymbol{V}_\mathrm{c}}{c^2}$
- $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{c}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) = \sum_a \frac{\partial \gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\partial \boldsymbol{V}_\mathrm{c}}\frac{\partial L^*}{\partial \gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} = \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\frac{\partial\left(1-\frac{V_\mathrm{c}^2}{c^2}\right)^{-1/2}}{\partial \boldsymbol{V}_\mathrm{c}}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$

$$= \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\left(\frac{\frac{1}{2}2\frac{\boldsymbol{V}_\mathrm{c}}{\boldsymbol{c}^2}}{\left(1-\frac{V_\mathrm{c}^2}{\boldsymbol{c}^2}\right)^{3/2}}\right)\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} = \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\gamma^3(V_\mathrm{c})\frac{\boldsymbol{V}_\mathrm{c}}{\boldsymbol{c}^2}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$\boldsymbol{P_c} = L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\left(-\gamma(V_\mathrm{c})\frac{\boldsymbol{V}_\mathrm{c}}{\boldsymbol{c}^2}\right) + \frac{1}{\gamma(V_\mathrm{c})}\sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\gamma^3(V_\mathrm{c})\frac{\boldsymbol{V}_\mathrm{c}}{\boldsymbol{c}^2}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$= \gamma(V_\mathrm{c})\frac{\boldsymbol{V}_\mathrm{c}}{\boldsymbol{c}^2}\left(-L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) + \sum_a \gamma\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}\right)$$

$$= \gamma(V_\mathrm{c})\frac{\boldsymbol{V}_\mathrm{c}}{c^2}\left(\sum_a \gamma\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\right)$$

$$= \gamma(V_\mathrm{c})\frac{\boldsymbol{V}_\mathrm{c}}{c^2}\left(\sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right)\right) \text{since } \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} = \gamma(V_\mathrm{c})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}$$

$$\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_\mathbf{C}$$

where $E^* \equiv \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right)$ is the internal energy.

So we have our relation.

$E^*$ is relative to the hyperplane $t^*$=cte, the mass $M = \frac{E^*}{c^2}$ is dealing with events ( the spatio-temporal positions of the particles) simultaneous in the frame K* and not in the frame K. This is well defined since $t^* = \int_0^t \frac{dt}{\gamma(t\prime)}$.

$$M = M(t^*) = M\left(\int_0^t \frac{dt'}{\gamma(t')}\right)$$

We see that we don't need to talk about closed system hypothesis or to have a 4 vector momentum to demonstrate it (we don't even use the expression $L.dt = \sum_a -m_a.cds_a$).

We have to note, in the proof, the importance to freeze the right variable $\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}$ (and not $\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\}$ ) in order to have the good expression.

## 2.2. Momentum and energy of a material system

### 2.2.1. Momentum

We can also notice that $\boldsymbol{P_a} \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} = \gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*}$ , so $\boldsymbol{P_a} = \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$ which is surprising but reassuring for the intelligibility of this quantity: this is the same as the one we would have in the frame of the centre of mass K*.

More over the total momentum $\boldsymbol{P_{total}}$ associated to the Lagrangian $L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)$ is

$\boldsymbol{P_{total}} = \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} + \frac{\partial L'}{\partial V_C} = \sum_a \boldsymbol{P_a} + \boldsymbol{P_c} = \boldsymbol{P_c}$ since by definition of K*: $\sum_a \boldsymbol{P_a} \equiv 0$. This is interesting since despite considering the internal variables on the same level as the mass center, we obtain as it should the total momentum is the one associated to the mass center.

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**Proof:**

Indeed $L.dt = -\sum_a m_a . c ds_a$ => $L = -\sum_a m_a . c \frac{ds_a}{dt} = -\sum_a m_a . c \frac{ds_a}{dt^*}\frac{dt^*}{dt} = -\sum_a m_a . c^2 \frac{1}{\gamma_a^*}\frac{1}{\gamma}$

But $\frac{1}{\gamma . \gamma_a^*} = \frac{1}{\gamma}\sqrt{1-\frac{\left(\frac{d\boldsymbol{r}_a^*}{dt^*}\right)^2}{c^2}} = \sqrt{\frac{1}{\gamma^2}-\frac{\frac{1}{\gamma^2}\left(\frac{d\boldsymbol{r}_a^*}{dt^*}\right)^2}{c^2}} = \sqrt{\frac{1}{\gamma^2}-\frac{\left(\frac{d\boldsymbol{r}_a^*}{dt}\right)^2}{c^2}}$ since $\frac{d\boldsymbol{r}_a^*}{dt^*} = \gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r}_a^*}{dt}$

Moreover $\frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\left(\frac{1}{\gamma . \gamma_a^*}\right) = \frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\sqrt{\frac{1}{\gamma^2}-\frac{\left(\frac{d\boldsymbol{r}_a^*}{dt}\right)^2}{c^2}} = -\frac{1}{2}\frac{2\frac{d\boldsymbol{r}_a^*}{dt}}{c^2}\frac{1}{\sqrt{\frac{1}{\gamma^2}-\frac{\left(\frac{d\boldsymbol{r}_a^*}{dt}\right)^2}{c^2}}} = -\frac{\frac{d\boldsymbol{r}_a^*}{dt}}{c^2}\gamma . \gamma_a^*$

So $\boldsymbol{P_a} = -\frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\sum_{a'} m_{a'} . c^2 \frac{1}{\gamma_{a'}^*}\frac{1}{\gamma} = m_a . c^2 \frac{\frac{d\boldsymbol{r}_a^*}{dt}}{c^2}\gamma . \gamma_a^* = m_a . \frac{d\boldsymbol{r}_a^*}{dt^*}\gamma_a^*$

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### 2.2.2. Energy

By definition the energy associated to the Lagrangian $L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)$ is:

$$E' \equiv \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial L'}{\partial \boldsymbol{V}_C}\boldsymbol{V_c} - L'$$

We can re-express it as:

$E' = \sum_a \boldsymbol{P_a}\frac{d\boldsymbol{r}_a^*}{dt} + \boldsymbol{P_c}\boldsymbol{V_c} - \frac{L^{*\prime}}{\gamma}$ since $L' = \frac{L^{*\prime}}{\gamma}$

$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt} + \left(\gamma \frac{E^*}{c^2}.\boldsymbol{V_c}\right)\boldsymbol{V_c} - \frac{L^{*\prime}}{\gamma}$ since $\boldsymbol{P_a} \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} = \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$

$$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt} - \frac{L^{*\prime}}{\gamma} + \gamma \frac{E^*}{c^2}.\boldsymbol{V_c}^2$$

$$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{\gamma dt^*} - \frac{L^{*\prime}}{\gamma} + \gamma \frac{E^*}{c^2}.\boldsymbol{V_c}^2 = \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^{*\prime}\right)\frac{1}{\gamma} + \gamma \frac{E^*}{c^2}.\boldsymbol{V_c}^2$$

$$= \frac{E^*}{\gamma} + \gamma \frac{E^*}{c^2}.\boldsymbol{V_c}^2 = \frac{E^*}{\gamma} + \gamma \frac{E^*}{c^2}.\boldsymbol{V_c}^2 = \frac{E^* + \gamma^2 \frac{E^*}{c^2}.\boldsymbol{V_c}^2}{\gamma} = E^* \frac{1 + \gamma^2.\beta^2}{\gamma} = E^* \frac{1 + \frac{\beta^2}{1-\beta^2}}{\gamma}$$

$$= E^* \frac{\frac{1-\beta^2+\beta^2}{1-\beta^2}}{\gamma} = E^* \frac{\frac{1}{1-\beta^2}}{\gamma} = E^* \frac{\gamma^2}{\gamma} = \gamma E^*$$

We find that, as it should :

$$E' = \gamma E^*$$

With $E^* \equiv \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^*$

Indeed, it is the same relation that we had with the energy associated to the classical Lagrangian $L\left(\{\boldsymbol{r_a}\}, \left\{\frac{dr_a}{dt}\right\}\right)$

$$E \equiv \sum_a \frac{\partial L}{\partial \frac{d\boldsymbol{r_a}}{dt}} \frac{d\boldsymbol{r_a}}{dt} - L = \gamma E^*$$

We can conclude that $E' = E$
We can also conventionally note: $E = E^* + (\gamma - 1)E^*$ where we observe, for a closed system (E=cte), an exchange of Energy between the internal energy $E^*$ and the kinetic energy $(\gamma - 1)E^*$, the one depending of the center of mass.

### 2.3. The Euler-Lagrange equation for the internal particles and the mass center

The Euler-Lagrange equations are :

$$\frac{d}{dt}\frac{\partial}{\partial \boldsymbol{V_C}} L'\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right) = \frac{\partial}{\partial \boldsymbol{R_C}} L'\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right)$$

$$\forall a \quad \frac{d}{dt}\frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} L'\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right) = \frac{\partial}{\partial \boldsymbol{r}_a^*} L'\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right) = \frac{\partial}{\partial \boldsymbol{r}_a^*} \frac{L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_C)}$$

$$= \frac{1}{\gamma(\boldsymbol{V}_C)}\frac{\partial}{\partial \boldsymbol{r}_a^*} L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)$$

Taking account the momentum expression above we have therefore:

$$\frac{d}{dt}\left(\gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right)=\frac{\partial}{\partial\boldsymbol{R}_{\mathrm{C}}}L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)$$

$$\forall a\ \ \frac{d}{dt}\left(\gamma_a^* m_a.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right)=\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\frac{\partial}{\partial\boldsymbol{r}_{\boldsymbol{a}}^*}L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)$$

As $dt^*=\frac{dt}{\gamma(V_{\mathrm{C}})}$, the second equation can be re-write:

$$\frac{d}{dt^*}\left(\gamma_a^* m_a.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right)=\frac{\partial}{\partial\boldsymbol{r}_{\boldsymbol{a}}^*}L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right)$$

It is remarkable that we obtain the same equation that we should obtain for the dynamic in a K* frame. However, we should notice that, since the center of mass can a priori accelerate, this is the equation for a material point in a local Galilean frame. Indeed, $dt^*$ is not constant as it is equal to $dt^*=\frac{dt}{\gamma(V_{\mathrm{C}})}$ where dt is the true constant differential element. $\boldsymbol{V}_{\mathrm{C}}$ varies, so $dt^*$ has to vary also.

### 2.4. The material system seen as a material point: the reduced action

We can write:

$$S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\},\boldsymbol{R}_{\boldsymbol{c}}(t)]=\int_{t_1}^{t_2}L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)dt$$

$$=\int_{t_1}^{t_2}\left[\sum_a\boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}+\boldsymbol{P}_{\boldsymbol{c}}.\boldsymbol{V}_{\boldsymbol{c}}-E\right]dt$$

$$=\int_{t_1}^{t_2}\left[\sum_a\boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}+\left(\gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right).\boldsymbol{V}_{\boldsymbol{c}}-\gamma E^*\right]dt$$

$$=\int_{t_1}^{t_2}\left[\sum_a\boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}+\gamma E^*(\beta^2-1)\right]dt=\int_{t_1}^{t_2}\left[\sum_a\boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}-\frac{E^*}{\gamma}\right]dt$$

So

$$\boxed{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\},\boldsymbol{R}_{\boldsymbol{c}}(t)]=\int_{t_1}^{t_2}\left[\sum_a\boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}-\frac{E^*}{\gamma}\right]dt}$$

If we ignore the final position of the internal degree of freedom, we have like a "spatial Maupertuis principle" (instead of a temporal used in [2]):

$$\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\},\boldsymbol{R}_{\boldsymbol{c}}(t)]-\left(\sum_a\boldsymbol{P}_{\boldsymbol{a}}.\delta\boldsymbol{r}_{\boldsymbol{a}}^*\right)_{H_2}=0$$

We can see that if all the internal momentum are constant, it exists a reduced action principle:

$$\boxed{S_0[\boldsymbol{R}_{\boldsymbol{c}}(t)]=-\int_{t_1}^{t_2}\frac{E^*}{\gamma}dt}$$

We can surely generalize it for closed systems with internal separable variables where we've chosen well the variables with constant momentum. In this case, we see that for "stationary" system, in this restrict sense, the center of mass dynamic is the same as a material point.

Note: my idea to consider the quantity $\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}$ comes initially from the willingness to make appear the Lagrangian of the apparent material point with this reduced action (in the same manner we make appear the virtual work theorem: $\delta\int_{t_1}^{t_2}[\sum_a \boldsymbol{P_a}.\boldsymbol{dr} - H[\boldsymbol{P_a},\boldsymbol{r_a}]dt] + (\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r_a})_{H_2} = 0$ and $\boldsymbol{P_a} = \boldsymbol{cte}$=> $\delta\int_{t_1}^{t_2} H_{\boldsymbol{P_a}=\boldsymbol{cte},}(\{\boldsymbol{r_a}\})\,dt = 0$ ), cf .[2]).

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**Proof:**

Indeed (do the same that [2] but for space and not for time)):

$$\delta S[\{\boldsymbol{r}_a^*(t^*)\},\boldsymbol{R}_c(t)] - \left(\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r}_a^*\right)_{H_2} = 0$$

$$<=>\delta\int_{t_1}^{t_2} d\sum_a[\boldsymbol{P_a}.\boldsymbol{r}_a^*] + \delta\int_{t_1}^{t_2}\left[-\frac{E^*}{\gamma}\right]dt - \left(\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r}_a^*\right)_{H_2} = 0$$

$$<=>\delta\left[\sum_a \boldsymbol{P_a}.\boldsymbol{r}_a^*\right]_{H_2} + \delta\int_{t_1}^{t_2}\left[-\frac{E^*}{\gamma}\right]dt - \left(\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r}_a^*\right)_{H_2} = 0$$

$$<=>\delta\int_{t_1}^{t_2}\left[-\frac{E^*}{\gamma}\right]dt = 0$$

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### 2.5. The material system seen as a material point: the internal dynamic is known

As already written:

$$S[\{\boldsymbol{r}_a^*(t^*)\},\boldsymbol{R}_c(t)] = \int_{t_i}^{t_f} L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)dt$$

$$= \int_{t_i}^{t_f}\left[\sum_a \boldsymbol{P_a}.\frac{d\boldsymbol{r}_a^*}{dt} - \frac{E^*}{\gamma}\right]dt = \int_{t_i}^{t_f}\left[\sum_a \gamma_a^* m_a.\frac{d\boldsymbol{r}_a^*}{dt^*}\frac{d\boldsymbol{r}_a^*}{dt} - \frac{E^*}{\gamma}\right]dt$$

We **decide** to say that **we know** the internal dynamic of the system.

That is to say we know the maps:

- $\{\boldsymbol{r}_a^*(t^*)\}$
- $\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}(t^*)\right\}$

So, it results that the mass center is **in the field** (in the [2] terms) of the internal degree of freedom$\{\boldsymbol{r}_a^*\}$. We can inject this information $\{\boldsymbol{r}_a^*(t^*)\}$, $\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}(t^*)\right\}$ in the Action :

$$S[\{\boldsymbol{r}_a^*(t^*)\},\boldsymbol{R}_c(t)] = \int_{t_i}^{t_f} L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)dt$$

$$= \int_{t_i}^{t_f} \left[ \sum_a \gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*} d\boldsymbol{r}_a^* - \frac{E^*(t^*)}{\gamma} dt \right]$$

$$= \int_{\{\boldsymbol{t}_{a,i}^*\}}^{\{\boldsymbol{t}_{a,f}^*\}} \left[ \sum_a \gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*} d\boldsymbol{r}_a^* \right] + \int_{t_i}^{t_f} -\frac{E^*(t^*)}{\gamma} dt$$

$$= \int_{\{\boldsymbol{t}_{a,i}^*\}}^{\{\boldsymbol{t}_{a,f}^*\}} df\{t_a^*\} + \int_{t_i}^{t_f} -\frac{E^*(t^*)}{\gamma} dt$$

The least action principle can therefore be express with the following action:

$$S''[\boldsymbol{R}_c(t), t] = \int_{t_i}^{t_f} L''\,(t, \boldsymbol{R}_\text{C}, \boldsymbol{V}_\text{C}) dt = \int_{t_i}^{t_f} -\frac{E^*(t^*)}{\gamma(\boldsymbol{V}_\text{C})} dt$$

With $t^* = t^*(t) = \int_{t_i}^{t} \frac{dt\prime}{\gamma(t\prime)}$

It is important to not that <u>we a priori don't know the expression of $t^*$</u> although we know the internal dynamic express relative to it. Indeed, knowing $t^*$ required to know the map $\boldsymbol{V}_\text{C}(t)$ (part of the solution we are looking for) since $t^* = \int_{t_i}^{t} \frac{dt}{\gamma(t\prime)}$,which is absurd. Another proof: knowing $t^*$, implies the undesirable consequence that $\frac{E^*(t^*(t))}{\gamma} dt = E^*(t^*(t)) dt^*(t) = df(t^*(t)) = dg(t)$. . This would suppress (according to the least action principle) the only one term of the action that we want to maintain in order to find the trajectory. We see therefore that the center of mass is again <u>in the field</u> of a variable : his own proper time $t^*$, as for a material point.

It seems difficult to find any relevant way in order to take account the constraint $t^* = \int_{t_i}^{t} \frac{dt\prime}{\gamma(t\prime)}$ in the Lagrangian.

Despite this problem, we can make a stronger supposition that we know, in addition to the internal dynamic, the behaviour of the energy relative to t (and not): $E^*(t^*(t))$ noted abusively $E^*(t)$.

Indeed even if we don't know $t^*(t)$ we can pretend to know $E^*(t)$. More precisely

$E^*(t^*(t)) = (E^* o\ t^*)(t)$. Knowing the map $(E^* o\ t^*)$ is not sufficient to know the map $t^*$ since the inverse map $E^{*-1}$ could eventually not exist.

Knowing $E^*(t^*(t))$ and inject it in the Lagrangian, is equivalent to say that the center of mass is now in the field of the energy.

This situation is automatically realized in the classical case where we put $t^* \approx t$ in the Energy. However, we do not make the same approximation for $dt^*$, indeed we put $dt^* \approx dt\left(1 - \frac{1}{2}\frac{Vc^2}{c^2}\right)$. Otherwise, all the information would be lost:

we do $\frac{E^*(t^*(t))}{\gamma} dt \approx E^*(t) dt \left(1 + \frac{1}{2}\frac{Vc^2}{c^2}\right)$ but not $\frac{E^*(t^*(t))}{\gamma} dt \approx E^*(t) dt$

## 2.6. A strong link between the Einstein law and the dilatation of time

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} = \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)$$

Since in special relativity, the space is isotropic (≡the **laws of a material system** in a homogeneous & isotropic gravitational field are isotropic) $\gamma(\boldsymbol{V}_{\mathrm{C}})$ depends only on the norm of $\boldsymbol{V}_{\mathrm{C}}$ or equivalently on $\boldsymbol{V}_{\mathrm{C}}{}^2$.

- $\frac{\partial}{\partial V_{\mathrm{C,x}}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} = -\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}})}{\partial V_{\mathrm{C,x}}} = -\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial V_{\mathrm{C,x}}} = -\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}\frac{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}{\partial V_{\mathrm{C,x}}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2} = -\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}$
- $\frac{\partial}{\partial V_{\mathrm{C,x}}}L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) = \sum_a \frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\partial V_{\mathrm{C,x}}}\frac{\partial L^*}{\partial\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} = \sum_a\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}})}{\partial V_{\mathrm{C,x}}}\right)\frac{\partial L^*}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} =$
  $\sum_a\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\right)\frac{\partial L^*}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} = \sum_a\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\right)\frac{\partial L^*}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$

$$P_{c,x} = \frac{\partial L'}{\partial V_{\mathrm{C,x}}} = L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\left(-\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\right) + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\sum_a\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\right)\frac{\partial L^*}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$= 2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}\left[\gamma(\boldsymbol{V}_{\mathrm{C}})\sum_a\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)\frac{\partial L^*}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\right]$$

$$= 2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}\left[\sum_a\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\frac{\partial L^*}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\right]$$

$$= 2V_{\mathrm{C,x}}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\frac{E^*}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2} = V_{\mathrm{C,x}}\left(\frac{2c^2}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}\frac{\partial\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{\partial \boldsymbol{V}_{\mathrm{C}}{}^2}\right)\frac{E^*}{c^2}$$

Starting from $\boldsymbol{P_c} \equiv \frac{\partial L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)}{\partial \boldsymbol{V}_{\mathrm{C}}}$, the fact that the space is isotropic in special relativity

and without express explicitly $\gamma(\boldsymbol{V}_{\mathrm{C}})$, we have:

$$\boldsymbol{P_c} = \boldsymbol{V}_{\mathrm{C}}.\gamma^{eff}(\boldsymbol{V}_{\mathrm{C}}{}^2)\frac{E^*}{c^2}$$

With $\gamma^{eff}(\boldsymbol{V}_{\mathrm{C}}) \equiv \frac{2c^2}{\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)^2}\frac{d\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2)}{d\boldsymbol{V}_{\mathrm{C}}{}^2}$

And of course $\gamma(\boldsymbol{V}_{\mathrm{C}}{}^2) \equiv \frac{dt}{dt^*}$ is dilatation of time

This is the expression of the 3-momentum of a material system without knowing explicitly the relation between the dilatation of time and the speed of the mass center $\boldsymbol{V}_\mathrm{C}$.

Now using this general result, we want to know if the Einstein law is sufficient to obtain the right expression of the dilatation of time $\gamma$ relative to $\boldsymbol{V}_\mathrm{C}$, that is to say the expression $\gamma({\boldsymbol{V}_\mathrm{C}}^2)$

We start from $\frac{E^*}{c^2} = M$. This expression **means** that the form of the impulsion of a system, with internal energy $E^*$, is the same of a material point of mass M verifying $\frac{E^*}{c^2} = M$.

But for a material point we have $\boldsymbol{P_c} = \boldsymbol{V}_\mathrm{C}.\gamma({\boldsymbol{V}_\mathrm{C}}^2)M$, so the Einstein law implies

$$\frac{E^*}{c^2} = M \Rightarrow \gamma^{eff}({\boldsymbol{V}_\mathrm{C}}^2) = \gamma(\boldsymbol{V}_\mathrm{C})$$

$$\gamma(\boldsymbol{V}_\mathrm{C}) = \left(\frac{2c^2}{\gamma({V_\mathrm{C}}^2)^2}\frac{d\gamma({V_\mathrm{C}}^2)}{d{V_\mathrm{C}}^2}\right) \Rightarrow \frac{1}{2c^2} = \frac{1}{\gamma({V_\mathrm{C}}^2)^3}\frac{d\gamma({V_\mathrm{C}}^2)}{d{V_\mathrm{C}}^2} \Rightarrow \frac{d\gamma({V_\mathrm{C}}^2)}{\gamma({V_\mathrm{C}}^2)^3} = \frac{d{V_\mathrm{C}}^2}{2c^2} \Rightarrow -\frac{1}{2}d[\gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2}] = \frac{d{V_\mathrm{C}}^2}{2c^2}$$

$$\Rightarrow -\frac{1}{2}d[\gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2}] = \frac{d{V_\mathrm{C}}^2}{2c^2} \Rightarrow -\frac{1}{2}[\gamma({\boldsymbol{V}'_\mathrm{C}}^2)^{-2}]_0^{{V_\mathrm{C}}^2} = \frac{[{V'_\mathrm{C}}^2]_0^{{V_\mathrm{C}}^2}}{2c^2} \Rightarrow -[\gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2} - \gamma(0)^{-2}] = \frac{{V_\mathrm{C}}^2}{c^2}$$

$$\Rightarrow \gamma(0)^{-2} - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2} = \gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2} \Rightarrow \gamma({\boldsymbol{V}_\mathrm{C}}^2) = \frac{1}{\sqrt{\gamma(0)^{-2} - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2}}}$$

But $\gamma^{eff}(\boldsymbol{0}) = \gamma(\boldsymbol{0})$=1 => $1 = \frac{2c^2}{\gamma(\boldsymbol{0})^2}\left(\frac{d\gamma({V_\mathrm{C}}^2)}{d{V_\mathrm{C}}^2}\right)_{{V_\mathrm{C}}^2=0} = \frac{2c^2}{\gamma(\boldsymbol{0})^2}\left(-\frac{1}{c^2}\frac{-1}{2}\frac{1}{\left(\gamma(0)^{-2}-\frac{{V_\mathrm{C}}^2}{c^2}\right)^{3/2}}\right)_{{V_\mathrm{C}}^2=0} =$

$$\frac{c^2}{\gamma(\boldsymbol{0})^2}\frac{\boldsymbol{1}}{c^2}\frac{1}{(\gamma(0)^{-2})^{3/2}} = \frac{c^2}{\gamma(\boldsymbol{0})^2}\frac{\boldsymbol{1}}{c^2}\gamma(0)^3 = \gamma(0)$$

So $\frac{E^*}{c^2} = M$ with $\boldsymbol{P_c} = \boldsymbol{V}_\mathrm{C}.\gamma({\boldsymbol{V}_\mathrm{C}}^2)M \Rightarrow \gamma({\boldsymbol{V}_\mathrm{C}}^2) = \frac{1}{\sqrt{1-\frac{{V_\mathrm{C}}^2}{c^2}}}$

We have the final result:

Starting from $\boldsymbol{P_c} \equiv \frac{\partial L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)}{\partial \boldsymbol{V}_\mathrm{C}}$, the fact that the space is isotropic in special relativity and without express explicitly $\gamma(\boldsymbol{V}_\mathrm{C})$, we have the equivalence:

$$E^* = Mc^2 \Leftrightarrow \gamma({\boldsymbol{V}_\mathrm{C}}^2) = \frac{1}{\sqrt{1 - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2}}}$$

With the definition

$$\left\{\frac{E^*}{c^2} = M\right\} \equiv$$

≡ {the form of the **impulsion** of a system, with internal energy $E^*$,
is the same of a material point of mass M verifying $\frac{E^*}{c^2} = M$.}

Hence the Einstein law is **not only a necessary condition** of special relativity (via kinematic and least action principle), but also **a sufficient condition** for the dilatation factor expression $\gamma(\boldsymbol{V}_c{}^2)$.
In this sense, this theorem shows that the dilatation of time and the Einstein law are strongly related. So any proof of the dilatation of time, is a proof of the Einstein Law and inversely.
This can also be illustrated by showing that any empirical deviation of the Einstein law $\Delta \equiv \frac{E^*}{c^2} - M$ is linked to a deviation of the Special Relativity relation $\frac{1}{\gamma(\boldsymbol{V}_c{}^2)^2} = 1 - \frac{\boldsymbol{V}_c{}^2}{c^2}$.

$$\Delta \equiv \frac{E^*}{c^2} - M = \frac{\gamma(\boldsymbol{V}_c{}^2)}{\gamma^{eff}(\boldsymbol{V}_c{}^2)} M - M = M\left(\frac{\gamma(\boldsymbol{V}_c{}^2)}{\gamma^{eff}(\boldsymbol{V}_c{}^2)} - 1\right) = M\left(\frac{\gamma(\boldsymbol{V}_c{}^2)}{\frac{2c^2}{\gamma(\boldsymbol{V}_c{}^2)^2}\frac{d\gamma(\boldsymbol{V}_c{}^2)}{d\boldsymbol{V}_c{}^2}} - 1\right)$$

$$= M\left(\frac{\gamma(\boldsymbol{V}_c{}^2)^3}{2c^2\frac{d\gamma(\boldsymbol{V}_c{}^2)}{d\boldsymbol{V}_c{}^2}} - 1\right) = M\left(\frac{1}{\frac{2c^2}{-2}\frac{d[\gamma(\boldsymbol{V}_c{}^2)^{-2}]}{d\boldsymbol{V}_c{}^2}} - 1\right) = -M\left(1 + \frac{1}{c^2\frac{d[\gamma(\boldsymbol{V}_c{}^2)^{-2}]}{d\boldsymbol{V}_c{}^2}}\right)$$

So we have

$$\Delta \equiv \frac{E^*}{c^2} - M = -M\left(1 + \frac{1}{c^2\frac{d[\gamma(V_c{}^2)^{-2}]}{dV_c{}^2}}\right) \text{ or } c^2\frac{d[\gamma(V_c{}^2)^{-2}]}{dV_c{}^2} = \left(\frac{1}{\frac{\Delta}{-M} - 1}\right) = \left(\frac{-1}{1 + \frac{\Delta}{M}}\right) \approx -1 + \frac{\Delta}{M}$$

If we measures $\frac{1}{\gamma(V_c{}^2)^2}$ in function of $\boldsymbol{V}_c{}^2$ (like Bertozzi Experiment [5]), we can obtain an empiric law like

$\frac{1}{\gamma(V_c{}^2)^2} = \sum_{n=0}^{\infty}(a_n^{SR} + \varepsilon_n).\left\|\frac{V_c}{c}\right\|^{2n}$ with $a_n^{SR} = (1, -1, 0, 0, 0, \dots)$

<=> $\frac{1}{\gamma(V_c{}^2)^2} = (1 + \varepsilon_0) + (-1 + \varepsilon_1).\left(\frac{V_c}{c}\right)^2 + \sum_{n=2}^{\infty}\varepsilon_n.\left\|\frac{V_c}{c}\right\|^{2n}$

$$=> \frac{d[\gamma(\boldsymbol{V}_c{}^2)^{-2}]}{d\boldsymbol{V}_c{}^2} = \frac{1}{c^2}\frac{d[\gamma(\boldsymbol{V}_c{}^2)^{-2}]}{d\left(\frac{\boldsymbol{V}_c}{c}\right)^2} = \frac{1}{c^2}\left(\varepsilon_1 - 1 + 2\sum_{n=2}^{n}\varepsilon_n.\left\|\frac{\boldsymbol{V}_c}{c}\right\|^{2n-1}\right)$$

Thus we have the following relation between the empiric deviation of the 2 laws:

$$\varepsilon_1 + 2\sum_{n=2}^{n}\varepsilon_n.\left\|\frac{\boldsymbol{V}_c}{c}\right\|^{2n-1} = \frac{\Delta}{M}$$

Any deviation of the Einstein law is linked to a deviation of the expression of the dilatation of time :

$$\varepsilon_1 + 2\sum_{n=2}^{n} \varepsilon_n . \left\| \frac{\boldsymbol{V}_C}{c} \right\|^{2n-1} \approx \frac{\Delta}{M}$$

with

- $\Delta \equiv \frac{E^*}{c^2} - M$
- $\frac{1}{\gamma({V_c}^2)^2} = (1+\varepsilon_0) + (-1+\varepsilon_1).\left(\frac{V_C}{c}\right)^2 + \sum_{n=2}^{\infty} \varepsilon_n . \left\| \frac{V_C}{c} \right\|^{2n}$

This is another way to express the link between the 2 laws.

## 2.7. Questions about the meaning of events and physical quantities used in the proof

### 2.7.1. The meaning of a speed

There is a priori a problem with the speed $\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}$ since it combines 2 quantities that each relies to 2 different references frames: K* for $d\boldsymbol{r}_{\boldsymbol{a}}^*$ and K for $dt$. It may be thought to be ill-defined, which would break the demonstration.

In many textbook like in [5] we can "traditionally" write $\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\frac{dt^*}{dt}$, and according to the Lorentz Transformation $\frac{dt^*}{dt} = \frac{1}{\gamma\left(dt^* + \frac{\beta}{c}dx_a^*\right)} = \frac{1}{dt_1 + dt_2}$ with $dt_1 \equiv \gamma . dt^*$ and $dt_2 \equiv \gamma \frac{\beta}{c} dx_a^*$.

$$\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt_1 + dt_2} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{\gamma\left(dt^* + \frac{\beta}{c}dx_a^*\right)}$$

However **we don't use this textbook (or traditional) formula** above in this article but another instead (consequently $d\boldsymbol{r}_{\boldsymbol{a}}^*$ has also another meaning):

$$\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt_1} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{\gamma dt^*}$$

So what the 2 expressions really mean, why are we using the second whereas the first ? and is there any sense to use the second ? The latter question is important since my proof is totally based on it.

**a.** <u>In the first expression</u> $\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{\gamma\left(dt^* + \frac{\beta}{c}dx_a^*\right)}$, we are actually using the Lorentz transformation about **the 2 <u>same</u> events** seen in 2 different Galilean Frames K and K*:

- $a_1 = \left(ct_{K_1^*}, \boldsymbol{r}_{\boldsymbol{a},K_1^*}(t_{K_1^*})\right)_{K_1^*} = \left(ct_1, \boldsymbol{r}_{\boldsymbol{a}}(t_1)\right)_K$
- $a_2 = \left(c\left(t_{K_1^*} + dt_{K_1^*}\right), \boldsymbol{r}_{\boldsymbol{a},K_1^*}\left(t_{K_1^*} + dt_{K_1^*}\right)\right)_{K_1^*} = \left(c\left(t_1 + (t_2 - t_1)\right), \boldsymbol{r}_{\boldsymbol{a}}(t_1) + d\boldsymbol{r}_{\boldsymbol{a}}(t_1)\right)_K$

Indeed, at the time $t_1$ of K we associate to the center of mass C, at the position $x_c(t_1)$ and any coordinate $\left(ct_{K_1^*}, x_{K_1^*}\right)_{K_1^*}$ of the local (current Galilean ) reference frame K* is related to that of K $(ct, x)_K$ with the Lorentz transformation:

$$\begin{cases} c.t - c.t_1 = \gamma_{t_1}.\left(c\left(t_{K_1^*} - \int_0^{t_1}\frac{dt}{\gamma_t}\right) + \beta_{t_1}.\left(x_{K_1^*} - x_{C,K_1^*}\right)\right) \\ x - x_c(t_1) = \gamma_{t_1}.\left(\left(x_{K_1^*} - x_{C,K_1^*}\right) + \beta_{t_1}.\left(t_{K_1^*} - \int_0^{t_1}\frac{dt}{\gamma_t}\right)\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} c.t - c.t_1 = \gamma_{t_1}.\left(c\left(t_{K_1^*} - t^*_{C(t_1)}\right) + \beta_{t_1}.x_{K_1^*}\right) \\ x - x_c(t_1) = \gamma_{t_1}.\left(x_{K_1^*} + \beta_{t_1}.\left(t_{K_1^*} - t^*_{C(t_1)}\right)\right) \end{cases}$$

- $t^*_{C(t_1)} \equiv \int_0^{t_1} \frac{dt}{\gamma_t}$ the time seen from the clock in C ;
- $x_{C,K_1^*} \equiv 0$ by deciding that C is the spatial origin of the current K*.
- $\gamma_{t_1}, t^*_{C(t_1)}, \beta_{t_1}$ are **constants** associated to the Lorentz transformation at the time $t_1$.

So we simply apply this transformation for the 2 events:

- On one hand:
  $d(c.t - c.t_1) \equiv (c.t - c.t_1)_{a_2} - (c.t - c.t_1)_{a_1} = (c.t)_{a_2} - c.t_1 - (c.t)_{a_1} + c.t_1 = c.dt$
- On the other hand: $(c.t - c.t_1)_{a_2} - (c.t - c.t_1)_{a_1} =$
  $= \left[\gamma_{t_1}.\left(c\left(t_{K_1^*} - t^*_{C(t_1)}\right) + \beta_{t_1}.x_{K_1^*}\right)\right]_{a_2} - \left[\gamma_{t_1}.\left(c\left(t_{K_1^*} - t^*_{C(t_1)}\right) + \beta_{t_1}.x_{K_1^*}\right)\right]_{a_1}$

$= \gamma_{t_1}\left[\left(c\left(t_{K_1^*,a_2} - t_{K_1^*,a_1}\right) + \beta_{t_1}.\left(x_{K_1^*,a_2} - x_{K_1^*,a_1}\right)\right)\right]$ since $\gamma_{t_1}, t^*_C$ & $\beta_{t_1}$ are constant

$= \gamma_{t_1}\left(cdt_{K_1^*} + \beta_{t_1}.dx_{K_1^*}\right)$

So we got what we expected $\boxed{c.dt = \gamma_{t_1}\left(c.dt_{K_1^*} + \beta_{t_1}.dx_{K_1^*}\right)}$

**b.** Now what is the meaning of the second expression $\frac{dr^*_a}{dt} = \frac{dr^*_a}{dt_1} = \frac{dr^*_a}{\gamma dt^*}$ ?

The answer of the question need to clarify what we are actually doing in the reasoning of this article. First, we start to suppose the **knowledge of** the movement of the center of mass C, for each time t of K. This knowledge **imposes** the movement of the reference frame K* since we **choose to define** it such that, around each time t, it coincides with the family of Galilean reference frame $\left(K^*(t)\right)_{t\in R}$

- in a uniform rectilinear translation relative to K (with the speed of C: $\boldsymbol{V}_{C/K^*}$);
- and having for spatial origin the position of C.

So we have parameterized the reference frame K* with the time $t_i$ of K with a map, say g:

$$\boldsymbol{g}: t_i \to K^*(t_i) \text{ also noted } K_i^*$$

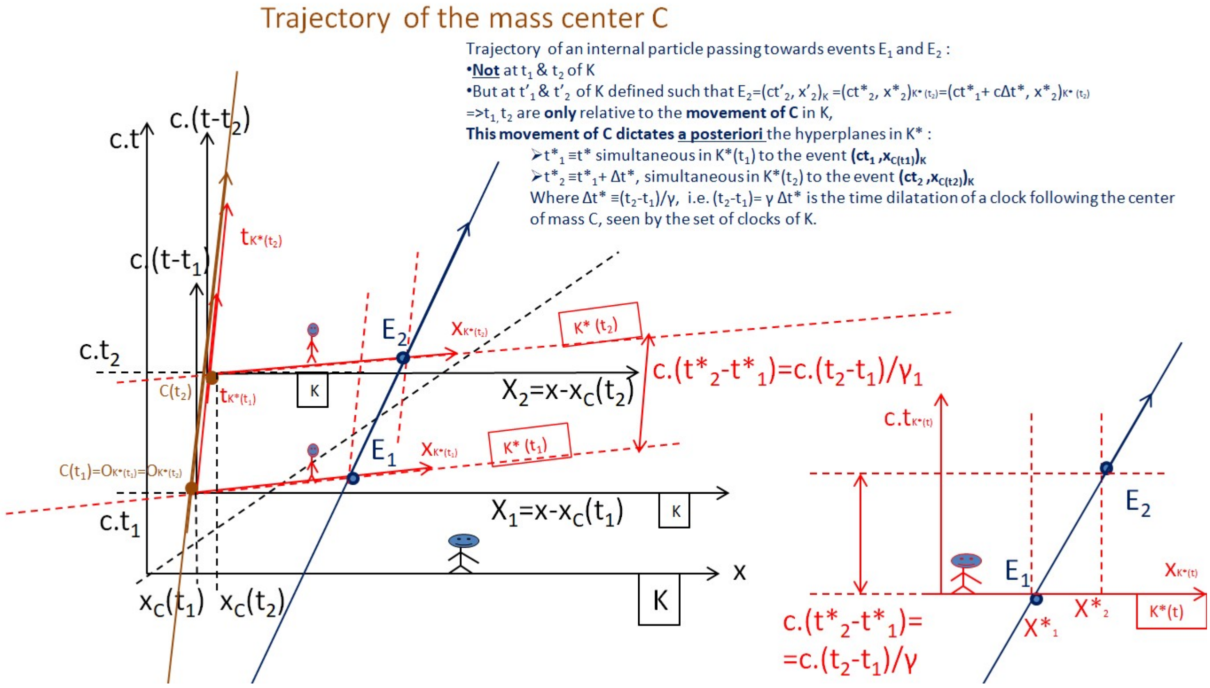
Secondly, what are the events involved in the two frames ? We are studying a particle “a” of a material system with C as its mass center. We can a priori think that, at the instant $t_1$ of K, since we study an event $\left(ct_1, x_a(t_1)\right)_K$ , we have to study in $K^*(t_1)$ the **same event** seen with the different coordinate due to the direct application of the Lorentz transformation to $\left(ct_1, x_a(t_1)\right)_K$ ...But it is actually **not the case.**

Indeed, at the instant $t_1$ of K we apply the map **g** defined above and we observe in $K^*(t_1)$ **all the elements which are simultaneous with the event associated to the spatio-temporal position of C**: $(ct_1, x_c)_K$.

So contrary to the case 1), in the case 2): we are not studying the same event (the same spatio-temporal position of the partcicle “a”) in two different frame but :

- An event $\left(ct_1, x_a(t_1)\right)_K$ in K;
- And an event $E_1 = \left(c.t^*_{C(t_1)}, x_{a,K_1^*}\right)_{K_1^*}$ in $K^*(t_1)$ defined by its **simultaneity** with $(ct_1, x_c)_K$.

By the relativity of the simultaneity, this event $E_1$ in $K^*(t_1)$ cannot be associated to the instant $t_1$ of K. In fact, only the event $(ct_1, x_c)_K$ is analysed with the two reference frame K & $K^*(t_1)$. So we understand why we cannot use the expression of the case a).
In order to visualize the situation, we show below the schematic view of what we are truly doing.




This schematic view use the 2 following expressions calculated in ANNEX:

- $c.t_{(x_{K_i^*}=cte)}(x) = c.t_i + \frac{x-x_c(t_i)}{\beta_{t_i}} - \frac{cte}{\gamma_{t_i}.\beta_{t_i}}$
- $c.t_{(ct_{K_i^*}=cte)}(x) = c.t_i + \beta_{t_i}.(x - x_c(t_i)) + \frac{c\left(cte - t^*_{C(t_i)}\right)}{\gamma_{t_i}}$

We also use the fact that, according to the definition of the reference frame of the centre of mass, the orientation all the hyperplane of simultaneity of $K^*(t_1)$ are (around $t_1$):

- the hyperplanes $t_{K_1^*} = t^*_{C(t_1)}$
- and all the other separated by $dt_{K_1^*} = \frac{dt}{\gamma_{t_1}}$

**Indeed**, thanks to the Lorentz transformation between the reference frame K and $K_i^* \equiv K^*(t_i)$

$$\begin{cases} c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) \\ x - x_c(t_i) = \gamma_{t_i}.\left(x_{K_i^*} + \beta_{t_i}.\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right) \end{cases}, \text{ we have}$$

$$c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) => c.t = c.t_i + \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right)$$

$$=> \boxed{c.t_{(x_{K_i^*}=c\quad)}(ct_{K_i^*}) \ = c.t_i + \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.cte\right)}$$

Its results that relative to K, events situated, at rest, at the origin of $K_1^*$ (that is to say C) and having the time $t_{K_i^*}$ are observed at the time $t_{(x_{K_1^*}=0)}(ct_{K_i^*}) \ = t_1 + \gamma_{t_1}.\left(t_{K_1^*} - t^*_{C(t_1)}\right)$.

This situation is of course relevant for the centre of mass C between the instant $t_1$ and $t_2$:

$$\boxed{t_2 - t_1 = \gamma_{t_1}.\left(t_{K_2^*} - t_{1,K_1^*}\right) <=> t_{K_1^*} - t^*_{C(t_1)} = \frac{t_2 - t_1}{\gamma_{t_1}}.}$$

This relation also relevant to all couples of events having the same position (at rest) in $K^*(t_1)$. So, we have the relation affirmed in 2) and showed in the picture above.

The particle event of the reference frame K* are also parameterized by the time $t$ of K
Indeed, we can define for a particle "a" a map:

$$\boldsymbol{g_a}: t_i \to \mathrm{E}_{t_i} = \left(c.t^*_{C(t_i)}, x_{a,K^*_i}\right)_{K^*_1}$$

That is to say, at each time $t_i$ of K, we associate a frame $K^*(t_i)$ , then the event $\mathrm{E}_{t_i}$ associated to the particle is the one localized in the hyperplane of $K^*(t_i)$ which **contain also C** at the instant $t_i$.

**We are not saying** that the particle "a" is seen at the instant $t_i$ in $K^*(t_i)$ (a non-sense in relativity) but instead it is associated to the instant $t_i$ in the map $\boldsymbol{g_a}$ sense: indeed, the hyperplane of simultaneity of $K^*(t_i)$ is parameterized by $t_i$.

In order to more untangle these relations, we give just below the explicit expression of $\mathrm{E}_i = \mathrm{E}_{t_i}$ in K. To insist in the fact that $\mathrm{E}_i$ is parameterized by the time $t_i$, I will always write it $\mathrm{E}_{t_i}$.

### 2.7.2. What is the coordinates of $\mathrm{E}_{t_1}$ in K?

We suppose the knowledge of the trajectory of C and the internal particle "a" relative to K $x_a(t)$.

At $t_1$, $\mathrm{E}_{t_1}$ has the same plane $c.t^* = ct_{K^*_1}$ than C which has the coordinate $\left(ct^*_{C(t_1)}, 0^*\right)_{K^*_1} = \left(c.\int_0^{t_1} \frac{dt}{\gamma_t}, 0^*\right)_{K^*_1}$ in $K^*(t_1)$.

Moreover at a given coordinate x of "a" in K we have:

$$c.t_{(c.t^*=ct^*_{C(t_1)})}(x) = c.t_1 + \beta_{t_1}.(x - x_c(t_1))$$

What can we choose for x ?

The expression was calculated for a particle "a" on the x-axis of K at a time of K where the function $x_a(t)$ is the x-coordinate associate to $c.t_{(c.t^*=ct^*_{C(t_1)})}$ which different from $t_1$ with a certain duration $\Delta t_1$. The time of K where $\mathrm{E}_{t_1}$ tooks place is :

$$c.t_{(c.t^*=ct^*_{C(t_1)})}\left(x_a(t_1 + \Delta t_1)\right) = c.t_1 + \beta_{t_1}.(x_a(t_1 + \Delta t_1) - x_c(t_1))$$

We can notice that, knowing the trajectories $x_a(t)$ , $x_c(t)$ , $\Delta t_1$ is a solution of the equation:

$$\boxed{\Delta t_1 = \frac{\beta_{t_1}}{c}.(x_a(t_1 + \Delta t_1) - x_c(t_1))}$$

- In a particular case where $x_a(t_1 + \Delta t_1)$ can be developed at the first order, the latter equation is reduced to:

$$\boxed{\Delta t_1{}^{(1)} \approx \frac{\beta_{t_1}}{c}.\left(x_a(t_1) + \frac{dx_a}{dt}(t_1)\Delta t_1{}^{(1)} - x_c(t_1)\right)}$$

$$\iff \Delta t_1{}^{(1)}\left(1 - \frac{\beta_{t_1}}{c}\frac{dx_a}{dt}(t_1)\right) \approx \frac{\beta_{t_1}}{c}.(x_a(t_1) - x_c(t_1))$$

$$\iff \boxed{\Delta t_1 \approx \Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1) - x_c(t_1)}{1 - \beta_{t_1}\frac{V_a}{c}(t_1)}}$$

- In a particular case where $x_a(t_1+\Delta t_1)$ can be developed at the second order, the latter equation is reduced to:

$$\Delta {t_1}^{(2)} \approx \frac{\beta_{t_1}}{c}.\left(x_a(t_1)+\frac{dx_a}{dt}(t_1).\Delta {t_1}^{(2)}+\frac{d^2x_a}{dt^2}(t_1).\frac{{\Delta {t_1}^{(2)}}^2}{2}-x_c(t_1)\right)$$

$$<=> 0 \approx \left[\frac{\beta_{t_1}}{2c}\frac{d^2x_a}{dt^2}(t_1)\right]{\Delta {t_1}^{(2)}}^2+\left[\frac{\beta_{t_1}}{c}\frac{dx_a}{dt}(t_1)-1\right]\Delta {t_1}^{(2)}+\frac{\beta_{t_1}}{c}[x_a(t_1)-x_c(t_1)]$$

$$<=> \boxed{0 \approx \left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]{\Delta {t_1}^{(2)}}^2-\left[1-\beta_{t_1}\frac{V_a}{c}(t_1)\right]\Delta {t_1}^{(2)}+\Delta {t_1}^{(1)}\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}$$

We can try to solve it directly, using the standard solution of the second order equation, but it should be not useful since the solution will not be applicable in the usual case where there is no acceleration...However, there is another way to solve it with the perturbation $\varepsilon$ of the first order solution $\Delta {t_1}^{(1)}$: $\Delta {t_1}^{(2)}=\Delta {t_1}^{(1)}+\varepsilon$

$$0 \approx \left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]{\Delta {t_1}^{(2)}}^2-\left(\Delta {t_1}^{(2)}-\Delta {t_1}^{(1)}\right)\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)$$

$$<=> \Delta {t_1}^{(2)}-\Delta {t_1}^{(1)} \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}{\Delta {t_1}^{(2)}}^2$$

Using $\Delta {t_1}^{(2)}=\Delta {t_1}^{(1)}+\varepsilon$, we have:

$$\varepsilon \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left(\Delta {t_1}^{(1)}+\varepsilon\right)^2=\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left({\Delta {t_1}^{(1)}}^2+\varepsilon^2+2\varepsilon\Delta {t_1}^{(1)}\right)$$

$$<=> \ \varepsilon = \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left({\Delta {t_1}^{(1)}}^2+\varepsilon^2+2\varepsilon\Delta {t_1}^{(1)}\right)$$

$$<=> \ \varepsilon \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left({\Delta {t_1}^{(1)}}^2+2\varepsilon\Delta {t_1}^{(1)}\right) \text{ with } \Delta {t_1}^{(1)} \gg \varepsilon$$

$$<=> \ \varepsilon\left(1-2\Delta {t_1}^{(1)}\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\right) \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}{\Delta {t_1}^{(1)}}^2$$

$$<=> \ \varepsilon \approx \frac{\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}}{1-2\Delta {t_1}^{(1)}\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}}{\Delta {t_1}^{(1)}}^2$$

$$=> \varepsilon \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.\Delta t_1{}^{(1)^2}\left(1+2\Delta t_1{}^{(1)}\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\right) \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.\Delta t_1{}^{(1)^2}$$

$$\Delta t_1{}^{(2)} = \Delta t_1{}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\Delta t_1{}^{(1)^2}$$

With $\Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$

The traditional calculation gives:

$$\Delta = \left[1-\beta_{t_1}\frac{V_a}{c}(t_1)\right]^2 - 4.\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1).\Delta t_1{}^{(1)}\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)$$

$$<=> \boxed{\Delta = \left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)\left(1-\beta_{t_1}\left(\frac{V_a}{c}(t_1)+\frac{a_a}{c}(t_1).2\Delta t_1{}^{(1)}\right)\right)}$$

$$<=> \boxed{\Delta = \left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)\left(1-\beta_{t_1}\frac{V_a}{c}\left(t_1+2\Delta t_1{}^{(1)}\right)\right)}$$

$\Delta > 0 <=> 1 > \beta_{t_1}\frac{V_a}{c}\left(t_1+2.\Delta t_1{}^{(1)}\right)$ which is always true

$$=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right) \pm \sqrt{\left(1-\beta_{t_1}V_a(t_1)\right)\left(1-\beta_{t_1}\frac{V_a}{c}\left(t_1+2\Delta t_1{}^{(1)}\right)\right)}}{\beta_{t_1}\frac{a_a}{c}(t_1)}$$

$$<=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right) \pm \sqrt{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)\left(1-\beta_{t_1}\left(\frac{V_a}{c}(t_1)+\frac{a_a}{c}(t_1).2\Delta t_1{}^{(1)}\right)\right)}}{\beta_{t_1}\frac{a_a}{c}(t_1)}$$

$$<=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}\left[1 \pm \sqrt{\frac{1-\beta_{t_1}\left(\frac{V_a}{c}(t_1)+\frac{a_a}{c}(t_1).2\Delta t_1{}^{(1)}\right)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}}\right]$$

$$<=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}V_a(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}\left[1 \pm \sqrt{1-\frac{\beta_{t_1}\frac{a_a}{c}(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.2\Delta t_1{}^{(1)}}\right]$$

$$<=> \Delta t_1{}^{(2)} \approx \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}\left[1 \pm 1 \mp \frac{\beta_{t_1}\frac{a_a}{c}(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.\Delta t_1{}^{(1)}\right]$$

$$<=> \Delta t_1{}^{(2)} \approx \mp\Delta t_1{}^{(1)} + (1 \pm 1)\frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}$$

$$\Delta t_1{}^{(2)} \approx \Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$$

$$or \ \approx -\Delta t_1{}^{(1)} + 2\frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}a_a(t_1)}$$

As explained, this solution relevant only when $a_a(t_1) \neq 0$

I will not use this one, I will use the first showed above.

-------------------------------------------------------

The position where $\mathrm{E}_{t_1}$ takes place in K is therefore $x_a(t_1 + \Delta t_1)$:

$$\text{With } \Delta t_1 \approx \Delta t_1{}^{(1)} + \frac{\left[\frac{\beta_{t_1} a_a}{2\ c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)} \Delta t_1{}^{(1)2}, \text{ and } \Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c} \cdot \frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$$

We have finally:

$$\boxed{\mathrm{E}_{t_1} = \left(c.\, t^*_{C(t_1)}, x_{a,K_1^*}\right)_{K_1^*} = \left(c(t_1 + \Delta t_1), x_a(t_1 + \Delta t_1)\right)_K}$$

with $\Delta t_1 = \frac{\beta_{t_1}}{c}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1)\right)$, that we can call it **the shift time** : the time to wait after $t_1$ in order to have the event “the particle “a” arrives on the hyper plane of $K^*(t_1)$ ".

We can notice that:

- $\mathrm{E}_{t_1} \neq (c.\, t_1, \dots)_K$
- $\boldsymbol{g_a}: t_i \rightarrow \mathrm{E}_{t_i} = \left(c.\,(t_i + \Delta t_i), x_a(t_i + \Delta t_i)\right)_K$

We clearly see that $\mathrm{E}_{t_1}$ is parameterized by $t_1$ although it is not seen at this instant in K but at the instant $t = t_1 + \Delta t_1$.

Another interesting point is that, at the $t_1$ , the internal events that take place in $K^*(t_1)$ are not of the kind $\left(c(t_1), x_a(t_1)\right)_K$ but the “shifted” version$\left(c(t_1 + \Delta t_1), x_a(t_1 + \Delta t_1)\right)_K$. That is to say the internal events considered (spatio-temporal position of particle) will happen in the future (or the past, depending the position compared to the mass centre). The weird consequence (another one of relativity...) is that **the internal energy and so the mass, is relative to the future and the past of the material system** (and also field as we will see below), in the point of view of K.

### 2.7.3. What is the difference of coordinates of the particle for infinitesimal interval dt, seen in K ?

With the same reasoning, we have at the instant $t_2$ just after $t_1$:

$$\mathrm{E}_{t_2} = \left(c.\, t^*_{C(t_2)}, x_{a,K_2^*}\right)_{K_2^*} = \left(c(t_2 + \Delta t_2), x_a(t_2 + \Delta t_2)\right)_K$$

$$\text{With } \Delta t_2 \approx \Delta t_2{}^{(1)} + \frac{\left[\frac{\beta_{t_2} a_a}{2\ c}(t_2)\right]}{1-\beta_{t_2}\frac{V_a}{c}(t_2)} \Delta t_2{}^{(1)2}, \text{ and } \Delta t_2{}^{(1)} \equiv \frac{\beta_{t_2}}{c} \cdot \frac{x_a(t_2)-x_c(t_2)}{1-\beta_{t_2}\frac{V_a}{c}(t_2)}$$

So by doing the simple algebraic difference in K, we have:

$$\mathrm{E}_{t_2} - \mathrm{E}_{t_1} = \left(c.\,(t_2 + \Delta t_2), x_a(t_2 + \Delta t_2)\right)_K - \left(c.\,(t_1 + \Delta t_1), x_a(t_1 + \Delta t_1)\right)_K$$

$$= \left(c.\,(t_2 - t_1) + c(\Delta t_2 - \Delta t_1), [x_a]_{t_1 + \Delta t_1}^{t_2 + \Delta t_2}\right)_K$$

With $[x_a]_{t_1 + \Delta t_1}^{t_2 + \Delta t_2} \equiv x_a(t_2 + \Delta t_2) - x_a(t_1 + \Delta t_1)$

When $(t_2 - t_1)$ tends to dt (no 2[nd] degree), we have:

- $\beta_{t_2} = \beta_{t_1} + \left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$
- $\Delta t_2 = \Delta t_1 + (t_2 - t_1)\left(\frac{d}{dt}\Delta t\right)_{t_1}$

With:

- $\Delta t_1 \approx \Delta t_1{}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\Delta t_1{}^{(1)^2}$
- $\Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$
- $\left(\frac{d}{dt}\Delta t_1\right)_{t_1} = \frac{d}{dt}\left(\Delta t_1{}^{(1)} + \Delta t_1{}^{(1)^2}\frac{1}{2c}\frac{\beta_{t_1}a_a}{1-\frac{1}{c}\beta_{t_1}V}\right)$

Moreover $[x_a]_{t_1+\Delta t_1}^{t_2+\Delta t_2} = x_a(t_2 + \Delta t_2) - x_a(t_1 + \Delta t_1) = x_a(t_1 + (t_2 - t_1) + \Delta t_2) - x_a(t_1 + \Delta t_1)$

$= x_a(t_1 + (t_2 - t_1) + \Delta t_2) - x_a(t_1 + \Delta t_1)$

$$= x_a\left(t_1 + (t_2 - t_1) + \Delta t_1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}(t_2 - t_1)\right) - x_a(t_1 + \Delta t_1)$$

$$= x_a\left(t_1 + \Delta t_1 + (t_2 - t_1)\left[1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right]\right) - x_a(t_1 + \Delta t_1)$$

$$= x_a(t_1 + \Delta t_1) + \left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}(t_2 - t_1)\left[1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right] - x_a(t_1 + \Delta t_1)$$

$$\boxed{[x_a]_{t_1+\Delta t_1}^{t_2+\Delta t_2} = (t_2 - t_1).\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}.\left[1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right]}$$

$$=> \mathrm{E}_{t_2} - \mathrm{E}_{t_1} = \left(c.(t_2 - t_1) + c(\Delta t_2 - \Delta t_1), [x_a]_{t_1+\Delta t_1}^{t_2+\Delta t_2}\right)_K$$

$$= \left(c.(t_2 - t_1) + c\left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}(t_2 - t_1), (t_2 - t_1).\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}.\left(1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right)\right)_K$$

$$= c.(t_2 - t_1)\left(1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right).\left(1, \frac{1}{c}\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}\right)_K$$

$$=> \boxed{\mathrm{E}_{t_2} - \mathrm{E}_{t_1} = c.(t_2 - t_1)\left(1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right).\left(1, \frac{1}{c}\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}\right)_K}$$

With:

- $\Delta t_1 \approx \Delta t_1{}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\Delta t_1{}^{(1)^2}$
- $\Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$

- $\left(\frac{d}{dt}\Delta t_1\right)_{t_1} = \frac{d}{dt}\left(\Delta t_1{}^{(1)} + \Delta t_1{}^{(1)2}\,\frac{1}{2c}\frac{\beta_{t_1}a_a}{1-\frac{1}{c}\beta_{t_1}V_a}\right)$

We can use this difference of events in order to calculate the speed of a particle "a" with these 2 events, we have:

$$\boxed{\left(\frac{x_{\mathrm{E}_{t_2}} - x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}} - t_{\mathrm{E}_{t_1}}}\right)_{t_1,K} = V_a(t_1 + \Delta t_1)}$$

The speed associated to the 2 events $\mathrm{E}_{t_2}$ & $\mathrm{E}_{t_1}$ is actually different than the one associated to the speed measured by K in the standard way. It is of course different to study in K 2 events observed at the instant $t_1$ & $t_1 + dt$ than the 2 others at $t_1 + \Delta t_1$ & $t_2 + \Delta t_2$.

We recover the standard speed at a given time t when the particle is sufficiently close to the mass centre C =>$\Delta t_1{}^{(1)} \approx 0$.

### 2.7.4. What is the difference of coordinates of the particle for infinitesimal interval dt, seen in K*

The first event is:

$$\mathrm{E}_{t_1} = \left(ct_{a_{(t_1)},K_1^*}, x_{a_{(t_1)},K_1^*}\right)_{K_1^*} = \left(ct^*_{C(t_1)}, x_{a_{(t_1)},K_1^*}\right)_{K_1^*} = \left(c.t_1 + c.\Delta t_1, x_a(t_1 + \Delta t_1)\right)_K$$

Remark: we use the expression $x_{a_{(t_1)},K_1^*}$ as we have explained above that the events in $K^*(t_1)$ are parameterized via the map **g**$_a$.

According to Lorentz $\begin{cases}(c.t_1 + c.\Delta t_1) - c.t_1 = \gamma_{t_i}.\left(\mathrm{c}\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) + \beta_{t_1}.x_{a_{(t_1)},K_1^*}\right)\\ x_a(t_1 + \Delta t_1) - x_c(t_1) = \gamma_{t_1}.\left(x_{a_{(t_1)},K_1^*} + \beta_{t_1}.\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right)\right)\end{cases}$

<=>

$$\begin{cases}c\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) = \gamma_{t_1}.\left((c.t_1 + c.\Delta t_1 - c.t_1) - \beta_{t_1}.(x_a(t_1 + \Delta t_1) - x_c(t_1))\right)\\ x_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1) - \beta_{t_1}.(c.t_1 + c.\Delta t_1 - c.t_1)\right)\end{cases}$$

$$<=>\begin{cases}t_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(\beta_{t_1}.(x_a(t_1 + \Delta t_1) - x_c(t_1)) - \beta_{t_1}.(x_a(t_1 + \Delta t_1) - x_c(t_1))\right)\\ x_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1) - \beta_{t_1}.c.\Delta t_1\right)\end{cases}$$

$$<=>\begin{cases}c\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) = 0 => \textit{ as it should}\\ x_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1) - \beta_{t_1}.\beta_{t_1}.x_a(t_1 + \Delta t_1) - x_c(t_1)\right)\end{cases}$$

$$\boxed{<=>\begin{cases}\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) = 0\\ x_{a_{(t_1)},K_1^*} = \frac{x_a(t_1 + \Delta t_1) - x_c(t_1)}{\gamma_{t_1}} = \frac{c\Delta t_1}{\beta_{t_1}.\gamma_{t_1}}\end{cases}}$$

We use $\Delta t_1 = \frac{\beta_{t_1}}{c}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1)\right)$

The second event is:

$$\mathrm{E}_2 = \left(ct_{a_{(t_2)},K_2^*}, x_{a_{(t_2)},K_2^*}\right)_{K_1^*} = \left(c.t^*_{C(t_2)}, x_{a_{(t_2)},K_2^*}\right)_{K_2^*} = \left(c.(t_2 + \Delta t_2), x_a(t_2 + \Delta t_2)\right)_K$$

But, in point of view of $K_1^*$ we have also

$$\mathrm{E}_2 = \left(c.t_{a_{(t_2)},K_1^*}, x_{a_{(t_2)},K_1^*}\right)_{K_1^*} = \left(c.t_2 + \beta_{t_2}.(x_a(t_2) - x_c(t_2)), x_a(t_2)\right)_K$$

Remark:

- In the notation $t_{a_{(t_2)},K_1^*}$ we have to note the small change: this is the event in the hyperperplane of $K^*(t_2)$ parametrized at $t_2$ but seen by an observatory in the frame$K^*(t_1)$.
- $c.t_{a_{(t_2)},K_1^*} \neq t_{C(t_1)}^*$ a priori

$$\begin{cases} c\left(t_{a_{(t_2)},K_1^*} - t_{C(t_1)}^*\right) = \gamma_{t_1}.\left(c.(t_2+\Delta t_2) - c.t_1 - \beta_{t_1}.(x_a(t_2+\Delta t_2) - x_c(t_1))\right) \\ x_{a_{\mathrm{E}_2},K_1^*} = \gamma_{t_1}.\left(x_a(t_2+\Delta t_2) - x_c(t_1) - \beta_{t_1}.(c.(t_2+\Delta t_2) - c.t_1)\right) \end{cases}$$

$$c\left(t_{a_{(t_2)},K_1^*} - t_{C(t_1)}^*\right) = \gamma_{t_1}.\left(c.(t_2+\Delta t_2) - c.t_1 - \beta_{t_1}.(x_a(t_2+\Delta t_2) - x_c(t_2) - x_c(t_1) + x_c(t_2))\right)$$

$$= \gamma_{t_1}.\left(c.(t_2+\Delta t_2) - c.t_1 - c\frac{\Delta t_2}{\beta_{t_2}}\beta_{t_1} - \beta_{t_1}(-x_c(t_1) + x_c(t_2))\right)$$

$$= \gamma_{t_1}.\left(c.t_2 + c.\Delta t_2 - c.t_1 - c\frac{\Delta t_2}{\beta_{t_2}}\beta_{t_1} - (t_2 - t_1)\frac{\beta_{t_1}}{c}V_c(t_1)\right)$$

$$= \gamma_{t_1}.\left(c.(t_2 - t_1) + c.\Delta t_2 - c\frac{\Delta t_2}{\beta_{t_2}}\beta_{t_1} - (t_2 - t_1){\beta_{t_1}}^2\right)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(1 - \frac{\beta_{t_1}}{\beta_{t_2}}\right) + (t_2 - t_1)\left(1 - \frac{{\beta_{t_1}}^2}{c}\right)\right)$$

because $\Delta t_2 = \Delta t_1 + (t_2 - t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}$

and $\Delta t_2 = \frac{\beta_{t_2}}{c}.(x_a(t_2 + \Delta t_2) - x_c(t_2))$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(1 - \frac{\beta_{t_1}}{\beta_{t_1} + \left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)}\right) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$\frac{1}{\beta_{t_1} + \left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)} = \frac{1}{\beta_{t_1}} + \left(\frac{d}{dX}\left(\frac{1}{X}\right)\right)_{X=\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1) = \frac{1}{\beta_{t_1}} + \left(\frac{-1}{X^2}\right)_{X=\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$= \frac{1}{\beta_{t_1}} - \frac{1}{{\beta_{t_1}}^2}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(1 - \beta_{t_1}\left(\frac{1}{\beta_{t_1}} - \frac{1}{{\beta_{t_1}}^2}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)\right)\right) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)\right) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$= c.\frac{(t_2 - t_1)}{\gamma_{t_1}} + \gamma_{t_1}.c.\Delta t_2\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$= c.\frac{(t_2 - t_1)}{\gamma_{t_1}} + \gamma_{t_1}.c.\left(\Delta t_1 + (t_2 - t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$\boxed{c\left(t_{a_{(t_2)},K^*_1}-t^*_{C(t_1)}\right)_{K*not\ Galilean}=c.\frac{(t_2-t_1)}{\gamma_{t_1}}+\gamma_{t_1}.\frac{c.\Delta t_1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2-t_1)}$$

But, since we use at each time a local Galilean frame, there are non acceleration for this frame (the condition for the use of Lorentz transformation): $\left(\frac{d\beta_t}{dt}\right)_{t_1,Galilean}\equiv 0$

$$\boxed{c\left(t_{a_{(t_2)},K^*_1}-t^*_{C(t_1)}\right)=c.\frac{(t_2-t_1)}{\gamma_{t_1}}}$$

$$x_{a_{(t_2)},K^*_1}=\gamma_{t_1}.\left(x_a(t_2+\Delta t_2)-x_c(t_1)-\beta_{t_1}.(c.(t_2+\Delta t_2)-c.t_1)\right)$$

$$=\gamma_{t_1}.\left(x_a(t_1+(t_2-t_1)+\Delta t_2)-x_c(t_1)-\beta_{t_1}.(c.\Delta t_2+c.(t_2-t_1))\right)$$

$$=\gamma_{t_1}.\left(x_a\left(t_1+(t_2-t_1)+\Delta t_1+(t_2-t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)-x_c(t_1)-\beta_{t_1}.c\left(\Delta t_1+(t_2-t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)-\beta_{t_1}.(c.(t_2-t_1))\right)$$

$$=\gamma_{t_1}.\left[x_a\left(t_1+\Delta t_1+(t_2-t_1)\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right)-x_c(t_1)-\beta_{t_1}.c\left(\Delta t_1+(t_2-t_1)\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right)\right]$$

$$=\gamma_{t_1}.\left[x_a(t_1+\Delta t_1)+\left(\frac{d}{dt}x_a\right)_{t_1+\Delta t_1}.(t_2-t_1)\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)-x_c(t_1)-\beta_{t_1}.c\left(\Delta t_1+(t_2-t_1)\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right)\right]$$

$$=\gamma_{t_1}.\left[x_a(t_1+\Delta t_1)-x_c(t_1)+V_a(t_1+\Delta t_1).(t_2-t_1)\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)-\beta_{t_1}.c\Delta t_1-\beta_{t_1}.c(t_2-t_1)\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right]$$

$$=\gamma_{t_1}.\left(c\frac{\Delta t_1}{\beta_{t_1}}-\beta_{t_1}.c\Delta t_1+(t_2-t_1).\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\left[V_a(t_1+\Delta t_1)-\beta_{t_1}.c\right]\right)$$

$$=\gamma_{t_1}.\left(c\Delta t_1\left(\frac{1-\beta_{t_1}{}^2}{\beta_{t_1}}\right)+(t_2-t_1).\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\left[V_a(t_1+\Delta t_1)-\beta_{t_1}.c\right]\right)$$

$$=\gamma_{t_1}.\left(\frac{c\Delta t_1}{\gamma_{t_1}{}^2}\frac{1}{\beta_{t_1}}+(t_2-t_1).\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\left[V_a(t_1+\Delta t_1)-\beta_{t_1}.c\right]\right)$$

$$=\gamma_{t_1}(t_2-t_1).\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).c\left(\frac{V_a}{c}(t_1+\Delta t_1)-\beta_{t_1}\right)$$

Because $x_{a_{(t_1)},K^*_1}=\frac{x_a(t_1+\Delta t_1)-x_c(t_1)}{\gamma_{t_1}}=\frac{c\Delta t_1}{\beta_{t_1}.\gamma_{t_1}}$

The expression of the

$$\boxed{\begin{gathered}x_{a_{(t_2)},K^*_1}-x_{a_{(t_1)},K^*_1}=\gamma_{t_1}(t_2-t_1).\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).c\left(\frac{V_a}{c}(t_1+\Delta t_1)-\beta_{t_1}\right)\\ c\left(t_{a_{(t_2)},K^*_1}-t^*_{C(t_1)}\right)=c.\frac{(t_2-t_1)}{\gamma_{t_1}}\end{gathered}}$$

### 2.7.5. What is the expression of the speed in K and K*and what are their relation (velocity addition formula)?

Using the expression above, we calculate different speed for different frame.

- Relative to the internal frame $K^*(t_1)$

$$\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} = \frac{\gamma_{t_1}(t_2 - t_1).\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).c\left[\frac{V_a}{c}(t_1 + \Delta t_1) - \beta_{t_1}\right]}{c.\frac{(t_2 - t_1)}{\gamma_{t_1}}}$$

$$\Leftrightarrow \boxed{\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} = {\gamma_{t_1}}^2\left(1 + \left(\frac{d}{dt}\Delta t\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)}$$

$$\Rightarrow \boxed{\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_2 - t_1} = \gamma_{t_1}.\left(1 + \left(\frac{d}{dt}\Delta t\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)}$$

- A modified velocity addition formula

Since $\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K} = V_a(t_1 + \Delta t_1)$, we have

$$\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} = {\gamma_{t_1}}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)$$

$$= {\gamma_{t_1}}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).\left(\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K} - V_C(t_1)\right)$$

But the "shift time" is:

$$\Delta t_1 = \frac{\beta_{t_1}}{c}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1)\right)$$

$$\Leftrightarrow \Delta t = \frac{\beta_t}{c}.\left(x_a(t + \Delta t) - x_c(t)\right)$$

$$\Rightarrow \frac{d\Delta t}{dt} = \frac{\beta_t}{c}.\left(\frac{dt + \Delta t}{dt}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)} - \frac{d}{dt}x_c(t)\right)$$

$$\Leftrightarrow \frac{d\Delta t}{dt} = \frac{\beta_t}{c}.\left(\left(1 + \frac{d\Delta t}{dt}\right)\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)} - \frac{d}{dt}x_c(t)\right)$$

$$\Leftrightarrow \frac{d\Delta t}{dt}\left(1 - \frac{\beta_t}{c}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}\right) = \frac{\beta_t}{c}.\left(\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)} - \frac{d}{dt}x_c(t)\right)$$

$$\Leftrightarrow \boxed{\frac{d\Delta t}{dt} = \frac{\frac{\beta_t}{c}.\left(\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)} - \frac{d}{dt}x_c(t)\right)}{1 - \frac{\beta_t}{c}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}}}$$

$$=> 1+\frac{d\Delta t}{dt}=\frac{1-\frac{\beta_t}{c}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}+\frac{\beta_t}{c}.\left(\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}-\frac{d}{dt}x_c(t)\right)}{1-\frac{\beta_t}{c}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}}$$

$$=\frac{1-\frac{{\beta_t}^2}{c^2}}{1-\frac{\beta_t}{c}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}}=\frac{1}{{\gamma_{t_1}}^2}\frac{1}{1-\frac{\beta_t}{c}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}}$$

$$=> \frac{x_{a_{(t_2)},K_1^*}-x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*}-t_{a_{(t_1)},K_1^*}}={\gamma_{t_1}}^2\left(1+\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).\left(\left(\frac{x_{\mathrm{E}_{t_2}}-x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}}-t_{\mathrm{E}_{t_1}}}\right)_{t_1,K}-V_C(t_1)\right)$$

$$=\frac{\left(\frac{x_{\mathrm{E}_{t_2}}-x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}}-t_{\mathrm{E}_{t_1}}}\right)_{t_1,K}-V_C(t_1)}{1-\frac{\beta_t}{c}\left(\frac{dx_a}{dt}\right)_{(t+\Delta t)}}$$

$$=> \boxed{\frac{x_{a_{(t_2)},K_1^*}-x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*}-t_{a_{(t_1)},K_1^*}}=\frac{\left(\frac{x_{\mathrm{E}_{t_2}}-x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}}-t_{\mathrm{E}_{t_1}}}\right)_{t_1,K}-V_C(t_1)}{1-\frac{\beta_{t_1}}{c}\left(\frac{dx_a}{dt}\right)_{(t_1+\Delta t_1)}}}$$

We revover the Einstein-Poincaré formula when the system is close to its center of mass ($\Delta t_1 \approx 0$)or otherwise for particles without acceleration.

In the case of an accelerated particule in K, we have:

$$\left(\frac{dx_a}{dt}\right)_{(t_1+\Delta t_1)}\approx\left(\frac{dx_a}{dt}\right)_{(t_1)}+\left(\frac{d^2x_a}{dt^2}\right)_{(t_1)}\Delta t_1$$

$$=> \boxed{\frac{x_{a_{(t_2)},K_1^*}-x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*}-t_{a_{(t_1)},K_1^*}}\approx\frac{\left(\frac{x_{\mathrm{E}_{t_2}}-x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}}-t_{\mathrm{E}_{t_1}}}\right)_{t_1,K}-V_C(t_1)}{1-\frac{\beta_{t_1}}{c}\left(\left(\frac{dx_a}{dt}\right)_{(t_1)}+\left(\frac{d^2x_a}{dt^2}\right)_{(t_1)}\Delta t_1\right)}}$$

$\approx\left[\left(\frac{x_{\mathrm{E}_{t_2}}-x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}}-t_{\mathrm{E}_{t_1}}}\right)_{t_1,K}-V_C(t_1)\right]\left(1+\frac{\beta_{t_1}}{c}\left(\frac{dx_a}{dt}\right)_{(t_1)}+\frac{\beta_{t_1}}{c}\left(\frac{d^2x_a}{dt^2}\right)_{(t_1)}\Delta t_1\right)$ for sufficiently low speed and/or low acceleration and/or low dimension.

Interestingly, we see that if we cannot neglect the dimension of the system, a gravitational field $g=\left(\frac{d^2x_a}{dt^2}\right)_{(t_1)}$, seen in K, modifies the speed addiction formula as:

$$\boxed{\frac{x_{a_{(t_2)},K_1^*}-x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*}-t_{a_{(t_1)},K_1^*}}\approx\frac{\left(\frac{x_{\mathrm{E}_{t_2}}-x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}}-t_{\mathrm{E}_{t_1}}}\right)_{t_1,K}-V_C(t_1)}{1-\frac{\beta_{t_1}}{c}\left(\left(\frac{dx_a}{dt}\right)_{(t_1)}+g_{(t_1)}\Delta t_1\right)}}$$

The characteristic acceleration g verifies:

$$1 \approx \frac{\beta_{t_1}}{c} g \Delta t_1$$

<=> $\frac{c}{\Delta t_1 \beta_{t_1}} \approx g$ with $\Delta t_1 = \frac{\beta_{t_1}}{c}.(x_a(t_1 + \Delta t_1) - x_c(t_1)) \approx \frac{\beta_{t_1}}{c} L$

$$=> \boxed{g \approx \frac{c^2}{L{\beta_{t_1}}^2} = \frac{c^4}{L{V_C}^2}}$$

More the system is a point, compared to other dimension of the context, less the dynamic is affected. We can also check that if one of the internal particle has the speed c, the apparent speed is no more the invariant speed c.

$$\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} \approx \frac{c - V_C(t_1)}{1 - \frac{\beta_{t_1}}{c}\left(c + g_{(t_1)}\Delta t_1\right)} = c\frac{1 - \beta_{t_1}}{1 - \beta_{t_1}\left(1 + g_{(t_1)}\frac{\Delta t_1}{c}\right)} = c\frac{1 - \beta_{t_1}}{1 - \beta_{t_1} - \beta_{t_1} g_{(t_1)}\frac{\Delta t_1}{c}}$$

$$= c\frac{1}{1 - \frac{\beta_{t_1} g_{(t_1)}}{1 - \beta_{t_1}}\frac{\Delta t_1}{c}} \approx c\left(1 + \frac{\beta_{t_1} g_{(t_1)}}{1 - \beta_{t_1}}\frac{\Delta t_1}{c}\right) = c + \frac{\beta_{t_1} g_{(t_1)}}{1 - \beta_{t_1}}\Delta t_1$$

It is of course an <u>artefact</u> due to the fact that the particles events considered in K* are <u>not the same</u> as the one treated in K, hence the Lorentz Transformation is not applied in a standard manner.

- A second modified velocity addition formula

Since $\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_2 - t_1} = \gamma_{t_1}.\left(1 + \left(\frac{d}{dt}\Delta t\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)$

With the same reasoning we have

$$\boxed{\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} = \frac{1}{\gamma_{t_1}.}\frac{\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K} - V_C(t_1)}{1 - \frac{\beta_{t_1}}{c}\left(\frac{dx_a}{dt}\right)_{(t_1+\Delta t_1)}}}$$

With

- $\Delta t_1 \approx {\Delta t_1}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1 - \beta_{t_1}\frac{V_a}{c}(t_1)}{{\Delta t_1}^{(1)}}^2$
- ${\Delta t_1}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1) - x_c(t_1)}{1 - \beta_{t_1}\frac{V_a}{c}(t_1)}$
- $\left(\frac{d}{dt}\Delta t_1\right)_{t_1} = \frac{d}{dt}\left({\Delta t_1}^{(1)} + {{\Delta t_1}^{(1)}}^2 \frac{1}{2c}\frac{\beta_{t_1} a_a}{1 - \frac{1}{c}\beta_{t_1} V_a}\right)$

#### 2.7.6. Conclusion about the proof

We can conclude that although during the proof we use a particular duration of time $dt_1 = \gamma dt^*$ , it is well defined as I try to convince the reader in this paragraph 2.6. We should carefully take care to the events implied by this way of reasoning.

## 3. Free field

### 3.1. The proof for a field

Now, I will repeat the same method for a field theory (a scalar field $\varphi$ for simplify), and again:

The important point to keep in mind is that we are not considering the variation of the internal degree of freedom $\varphi^*$ **:**

- relative to the internal time $t^*$ of K* : $\frac{\partial \varphi^*}{\partial t^*}$;
- **but instead relative to time t of K** : $\frac{\partial \varphi^*}{\partial t}$.

So without comments, we have successively:

$$S[\{\varphi(x,t)\}] = \frac{1}{c}\int \iiint \Lambda\left(\varphi, \frac{\partial \varphi}{\partial \boldsymbol{r}}, \frac{\partial \varphi}{\partial t}\right) d\Omega$$

$$= \frac{1}{c}\int \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) d\Omega^* = \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right] dt^*$$

$$= \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right] \frac{dt}{\gamma}$$

=>

$$S[\{\varphi^*(x^*,t^*)\}, \boldsymbol{R}_\mathrm{C}(\boldsymbol{t})] = \int L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right] dt$$

With $L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right] = \frac{1}{\gamma}\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$

So we can calculate the 3-momentum as:

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\left[\frac{1}{\gamma}\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right]$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma} + \frac{1}{\gamma}\iiint \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \left(-\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)$$

$$+ \frac{1}{\gamma}\iiint \frac{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} \frac{\partial}{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

But $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma} = -\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}$ ; $\gamma\frac{\partial \varphi^*}{\partial t} = \frac{\partial \varphi^*}{\partial t^*}$

And $\frac{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{\partial \gamma}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{\partial\frac{1}{\sqrt{1-\frac{\boldsymbol{V}_\mathrm{C}^2}{\boldsymbol{c}^2}}}}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{-1}{2}\left(-2\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)\frac{1}{\left(1-\frac{\gamma^2\left(\frac{d\boldsymbol{r}_a^*}{dt}\right)^2}{\boldsymbol{c}^2}\right)^{3/2}} = \frac{\partial \varphi^*}{\partial t}\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\gamma^3$

$$\boldsymbol{P_c} = \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* \left(-\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\right)$$
$$+\frac{1}{\gamma}\iiint \left(\frac{\partial \varphi^*}{\partial t}\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma^3\right)\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* (-1)$$
$$+\iiint \left(\frac{\partial \varphi^*}{\partial t}\gamma\right)\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma\left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* (-1) + \iiint \frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*\right]$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma \iiint \left[\frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right] dV^*$$

So we have again:

$$\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{C}}$$

where $E^* \equiv \iiint \left(\frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right) dV^*$ is the internal energy (associated to the hyperplane $t^* = cte$)

And also:

$$M = M(t^*) = M\left(\int_0^t \frac{dt'}{\gamma(t')}\right) = \frac{\iiint \left(\frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right) dV^*}{c^2}$$

We see that we don't need to talk about closed system hypothesis or to have a 4 vector to demonstrate it (we don't even use the expression of any density Lagrangian).

We have to note, in the proof, the importance to freeze the right variable $\frac{\partial \varphi^*}{\partial t}$ (and not $\frac{\partial \varphi^*}{\partial t^*}$ ) in order to have the good expression.

## 3.2. Momentum and energy for a field

### 3.2.1. Momentum

We can also notice that $\boldsymbol{p}_{\varphi^*} \equiv \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} = \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)}$, so $\boldsymbol{p}_{\varphi^*} = \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)}$ as for material syste this is the same as the one we would have in the frame of the centre of mass K*.

More over the total momentum $\boldsymbol{P}_{total}$ associated to the Lagrangian $L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial r^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right]$ is

$\boldsymbol{P}_{total} = \iiint \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} dV^* + \frac{\partial L'}{\partial \boldsymbol{v}_\mathrm{C}} = \iiint \boldsymbol{p}_{\varphi^*} dV^* + \boldsymbol{P}_c = \boldsymbol{P}_c$ since by definition of K*: $\iiint \boldsymbol{p}_{\varphi^*} dV^* \equiv 0$.

As the material system above, we obtain as it should the total momentum is the one associated to the mass center.

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**Proof:**

$$\boldsymbol{p}_{\varphi^*} \equiv \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} = \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \frac{\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial r^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)}{\gamma}}{\partial \frac{\partial \varphi^*}{\partial t}} = \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \left(\frac{\partial \left(\gamma \frac{\partial \varphi^*}{\partial t}\right)}{\partial \frac{\partial \varphi^*}{\partial t}}\right) \left(\frac{\partial \frac{\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial r^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)}{\gamma}}{\partial \left(\gamma \frac{\partial \varphi^*}{\partial t}\right)}\right) = \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} (\gamma) \left(\frac{1}{\gamma} \frac{\partial \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial r^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)}\right)$$

$$= \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial r^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)} = \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)}, \text{ so } \boldsymbol{p}_{\varphi^*} = \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$$

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### 3.2.2. Energy

By definition the energy is: $E' \equiv \iiint \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} \frac{\partial \varphi^*}{\partial t} dV^* + \frac{\partial L'}{\partial \boldsymbol{v}_\mathrm{C}} \boldsymbol{V}_c - L'$

We can re-express it as:

$E' = \iiint \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} \frac{\partial \varphi^*}{\partial t} dV^* + \boldsymbol{P}_c \boldsymbol{V}_c - \frac{L^{*\prime}}{\gamma}$ since $L' = \frac{L^{*\prime}}{\gamma}$

$= \iiint \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)} \frac{\partial \varphi^*}{\partial t} dV^* + \left(\gamma \frac{E^*}{c^2} . \boldsymbol{V}_c\right) \boldsymbol{V}_c - \frac{L^{*\prime}}{\gamma}$ since $\boldsymbol{p}_{\varphi^*} \equiv \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} = \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)}$

$$= \iiint \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)} \frac{\partial \varphi^*}{\partial t} dV^* - \frac{L^{*\prime}}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V}_c^{\,2}$$

$$= \iiint \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)} \frac{\partial \varphi^*}{\gamma \partial t^*} dV^* - \frac{L^{*\prime}}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V}_c^{\,2} = \left(\iiint \frac{\partial \Lambda^*}{\partial \left(\frac{\partial \varphi^*}{\partial t^*}\right)} \frac{\partial \varphi^*}{\partial t^*} dV^* - L^{*\prime}\right) \frac{1}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V}_c^{\,2}$$

$$= \frac{E^*}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V}_c^{\,2} = \frac{E^*}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V}_c^{\,2} = \frac{E^* + \gamma^2 \frac{E^*}{c^2} . \boldsymbol{V}_c^{\,2}}{\gamma} = E^* \frac{1 + \gamma^2 . \beta^2}{\gamma} = E^* \frac{1 + \frac{\beta^2}{1 - \beta^2}}{\gamma}$$

$$= E^* \frac{\frac{1 - \beta^2 + \beta^2}{1 - \beta^2}}{\gamma} = E^* \frac{\frac{1}{1 - \beta^2}}{\gamma} = E^* \frac{\gamma^2}{\gamma} = \gamma E^*$$

So we have, as it should:

$$\boxed{E' = E = \gamma E^*}$$

We can also conventionally note: $E = E^* + (\gamma - 1)E^*$ where we observe, for a closed system (E=cte), an exchange of Energy between the internal energy $E^*$ and the kinetic energy $(\gamma - 1)E^*$, the one depending of the center of mass.

### 3.3. The Euler-Lagrange equation for the internal field and the mass center

The Euler-Lagrange equations in K are :

$$\frac{d}{dt}\frac{\partial}{\partial \boldsymbol{V}_{\mathbf{C}}} L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right] = \frac{\partial}{\partial \boldsymbol{R}_{\mathrm{C}}} L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right]$$

$$\Leftrightarrow \frac{d}{dt}\left(\gamma \frac{E^*}{c^2} \boldsymbol{V}_{\mathrm{C}}\right) = \frac{\partial}{\partial \boldsymbol{R}_{\mathrm{C}}} L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial r^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right]$$

And we find in K*:

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}}\right) + \frac{\partial}{\partial \boldsymbol{r}^*}\left(\frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}}\right) = \frac{\partial \Lambda'^*}{\partial \varphi^*} = \frac{1}{\gamma}\frac{\partial \Lambda^*}{\partial \varphi^*}$$

That we can show is equivalent of the 4-dimensional equation in K*:

$$<=> \frac{\partial}{\partial \boldsymbol{x}^i}\left(\frac{\partial \Lambda^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{x}^{i*}}}\right) = \frac{\partial \Lambda^*}{\partial \varphi^*}$$

As above for the material system, we obtain the same equation that we should obtain for the dynamic in a K* frame: the local Galilean frame.

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Proof:

We start from the general Lagrangian

$$S[\{\varphi(x,t)\}] = \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*\right]\frac{dt}{\gamma} = \int \left[\iiint \Lambda'^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*\right] dt$$

With $\Lambda'^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial r^*}, \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) \equiv \frac{\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial r^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right)}{\gamma}$

The variation of the action gives:

$$\delta S[\{\varphi(x,t)\}] = \int \left[\iiint \frac{\partial \Lambda'^*}{\partial \varphi^*}\delta\varphi^* + \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}}\delta\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*} + \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}}\delta\frac{\partial \varphi^*}{\partial t}\right] dV^*\, dt$$

$$= \int \left[\iiint \frac{\partial \Lambda'^*}{\partial \varphi^*}\delta\varphi^* + \left\{\frac{\partial}{\partial \boldsymbol{r}^*}\left(\frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}}\delta\varphi^*\right) - \frac{\partial}{\partial \boldsymbol{r}^*}\left(\frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}}\right)\delta\varphi^*\right\} + \left\{\frac{\partial}{\partial t}\left(\frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}}\delta\varphi^*\right) - \frac{\partial}{\partial t}\left(\frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}}\right)\delta\varphi^*\right\}\right] dV^*\, dt$$

$$= \int \left[ \iiint \left\{ \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \delta\varphi^* \right) + \frac{\partial}{\partial t} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \delta\varphi^* \right) \right\} + \delta\varphi^* \left\{ \frac{\partial \Lambda'^*}{\partial \varphi^*} - \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right) - \frac{\partial}{\partial t} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} \right) \right\} \right] dV^* \, dt$$

The least action principle tells us that

$$\delta S[\{\varphi(x,t)\}] = 0$$

$$=> \frac{\partial}{\partial t} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} \right) + \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right) = \frac{\partial \Lambda'^*}{\partial \varphi^*} = \frac{1}{\gamma} \frac{\partial \Lambda^*}{\partial \varphi^*}$$

$$<=> \gamma \frac{\partial}{\partial t} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} \right) + \gamma \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right) = \frac{\partial \Lambda^*}{\partial \varphi^*}$$

$<=> \gamma \frac{\partial}{\partial t} \left( \frac{\partial \Lambda^*}{\partial \left( \frac{\partial \varphi^*}{\partial t^*} \right)} \right) + \gamma \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right) = \frac{\partial \Lambda^*}{\partial \varphi^*}$ since $\frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial t}} = \frac{\partial \Lambda^*}{\partial \left( \frac{\partial \varphi^*}{\partial t^*} \right)}$

$$Moreover\ \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda'^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right) = \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \frac{\Lambda^* \left( \varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathbf{C}}, \boldsymbol{V}_{\mathbf{C}} \right)}{\gamma}}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right) = \frac{1}{\gamma} \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right)$$

And $\gamma \frac{\partial}{\partial t} \left( \frac{\partial \Lambda^*}{\partial \left( \frac{\partial \varphi^*}{\partial t^*} \right)} \right) = \frac{\partial}{\partial t^*} \left( \frac{\partial \Lambda^*}{\partial \left( \frac{\partial \varphi^*}{\partial t^*} \right)} \right)$

$$\frac{\partial}{\partial t^*} \left( \frac{\partial \Lambda^*}{\partial \left( \frac{\partial \varphi^*}{\partial t^*} \right)} \right) + \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial \Lambda^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}} \right) = \frac{\partial \Lambda^*}{\partial \varphi^*}$$

$$<=> \frac{\partial}{\partial \boldsymbol{x}^i} \left( \frac{\partial \Lambda^*}{\partial \frac{\partial \varphi^*}{\partial \boldsymbol{x}^{i*}}} \right) = \frac{\partial \Lambda^*}{\partial \varphi^*}$$

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## 4. Example: Application to the Einsteinian gravitational field

According to [1]:

$$S[\{g_{ik}(x,t)\}] = \frac{-c^3}{16\pi k}\int\iiint (R-2\Lambda)\sqrt{-g}d\Omega$$

Where R is the Ricci scalar.

$$\delta S[\{g_{ik}(x,t)\}] = \frac{-c^3}{16\pi k}\delta\int\iiint (R-2\Lambda)\sqrt{-g}d\Omega = \frac{-c^3}{16\pi k}\delta\int\iiint (G-2\Lambda)\sqrt{-g}d\Omega$$

With :

- $G\left(g_{ik},\frac{\partial g_{ik}}{\partial r},\frac{\partial g_{ik}}{\partial t}\right) = g^{ik}\left(\Gamma_{il}^{m}\Gamma_{km}^{l} - \Gamma_{ik}^{l}\Gamma_{lm}^{m}\right)$
- $\Lambda$ the Einstein-(Lemaître) cosmological constant

For linear transformation $\Gamma_{il}^{m}$ behave like tensor, so G behaves as a scalar.

We consider a context where the space is Lorentzian to infinity. The (linear) Lorentz transformation means, in this case, a modification of the speed for the part of the of the observers (associated to the current frame) to infinity. The modification of the coordinate system for other observers are meanwhile not directly evident but allowed.

$$S[\{g_{ik}(x,t)\}] = \frac{-c^3}{16\pi k}\int\iiint \left(G\left(g_{ik},\frac{\partial g_{ik}}{\partial \boldsymbol{r}},\frac{\partial g_{ik}}{\partial t}\right)-2\Lambda\right)\sqrt{-g}d\Omega.$$

$$= \frac{-c^3}{16\pi k}\int\iiint \left(G^*\left({g_{ik}}^*,\frac{\partial {g_{ik}}^*}{\partial \boldsymbol{r}^*},\frac{\partial {g_{ik}}^*}{\partial t^*},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)-2\Lambda\right)\sqrt{-g^*}d\Omega^*$$

$$= \frac{-c^4}{16\pi k}\int\left[\iiint \left(G^*\left({g_{ik}}^*,\frac{\partial {g_{ik}}^*}{\partial \boldsymbol{r}^*},\frac{\partial {g_{ik}}^*}{\partial t^*},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)-2\Lambda\right)\sqrt{-g^*}dV^*\right]dt^*$$

$$= \frac{-c^4}{16\pi k}\int\left[\iiint \left(G^*\left({g_{ik}}^*,\frac{\partial {g_{ik}}^*}{\partial \boldsymbol{r}^*},\gamma\frac{\partial {g_{ik}}^*}{\partial t},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)-2\Lambda\right)\sqrt{-g^*}dV^*\right]\frac{dt}{\gamma}$$

=>

$$S[\{\varphi^*(x^*,t^*)\},\boldsymbol{R}_\mathrm{C}(\boldsymbol{t})] = \frac{-c^4}{16\pi k}\int L'\left[\{{g_{ik}}^*\},\left\{\frac{\partial {g_{ik}}^*}{\partial \boldsymbol{r}^*}\right\},\left\{\frac{\partial {g_{ik}}^*}{\partial t}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right]dt$$

With $L'\left[\{{g_{ik}}^*\},\left\{\frac{\partial {g_{ik}}^*}{\partial r^*}\right\},\left\{\frac{\partial {g_{ik}}^*}{\partial t}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right] = \frac{1}{\gamma}\iiint\left(G^*\left({g_{ik}}^*,\frac{\partial {g_{ik}}^*}{\partial r^*},\gamma\frac{\partial {g_{ik}}^*}{\partial t},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)-2\Lambda\right)\sqrt{-g^*}dV^*$

Repeating the same calculation for the scalar field we have:

$$\boldsymbol{P_c} = \gamma \frac{E^*}{c^2} \boldsymbol{V}_{\mathbf{C}}$$

where $E^* \equiv \frac{-c^4}{16\pi k} \iiint \left( \frac{\partial g_{ik}{}^*}{\partial t^*} \frac{\partial \left(G^* \sqrt{-g^*}\right)}{\partial \left(\frac{\partial g_{ik}{}^*}{\partial t^*}\right)} - (G^* - 2\Lambda)\sqrt{-g^*} \right) dV^*$ is the internal energy (associated to the hyperplane $t^* = cte$)

This is of course coherent with the 4-momentum of [1], paragraph 96.

We have therefore:

$$M = M(t^*) = M\left( \int_0^t \frac{dt'}{\gamma(t')} \right) = \frac{\frac{-c^4}{16\pi k} \iiint \left( \frac{\partial g_{ik}{}^*}{\partial t^*} \frac{\partial \left(G^* \sqrt{-g^*}\right)}{\partial \left(\frac{\partial g_{ik}{}^*}{\partial t^*}\right)} - (G^* - 2\Lambda)\sqrt{-g^*} \right) dV^*}{c^2}$$

The Euler-Lagrange equations:

- For the gravitational free particle of mass  in K:

$$\frac{d}{dt} \frac{\partial}{\partial \boldsymbol{V}_{\mathbf{C}}} L' \left[ \{g_{ik}{}^*\}, \left\{ \frac{\partial g_{ik}{}^*}{\partial \boldsymbol{r}^*} \right\}, \left\{ \frac{\partial g_{ik}{}^*}{\partial t} \right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}} \right] = \frac{\partial}{\partial \boldsymbol{R}_{\mathrm{C}}} L' \left[ \{g_{ik}{}^*\}, \left\{ \frac{\partial g_{ik}{}^*}{\partial \boldsymbol{r}^*} \right\}, \left\{ \frac{\partial g_{ik}{}^*}{\partial t} \right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}} \right]$$

$$<=> \frac{d}{dt}\left( \gamma \frac{E^*}{c^2} \boldsymbol{V}_{\mathbf{C}} \right) = \frac{\partial}{\partial \boldsymbol{R}_{\mathrm{C}}} L' \left[ \{g_{ik}{}^*\}, \left\{ \frac{\partial g_{ik}{}^*}{\partial \boldsymbol{r}^*} \right\}, \left\{ \frac{\partial g_{ik}{}^*}{\partial t} \right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}} \right]$$

- For the internal (Einsteinian) gravitational field in K* ([1]):

$$\frac{\partial}{\partial t} \left( \frac{\partial G^* \sqrt{-g^*}}{\partial \frac{\partial g_{ik}{}^*}{\partial t}} \right) + \frac{\partial}{\partial \boldsymbol{r}^*} \left( \frac{\partial G^* \sqrt{-g^*}}{\partial \frac{\partial g_{ik}{}^*}{\partial \boldsymbol{r}^*}} \right) = \frac{\partial (G^* - 2\Lambda)\sqrt{-g^*}}{\partial g_{ik}{}^*}$$

$$<=> \frac{\partial}{\partial x^{l*}} \left( \frac{\partial G^* \sqrt{-g^*}}{\partial \frac{\partial g_{ik}{}^*}{\partial x^{l*}}} \right) = \frac{\partial (G^* - 2\Lambda)\sqrt{-g^*}}{\partial g_{ik}{}^*}$$

$$<=> R_{ik}{}^* - \frac{1}{2} g_{ik}{}^*.R^* + g_{ik}{}^*.\Lambda = 0 <=> R_{ik}{}^* = 4\,\Lambda$$

- And also $E' = E = \gamma E^*$

Remark: We can observe the impact of the cosmological constant on the mass of every volume $V^*$studied which are increased/decreased by the value $\Delta M_\Lambda = \frac{-c}{8\pi k} \Lambda \iiint \left( \sqrt{-g^*} \right) dV^*$ (there is a divergence for an infinite space...). This is why we can pretend to say that the cosmological constant give a mass to the gravitation field. However it is not a kind of mass which is seen by gravitation waves, that is to say like a more conventional field theory. In the latter sense the gravitation has not a mass (“the graviton has no mass”).

## 5. Interaction between a field and a particle

We consider the simplified action:

$$S[\boldsymbol{r_a}(t),\{\varphi(x,t)\}] = \int_{t_1}^{t_2}\left(\sum_a\left[-m_a.c\frac{ds_a}{dt}-\frac{e_a}{c}.\frac{ds_a}{dt}\varphi(\boldsymbol{r_a},t)\right]\right)dt+\frac{1}{c}\int\iiint\Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega$$

So we have also:

$$S=\int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a+\frac{e_a}{c^2}\varphi(\boldsymbol{r_a},t)\right).c.\frac{ds_a}{dt}\right]\right)dt+\frac{1}{c}\int\iiint\Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega$$

$$=\int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a+\frac{e_a}{c^2}\varphi^{K_m}\right).c^2\frac{1}{\gamma_a^{K_m}}\right]\right)\frac{dt}{\gamma_m}$$
$$+\int\left[\iiint\Lambda^{K_\varphi}\left(\varphi^{K_\varphi},\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}},\gamma\frac{\partial\varphi^{K_\varphi}}{\partial t},\boldsymbol{R}_{\mathrm{c}_\varphi},\boldsymbol{V}_{\mathrm{c}_\varphi}\right)dV^{K_\varphi}\right]\frac{dt}{\gamma_\varphi}$$

Where we have specified the quantities relative to:

- the frame $K_\varphi$ of the center of mass $\mathrm{c}_\varphi$ of the field $\varphi$ ;
- the frame $K_m$ of the center of mass $\mathrm{c}_m$ of the material system.

$$S\left[\{\boldsymbol{r_a}^{K_m}(t^{K_m}),\boldsymbol{R}_{\mathrm{c}_m}(t)\},\{\varphi^{K_\varphi}(x^{K_\varphi},t^{K_\varphi})\},\boldsymbol{R}_{\mathrm{c}_\varphi}(\boldsymbol{t})\right]$$
$$=\int_{t_1}^{t_2}L'\left(\{\boldsymbol{r_a}^{K_m}\},\left\{\frac{d\boldsymbol{r_a}^{K_m}}{dt}\right\},\boldsymbol{R}_{\mathrm{c}_m},\boldsymbol{V}_{\mathrm{c}_m},t\right)dt$$
$$+\int L'\left[\{\varphi^{K_\varphi}\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}}\right\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial t}\right\},\boldsymbol{R}_{\mathrm{c}_\varphi},\boldsymbol{V}_{\mathrm{c}_\varphi}\right]dt$$

So in this form, we can calculate the dynamic of the center of mass of one system and the other.
We can see that each system is not free at all, but we have again:

$$\boldsymbol{P}_{\mathrm{c}_m}=\gamma(\boldsymbol{V}_{\mathrm{c}_m})\frac{E^{K_m}}{c^2}\boldsymbol{V}_{\mathrm{c}_m}$$
$$\boldsymbol{P}_{\mathrm{c}_\varphi}=\gamma\left(\boldsymbol{V}_{\mathrm{c}_\varphi}\right)\frac{E^{K_\varphi}}{c^2}\boldsymbol{V}_{\mathrm{c}_\varphi}$$

So $M_m=\frac{E^{K_m}}{c^2}$**,** $M_\varphi=\frac{E^{K_\varphi}}{c^2}$

With the same method we can consider any set of systems.

## 6. Does the mass of a body depend on the indeterminacy of the origin of energy?

We have showed the generality of the Einstein law (without Momentum tensor). A question frequently come in mind when we derive this law is (see [1]) :

- the accordance between a <u>characteristic</u> quantities of an (apparent) particle, the mass M;
- and the a priori <u>indeterminacy</u> of the origin of the internal energy which is linked with.

How can we reconcile the 2 different aspects of theses quantities?

Moreover, in [1] it is stated that the mass sets the origin of the energy [scale] in relativity, what does that mean ?

### 6.1. A free material system

As above we start from the Least Action Principle:

$$S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$$

And again:

$$S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*), \boldsymbol{R}_c(t)\}] = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dt^*$$

Taking account $dt^* = \frac{dt}{\gamma(t)}$ and returnig to the Galilean frame K we have:

$$S = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{dt}{dt^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right)\frac{dt^*}{dt} dt = \int_{t_1}^{t_2} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})} dt$$

We want to calculate $\boldsymbol{P_c} = \frac{\partial}{\partial V_C}\frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(V_C)\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)}{\gamma(V_C)}$

Thanks to the indeterminacy of the action we can also physically work with the equivalent action:

$$S_{modif}[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L_{modif}\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt = \int_{t_1}^{t_2}\left(L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) + \frac{df(t,\{\boldsymbol{r_a}\})}{dt}\right) dt$$

Which gives now:

$$S_{modif}[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*), \boldsymbol{R}_c(t)\}] = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}}\left(L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) + \frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt^*}\right) dt^*$$

$$= \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}}\left(L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) + \frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt^*}\right)\frac{dt^*}{dt} dt$$

$$= \int_{t_1}^{t_2} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) + \frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt^*}}{\gamma(\boldsymbol{V}_{\mathrm{C}})} dt$$

So we can work with the modified Lagrangian:

$$L'_{modif}\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)=\frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)+\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt}}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$=\frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})}+\frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt}$$

But $\frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt}=\frac{\sum_a\frac{\partial f\left(x_{j,a}{}^*\right)}{\partial x_{i,a}{}^*}dx_{i,a}{}^*}{dt}=\sum_a\left(\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{c.\partial t^*}c\frac{dt^*}{dt}+\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)=$

$$=\sum_a\left(\gamma\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t}\frac{1}{\gamma}+\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)=\left(\sum_a\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t}+\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)$$

$$=\left(\sum_a\frac{dt^*}{dt}\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}+\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)=\left(\sum_a\frac{1}{\gamma}\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}+\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)$$

$$=>\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt}=\left(\sum_a\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}\right)\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma}=\left(\sum_a\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}\right)\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma}$$

$=>\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{df(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{dt}=\left(\sum_a\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}\right)\left(-\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}\right)$ since $\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}=-\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}$

We see that:

$$\boxed{\boldsymbol{P}_{\boldsymbol{c}}=\frac{\partial L'_{modif}}{\partial \boldsymbol{V}_{\mathrm{C}}}=\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}-\frac{\gamma(\boldsymbol{V}_{\mathrm{C}})\left(\sum_a\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}\right)}{c^2}\boldsymbol{V}_{\mathrm{C}}=\gamma(\boldsymbol{V}_{\mathrm{C}})\left(M-\frac{1}{c^2}\sum_a\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}\right)\boldsymbol{V}_{\mathrm{C}}}$$

Hence, the mass is indeed a priori indeterminate as :

- the general mass has the form $\boxed{M_{modif}=M-\frac{1}{c^2}\sum_a\frac{\partial f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})}{\partial t^*}}$ ;
- with $f(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})$ a function which can be freely chosen.

Yet, there is a point which we have overlooked so far: the relativistic invariance.

We can always make the choice, permitted in relativity, to consider only expression in the Lagrangian which gives a relativistic invariant. This restrict us to choose $(t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\})=0$ , as the only relativistic invariant associated to free a particle is the 4D line element $ds$.

$$\boxed{\left(M_{modif}\right)_{L_a dt\ is\ chosen\ relativistic\ invariant}=M-\frac{1}{c^2}\sum_a\frac{\partial 0}{\partial t^*}=M}$$

By definition the energy is: $E\equiv\sum_a\frac{\partial L'}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}+\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\boldsymbol{V}_{\boldsymbol{c}}-L'$

$$E_{modif}\equiv\sum_a\frac{\partial {L_{modif}}'}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}+\frac{\partial {L_{modif}}'}{\partial \boldsymbol{V}_{\mathrm{C}}}\boldsymbol{V}_{\boldsymbol{c}}-{L_{modif}}'$$

$$= \sum_a \frac{\partial\left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)} + \frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial\left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)} + \frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\boldsymbol{V}_C}\boldsymbol{V}_c - L_{modif}{}'$$

$$= \sum_a \frac{\partial\left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)} + \frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial\left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)} + \frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\boldsymbol{V}_C}\boldsymbol{V}_c - \left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)} + \frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)$$

$$= \left[\sum_a \frac{\partial\left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)}\right)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial\left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)}\right)}{\partial\boldsymbol{V}_C}\boldsymbol{V}_c - \left(\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)}\right)\right] + \left[\sum_a \frac{\partial\left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial\left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\boldsymbol{V}_C}\boldsymbol{V}_c - \left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)\right]$$

$$= \gamma E^* + \sum_a \frac{\partial\left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial\left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)}{\partial\boldsymbol{V}_C}\boldsymbol{V}_c - \left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)$$

$$= \gamma E^* + \sum_a \left(\frac{\partial\left(\sum_a \frac{1}{\gamma}\frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*} + \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial\boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\right)\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial\left(\sum_a \frac{1}{\gamma}\frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*} + \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial\boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)}{\partial\boldsymbol{V}_C}\boldsymbol{V}_c - \left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)$$

$$= \gamma E^* + \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial\boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt} + \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\frac{\partial\left(\frac{1}{\gamma}\right)}{\partial\boldsymbol{V}_C}\boldsymbol{V}_c - \left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)$$

$$= \gamma E^* + \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial\boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt} + \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\left(-\gamma(\boldsymbol{V}_C)\frac{\boldsymbol{V}_C}{c^2}\right)\boldsymbol{V}_c - \left(\frac{df(t^*,\{\boldsymbol{r}_a^*\})}{dt}\right)$$

$$= \gamma E^* + \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial\boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt} - \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\gamma(\boldsymbol{V}_C)\frac{\boldsymbol{V}_C{}^2}{c^2} - \left(\sum_a \frac{1}{\gamma}\frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*} + \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial\boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)$$

$$= \gamma E^* - \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\gamma(\boldsymbol{V}_C)\frac{\boldsymbol{V}_C{}^2}{c^2} - \left(\sum_a \frac{1}{\gamma}\frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\right)$$

$$= \gamma E^* - \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\left[\frac{1+\gamma^2\beta^2}{\gamma}\right] = \gamma E^* - \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\left[\frac{\frac{1-\beta^2+\beta^2}{1-\beta^2}}{\gamma}\right] = \gamma E^* - \gamma\sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}$$

$$\boxed{E_{modif} = \gamma\left(E^* - \sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}\right)}$$

But we know that $M_{modif} = M - \frac{1}{c^2}\sum_a \frac{\partial f(t^*,\{\boldsymbol{r}_a^*\})}{\partial t^*}$

Thus we can also write

$$\boxed{E_{modif} = \gamma M_{modif} c^2}$$

We see that the origin of the Energy scale is the mass, even if the mass is modified by the indeterminacy of the Lagrangian.

It results from that, the requirement to working only with relativistic invariant "Lagrangian" L.dt, sets the value of the mass (by implying $f(t^*, \{\boldsymbol{r_a^*}\}) = \frac{1}{c^2}\sum_a \frac{\partial f(t^*,\{\boldsymbol{r_a^*}\})}{\partial t^*} = 0$) and consequently sets the origin of the scale of the total Energy, at a non arbitrary value.

<u>Relativistic invariance</u> in conjunction with <u>Lagrangian mechanic</u> =>

- $=> \frac{1}{c^2}\sum_a \frac{\partial f(t^*,\{\boldsymbol{r_a^*}\})}{\partial t^*} = 0$
- $=> E_{modif} = E \ \& \ M_{modif} = M$
- $=> E = \gamma(\boldsymbol{V_c})Mc^2$
- => "The origin of the Energy scale is fixed by the mass [1] at a non arbitrary value": If we put simply $\boldsymbol{V_c} = \boldsymbol{0}$ we have $E_0 = Mc^2$.

Indeed, like Landau-Lifchitz [01] we can effectively say that in Special Relativity, the origin of the Energy scale is fixed by the mass (at least for free material system): If we put simply $\boldsymbol{V_c} = \boldsymbol{0}$ we have $E_0 = Mc^2$.

Another important point. In textbook we give easily to proof for the formula $E = \gamma(\boldsymbol{V_c})Mc^2$. Starting from this :

- we compute what we call the rest energy $E_0 \equiv (E)_{\boldsymbol{V_c}=0} = Mc^2$.
- Then we say "The mass is the Energy at rest", having actually in mind that this rest energy $E_0$ is the internal energy $E^* \equiv \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}} \frac{d\boldsymbol{r_a^*}}{dt^*} - L^{*\prime}$
- But...where is the proof of $E_0 = E^*$? Without an analysis à la Landau-Lifchitz [1] using the momentum tensor this affirmation has no fondation. In this article I have given another equivalent proof and I think more easy to understand (Einstein himself was helped by Klein in order to improve his proof using the momentum tensor, indeed he failed to give a totally general one, cf. [7], despite his intuitively convincing different demonstrations).

It is not at all a trivial statement as the way to prove it was not so easy, even if the elementary relativistic formula permit us indeed to guess it. But to guess is not to prove, that is to say to totally understand.

### 6.2. A material system in an external Electromagnetic field

- **<u>The momentum & the mass for material system in an external Electromagnetic field</u>**

Although the relativistic invariance gives a clear criteria to set the mass of a free material system, the same material system seen an external field (electromagnetic field, gravitational field) has its origin broken by other kind of invariance of physics law (in point of view of the least action principle): gauge invariance in Electromagnetism & al, transformation of coordinate system in General Relativity.

In the case of electromagnetism the action of a particle in a given 4-potential is ([1]):

$$S[\{\boldsymbol{r_a}(t)\},t] = \int_{s_1}^{s_2} \sum_a \left[-m_a.c ds_a - \frac{e_a}{c}.A^i(x_{ai})dx_{ai}\right]$$

$$= \int_{t_1}^{t_2} \sum_a \left[-m_a.c\frac{ds_a}{dt} - \frac{e_a}{c}.A^i(x_{ai})\frac{dx_{ai}}{dt}\right]dt$$

$$=> L\left(\{\boldsymbol{r_a}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},t\right) = \sum_a \left[-m_a.c\frac{ds_a}{dt} - \frac{e_a}{c}.A^i(x_{ai})\frac{dx_{ai}}{dt}\right]$$

Like above, we have also:

$$S[\{\boldsymbol{r_a^*}(t)\},\boldsymbol{R}_{\mathrm{C}}(t)] = \int_{t_1}^{t_1} \left(\frac{\sum_a \left[-m_a.c\frac{d{s_a}^*}{dt^*} - \frac{e_a}{c}.{A^i}^*(x_{ai}^*)\frac{dx_{ai}^*}{dt^*}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\right)dt$$

$$=> L'\left(\{\boldsymbol{r_a^*}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right) = \frac{\sum_a \left[-m_a.c\frac{d{s_a}^*}{dt^*} - \frac{e_a}{c}.{A^i}^*(x_{ai}^*)\frac{dx_{ai}^*}{dt^*}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

If we develop the expression

$$L'\left(\{\boldsymbol{r_a^*}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right) = \frac{\sum_a \left[-m_a.c^2\frac{d{\tau_a}^*}{dt^*} - \frac{e_a}{c}.\left(\varphi^*(\boldsymbol{r_a^*},t^*)c\frac{dt^*}{dt^*} - \boldsymbol{A}^*(\boldsymbol{r_a^*},t^*)\frac{d\boldsymbol{r_a^*}}{dt^*}\right)\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= \frac{\sum_a \left[-{\gamma_a}^*.m_a.c^2 - \frac{e_a}{c}.\left(\varphi^*(\boldsymbol{r_a^*},t^*)c - \boldsymbol{A}^*.\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r_a^*}}{dt}\right)\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= \frac{\sum_a \left[-\left({\gamma_a}^*.m_a.c^2 + e_a.\varphi^*(\boldsymbol{r_a^*},t^*)\right) + \frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r_a^*},t^*).\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r_a^*}}{dt}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= -\frac{\sum_a[{\gamma_a}^*.m_a.c^2 + e_a.\varphi^*(\boldsymbol{r_a^*},t^*)]}{\gamma(\boldsymbol{V}_{\mathrm{C}})} + \sum_a \frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r_a^*},t^*).\frac{d\boldsymbol{r_a^*}}{dt}$$

$$=> \boxed{L'\left(\{\boldsymbol{r_a^*}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{V}_{\mathbf{C}},t\right) = -\frac{\sum_a[{\gamma_a}^*.m_a.c^2 + e_a.\varphi^*(\boldsymbol{r_a^*},t^*)]}{\gamma(\boldsymbol{V}_{\mathbf{C}})} + \sum_a \frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r_a^*},t^*).\frac{d\boldsymbol{r_a^*}}{dt}}$$

$$=> \boxed{\boldsymbol{P_c} = \frac{\partial L'}{\partial \boldsymbol{V}_{\mathbf{C}}} = \gamma(\boldsymbol{V}_{\mathrm{C}})\left(\frac{\sum_a[{\gamma_a}^*.m_a.c^2 + e_a.\varphi^*(\boldsymbol{r_a^*},t^*)]}{c^2}\right)\boldsymbol{V}_{\mathrm{C}}}$$

Which gives a mass:

$$M = \frac{\sum_a[{\gamma_a}^*.m_a.c^2 + e_a.\varphi^*(\boldsymbol{r_a^*},t^*)]}{c^2} = M_{free} + M_{interaction}$$

With $M_{free} \equiv \frac{\sum_a {\gamma_a}^*.m_a.c^2}{c^2} = \sum_a {\gamma_a}^*.m_a$

$$M_{interaction} \equiv \frac{\sum_a[e_a.\varphi^*(\boldsymbol{r_a^*},t^*)]}{c^2}$$

The gauge invariance permits us to write with the equivalent physical action:

$$L_{modif}\left(\{\boldsymbol{r_a}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},t\right)=\sum_a\left[-m_a.c\frac{ds_a}{dt}-\frac{e_a}{c}.\left[A^i(x_{ai})+\frac{\partial f(x_{aj})}{\partial x_{ai}}\right]\frac{dx_{ai}}{dt}\right]$$

If we now take account the possibility to change of the gauge

$$L'_{modif}\left(\{\boldsymbol{r_a^*}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C},t\right)=\frac{\sum_a\left[-m_a.c\frac{ds_a{}^*}{dt^*}-\frac{e_a}{c}.A_{modif}{}^{i^*}(x_{aj}^*)\frac{dx_{ai}^*}{dt^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$=\frac{\sum_a\left[-m_a.c\frac{ds_a{}^*}{dt^*}-\frac{e_a}{c}.\left[A^{i^*}(x_{aj}^*)+\frac{\partial f(x_{aj}^*)}{\partial x_{ai}^*}\right]\frac{dx_{ai}^*}{dt^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$=\frac{\sum_a\left[\left(-m_a.c\frac{ds_a{}^*}{dt^*}-\frac{e_a}{c}.A^{i^*}(x_{aj}^*)\frac{dx_{ai}^*}{dt^*}\right)-\frac{e_a}{c}.\frac{\partial f(x_{aj}^*)}{\partial x_{ai}^*}\frac{dx_{ai}^*}{dt^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$=\frac{\sum_a\left[\left(-(\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r_a^*},t^*))+\frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r_a^*},t^*).\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r_a^*}}{dt}\right)-\frac{e_a}{c}.\frac{\partial f(x_{aj}^*)}{\partial x_{ai}^*}\frac{dx_{ai}^*}{dt^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$=-\frac{\sum_a[\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r_a^*},t^*)]}{\gamma(\boldsymbol{V}_\mathrm{C})}+\sum_a\frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r_a^*},t^*)\frac{d\boldsymbol{r_a^*}}{dt}-\frac{\sum_a\frac{e_a}{c}.\left(\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial t^*}-\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial \boldsymbol{r_a^*}}\frac{d\boldsymbol{r_a^*}}{dt^*}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$=-\frac{\sum_a\left[\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r_a^*},t^*)+\frac{e_a}{c}\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}+\sum_a\frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r_a^*},t^*)\frac{d\boldsymbol{r_a^*}}{dt}+\sum_a\frac{e_a}{c}\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial \boldsymbol{r_a^*}}\frac{d\boldsymbol{r_a^*}}{dt}$$

*Thus*

$$\boxed{L'_{modif}\left(\{\boldsymbol{r_a^*}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C},t\right)=-\frac{\sum_a\left[\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r_a^*},t^*)+\frac{e_a}{c}\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}+\sum_a\frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r_a^*},t^*)\frac{d\boldsymbol{r_a^*}}{dt}+\sum_a\frac{e_a}{c}\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial \boldsymbol{r_a^*}}\frac{d\boldsymbol{r_a^*}}{dt}}$$

$$=>\boxed{\boldsymbol{P_c}=\frac{\partial L'_{modif}}{\partial \boldsymbol{V}_\mathrm{C}}=\gamma(\boldsymbol{V}_\mathrm{C})\left(\frac{\sum_a[\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r_a^*},t^*)]+\frac{e_a}{c}\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial t^*}}{c^2}\right)\boldsymbol{V}_\mathrm{C}}$$

Which gives a mass:

$$\boxed{\begin{aligned}M_{modif}&=\frac{\sum_a[\gamma_a{}^*.m_a.c^2+e_a.\varphi_{modif}{}^*(\boldsymbol{r_a^*},t^*)]}{c^2}=\frac{\sum_a\left[\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r_a^*},t^*)+\frac{e_a}{c}\frac{\partial f(\boldsymbol{r_a^*},t^*)}{\partial t^*}\right]}{c^2}\\&=M_{free}+M_{interaction,modif}\\&=M_{free}+M_{interaction}+M_{gauge}\end{aligned}}$$

With $M_{free} \equiv \frac{\sum_a \gamma_a{}^*.m_a.c^2}{c^2}$ ; $M_{interaction} \equiv \frac{\sum_a e_a.\varphi^*(\boldsymbol{r}_a^*,t^*)}{c^2}$ ; $M_{gauge} \equiv \frac{\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}}{c^2}$

In this case we cannot evacuate the mass $M_{gauge}$ by using the relativistic invariance argument since the additive gauge term $\frac{\partial f(x_{aj}^*)}{\partial x_{ai}^*} dx_{ai}^*$ is an existing relativistic invariant. Indeed, contrary to the case of the free material system (where we cannot express the additive term due to the non-existence of other relativistic expression than the one for the line element ds) the addidive gauge term is not a material term but a field term whose the expression or the existence is guarantee by the field equation of the Electromagnetism: $f(x_{aj}^*)$ is a dynamical variable, a scalar field, a direct invariant which is not needed to be express via other material terms invariant.

The arbitrariness of the value of the mass is here of course not a physical problem since the least action principle ensure us that the modification of the mass in this way do not modify the equation of the dynamic, at least in a observable way.

However, we see that the mass is in general affected by the gauge chosen...but:

- This is only proven here for the specific case where the system considered is in an external field;
- The value of the mass is well define for a free (material) system thanks to the relativistic invariant requirement.

- **The origin of the Energy scale for material system in an external Electromagnetic field**

By definition the energy is:

$$E' \equiv \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} \frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial L'}{\partial v_c} \boldsymbol{V_c} - L'$$

$$E_{modif} \equiv \sum_a \frac{\partial L_{modif}{}'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} \frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial L_{modif}{}'}{\partial \boldsymbol{V}_C} \boldsymbol{V_c} - L_{modif}{}'$$

$$L'_{modif}\left(\{\boldsymbol{r}_a^*(t)\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C,t\right) = -\frac{\sum_a\left[\gamma_a{}^*.m_a.c^2 + e_a.\varphi^*(\boldsymbol{r}_a^*,t^*) + \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_C)} + \sum_a \frac{e_a}{c}\frac{d\boldsymbol{r}_a^*}{dt} + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}$$

$$= L'\left(\{\boldsymbol{r}_a^*(t)\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C,t\right) - \frac{\sum_a\left[\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_C)} + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}$$

$$E_{modif} \equiv \sum_a \frac{\partial\left(L'\left(\{\boldsymbol{r}_a^*(t)\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C,t\right) - \frac{\sum_a\left[\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_C)} + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} \frac{d\boldsymbol{r}_a^*}{dt}$$
$$+ \left(\gamma(\boldsymbol{V}_C)\left(\frac{\sum_a[\gamma_a{}^*.m_a.c^2 + e_a.\varphi^*(\boldsymbol{r}_a^*,t^*)] + \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}}{c^2}\right)\boldsymbol{V}_C\right)\boldsymbol{V}_c$$
$$- \left(L'\left(\{\boldsymbol{r}_a^*(t)\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C,t\right) - \frac{\sum_a\left[\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_C)} + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)$$

$$= E + \sum_a \frac{\partial\left(-\frac{\sum_a\left[\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})} + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \left(\gamma(\boldsymbol{V}_\mathrm{C})\left(\frac{\sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}}{c^2}\right)\boldsymbol{V}_\mathrm{C}\right)\boldsymbol{V}_c$$

$$- \left(-\frac{\sum_a\left[\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})} + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)$$

$$= E + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt} + \left(\gamma(\boldsymbol{V}_\mathrm{C})\left(\frac{\sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}}{c^2}\right)\boldsymbol{V}_\mathrm{C}\right)\boldsymbol{V}_c - \left(-\frac{\sum_a\left[\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})} + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial \boldsymbol{r}_a^*}\frac{d\boldsymbol{r}_a^*}{dt}\right)$$

$$= E + \left(\sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right)\left[\gamma(\boldsymbol{V}_\mathrm{C})\beta^2 + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\right] = E + \left(\sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right)\left[\frac{\gamma(\boldsymbol{V}_\mathrm{C})^2\beta^2 + 1}{\gamma(\boldsymbol{V}_\mathrm{C})}\right] = E + \left(\sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right)\gamma$$

$$= \gamma\left(E^* + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right)$$

$$\Rightarrow \boxed{E_{modif} = \gamma\left(E_{free}{}^* + E_{interaction}{}^* + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right)}$$

But we know that

$$M_{modif} = M_{free} + M_{interaction} + \frac{\frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}}{c^2}$$

Thus we can also write

$$\boxed{E_{modif} = \gamma M_{modif}c^2}$$

As above, the value of the mass sets the origin of the Energy scale even if :

- The system is in an external field;
- And the gauge was modified.

We are again in accordance with Landau-Lifchitz [1].
If one decides to define the mass only for free system we have

$$E_{modif} = \gamma Mc^2 + \gamma\left(E_{interaction}{}^* + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}\right)$$

In this case the origin of the Energy scale is

$$E_{modif,0} = Mc^2 + E_{interaction}{}^* + \sum_a \frac{e_a}{c}\frac{\partial f(\boldsymbol{r}_a^*,t^*)}{\partial t^*}$$

We see that with this definition of the mass, the origin of the Energy scale $E_{modif,0}$ is no more the mass. So :

- Either we set the origin of the Energy scale by the mass but we work with a mass dependent of the gauge;
- Either we don't set the origin of the Energy scale by the mass in order to keep a fixed value for the mass.

### 6.3. A material system in a non-Minkowskian space-time (General Relativity)

- **<u>The momentum & the mass for material system in a non-Minkowskian space-time</u>**

The variability of the mass value appears also if we take account General relativity. But, contrary of the Electromagnetism & al, the gravitational field is always present. Indeed there is no sense to talk about space time without gravitation since we cannot put the metric equal to zero (even in Minkowsky space-time which is a particular gravitational field). This point is for me the fundamental idea of General Relativity, without that, any (directly) unobservable frame of reference would be considered as a possible cause for the acceleration of a body: it would be a come-back to the "ugly" pre-Machian (as understood by Einstein) privileged frame of reference...

So in general the mass of a material system in General Relativity always depends on the gravitational field, as stated by Einstein in his first article on cosmology in1917 [8]. Fundamentally, there is only a very particular case where we can define the mass in a systematic way in GR (independently to the context): the case where the system is infinitesimally small...since the Equivalence principle certifies us that the gravitation field is always sufficiently smooth to be sure to encounter in any infinitesimal space-time a quasi-Minskowskian space-time. However, in practice, we can also have a finite volume Minkowskian space-time ("Galilean domain") where we can set the mass. But again, it is fundamentally accidental (=not at all necessary) in the spirit of GR.

$$S[\{\boldsymbol{r_a}(t)\},t] = \int_{s_1}^{s_2}\sum_a[-m_a.cds_a] = \int_{t_1}^{t_2}\sum_a\left[-m_a.c\sqrt{g_{ik}(x_a^j)dx_a^i dx_a^k}\right]$$

$$= \int_{t_1}^{t_2}\sum_a\left[-m_a.c\sqrt{g_{ik}(x_a^j)\frac{dx_a^i}{dt}\frac{dx_a^k}{dt}}\right]dt$$

$$=> L\left(\{\boldsymbol{r_a}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},t\right) = \sum_a\left[-m_a.c\sqrt{g_{ik}(x_a^j)\frac{dx_a^i}{dt}\frac{dx_a^k}{dt}}\right]$$

Like above, we have also:

$$S[\{\boldsymbol{r_a^*}(t)\},\boldsymbol{R}_\mathrm{C}(t)] = \int_{t_1}^{t_1}\left(\frac{\sum_a\left[-m_a.c\sqrt{g_{ik}{}^*(x_a^{j*})\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}\right)dt$$

$$=> L'\left(\{\boldsymbol{r_a^*}(t)\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C},t\right) = \frac{\sum_a\left[-m_a.c\sqrt{g_{ik}{}^*(x_a^{j*})\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

If we develop the expression

$$L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}(t)\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right) = \frac{\sum_a\left[-m_a.c\sqrt{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})c^2+2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt^{*}}c+g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt^{*}}\frac{dx_a^{\beta*}}{dt^{*}}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= \frac{\sum_a\left[-m_a.c\sqrt{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})c^2+\gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c+\gamma(\boldsymbol{V}_{\mathrm{C}})^2g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= \frac{\sum_a\left[-m_a.c\sqrt{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})c^2+\gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}c}{dt}+\gamma(\boldsymbol{V}_{\mathrm{C}})^2g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$\boxed{=> L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}(t)\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right) = \frac{\sum_a\left[-m_a.c\sqrt{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})C^2+\gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c+\gamma(\boldsymbol{V}_{\mathrm{C}})^2g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}}$$

$$\boldsymbol{P_c} = \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} = \sum_a\left[-m_a.c\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\sqrt{\frac{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}c^2+\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c+g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right]$$

$$= \sum_a\left[-m_a.c\frac{1}{2}\frac{\left(g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})c^2\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}+2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\right)}{\sqrt{\frac{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}c^2+\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c+g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}}\right]$$

- $\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} = \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\sqrt{1-\frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{\boldsymbol{c}^2}} = \frac{-\frac{1}{2}2\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}}{\sqrt{1-\frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{\boldsymbol{c}^2}}} = -\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}$
- $\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2} = \frac{\partial\left(1-\frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{\boldsymbol{c}^2}\right)}{\partial \boldsymbol{V}_{\mathrm{C}}} = -2\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}$

$$\boldsymbol{P_c} = \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} = \sum_a\left[-m_a.c\frac{1}{2}\frac{\left(g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\left(-2\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\right)c^2+2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c\left(-\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\right)\right)}{\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\sqrt{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})c^2+\gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c+\gamma(\boldsymbol{V}_{\mathrm{C}})^2g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}}\right]$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}\gamma(\boldsymbol{V}_{\mathrm{C}})\sum_a\left[m_a.c\frac{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})c^2+g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\gamma(\boldsymbol{V}_{\mathrm{C}})c}{\sqrt{g_{00}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})c^2+\gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}c+\gamma(\boldsymbol{V}_{\mathrm{C}})^2g_{\alpha\beta}{}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},t^{*})\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}}\right]$$

$$= \frac{\boldsymbol{V}_{\mathbf{C}}}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})\sum_a\left[m_a.c\frac{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)c^2+g_{0\alpha}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)\frac{dx_a^{\alpha*}}{dt}\gamma(\boldsymbol{V}_{\mathbf{C}})c}{\frac{ds}{dt^*}}\right]$$

$$= \frac{\boldsymbol{V}_{\mathbf{C}}}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})\sum_a\left[m_a.c^2\frac{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)c\frac{dt^*}{dt^*}+g_{0\alpha}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)\frac{dx_a^{\alpha*}}{dt^*}}{\frac{ds}{dt^*}}\right]$$

$$= \frac{\boldsymbol{V}_{\mathbf{C}}}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})\sum_a\left[m_a.c^{2\frac{g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)\frac{dx_a^{i*}}{dt^*}}{\frac{ds}{dt^*}}}\right] = \frac{\boldsymbol{V}_{\mathbf{C}}}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})\sum_a\left[m_a.c^{2g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)\frac{dx_a^{i*}}{ds}}\right]$$

$$= \frac{\boldsymbol{V}_{\mathbf{C}}}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})\sum_a m_a.c^2g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)u_a^{i*}$$

$$\boldsymbol{P_c} = \frac{\partial L'}{\partial \boldsymbol{V}_{\mathbf{C}}} = \gamma(\boldsymbol{V}_{\mathbf{C}})\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{C}}$$

With

$$E^* = \sum_a E_a^* = \sum_a m_a.c^2g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)u_a^{i*}$$

According to Landau-Lifchitz in [1] in the paragraph 88:

- if we take correctly account the a priori de-synchronization between the temporal coordinates of different points of a coordinate system in a stationary metric field,
- the expression $E_a^* = m_a.c^2g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)u_a^{i*}$ can be express also

$$E_a^* = m_a.c^2g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)u_a^{i*} = \frac{m_a.c^2\sqrt{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*)}}{\sqrt{1-\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{ds_a}\right)^2}}$$

Which gives a mass:

$$M = \frac{\sum_a m_a.c^2g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)u_a^{i*}}{c^2} = \left(\sum_a\frac{m_a.\sqrt{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*)}}{\sqrt{1-\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{ds_a}\right)^2}}\right)_{if\ stationary}$$

$$= \left(\sum_a\gamma\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{ds_a}\right)^* m_a.c^2\sqrt{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*)}\right)_{if\ stationary}$$

- **The origin of the Energy scale for material system in non-Minkowskian space-time**

By definition the energy is:

$$E \equiv \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} \boldsymbol{V}_{\boldsymbol{c}} - L'$$

$$L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*(t)\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}, t\right) = \frac{\sum_a \left[-m_a.c\sqrt{g_{ik}{}^*(x_a^{j*})\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= \frac{\sum_a \left[-m_a.c\sqrt{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*)C^2 + \gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*)\frac{dx_a^{\alpha*}}{dt}c + \gamma(\boldsymbol{V}_{\mathrm{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*)\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$E \equiv \sum_a \frac{\partial\left(\frac{\sum_a \left[-m_a.c\sqrt{g_{ik}{}^*(x_a^{j*})\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\right)}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} \frac{d\boldsymbol{r}_a^*}{dt} + \left(\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right)\boldsymbol{V}_c - \left(\frac{\sum_a \left[-m_a.c\sqrt{g_{ik}{}^*(x_a^{j*})\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\right)$$

$$\frac{\partial\left(\frac{\left[-m_a.c\sqrt{g_{ik}{}^*(x_a^{j*})\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\right)}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}$$

$$= \frac{\partial\left(\frac{-m_a.c\sqrt{g_{00}{}^*(\boldsymbol{r}_a^*, t^*)C^2 + \gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}c + \gamma(\boldsymbol{V}_{\mathrm{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\right)}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}$$

$$= \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathrm{C}})} \frac{\partial\left(\sqrt{g_{00}{}^*(\boldsymbol{r}_a^*, t^*)C^2 + \gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}c + \gamma(\boldsymbol{V}_{\mathrm{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right)}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}$$

$$\frac{\partial\left(\sqrt{g_{00}{}^*(\boldsymbol{r}_a^*, t^*)C^2 + \gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}c + \gamma(\boldsymbol{V}_{\mathrm{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right)}{\partial \frac{dx_a^{\gamma*}}{dt}}$$

$$= \frac{1}{2} \frac{\gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^*(\boldsymbol{r}_a^*, t^*)\delta_{\alpha\gamma}c + 2\gamma(\boldsymbol{V}_{\mathrm{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_a^*, t^*)\delta_{\alpha\gamma}\frac{dx_a^{\beta*}}{dt}}{\sqrt{g_{00}{}^*(\boldsymbol{r}_a^*, t^*)C^2 + \gamma(\boldsymbol{V}_{\mathrm{C}})2g_{0\alpha}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}c + \gamma(\boldsymbol{V}_{\mathrm{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_a^*, t^*)\frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}}$$

$$= \frac{\gamma(\boldsymbol{V}_{\mathbf{C}}) g_{0\gamma}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) c + \gamma(\boldsymbol{V}_{\mathbf{C}})^2 g_{\gamma\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\beta*}}{dt}}{\frac{ds}{dt^*}}$$

$$\frac{\partial\left(\Sigma_a \frac{\left[-m_a.c\sqrt{g_{ik}{}^*\left(x_a^{j*}\right)\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\right)}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}$$

$$= \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})} \frac{\partial\left(\sqrt{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) c^2 + \gamma(\boldsymbol{V}_{\mathbf{C}}) 2 g_{0\alpha}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\alpha*}}{dt} c + \gamma(\boldsymbol{V}_{\mathbf{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\alpha*}}{dt}\frac{dx_a^{\beta*}}{dt}}\right)}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}$$

$$= \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{ds}{dt^*}} \begin{pmatrix} \gamma(\boldsymbol{V}_{\mathbf{C}}) g_{01}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) c + \gamma(\boldsymbol{V}_{\mathbf{C}})^2 g_{1\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\beta*}}{dt} \\ \gamma(\boldsymbol{V}_{\mathbf{C}}) g_{02}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) c + \gamma(\boldsymbol{V}_{\mathbf{C}})^2 g_{2\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\beta*}}{dt} \\ \gamma(\boldsymbol{V}_{\mathbf{C}}) g_{03}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) c + \gamma(\boldsymbol{V}_{\mathbf{C}})^2 g_{3\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\beta*}}{dt} \end{pmatrix}$$

$$E = \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{ds}{dt^*}} \left(\gamma(\boldsymbol{V}_{\mathbf{C}}) g_{0\alpha}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\alpha*}}{dt} c + \gamma(\boldsymbol{V}_{\mathbf{C}})^2 g_{\alpha\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\beta*}}{dt}\frac{dx_a^{\alpha*}}{dt}\right) + \left(\gamma(\boldsymbol{V}_{\mathbf{C}}) \frac{E^*}{c^2} \boldsymbol{V}_{\mathbf{C}}\right) \boldsymbol{V}_c - \left(\frac{\Sigma_a \left[-m_a.c\sqrt{g_{ik}{}^*\left(x_a^{j*}\right)\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\right)$$

$$= \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})} \left(g_{0\alpha}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\alpha*}}{ds} c + g_{\alpha\beta}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\beta*}}{dt^*}\frac{dx_a^{\alpha*}}{ds}\right) + \left(\gamma(\boldsymbol{V}_{\mathbf{C}}) \frac{E^*}{c^2} \boldsymbol{V}_{\mathbf{C}}\right) \boldsymbol{V}_c - \left(\frac{\Sigma_a \left[-m_a.c\sqrt{g_{ik}{}^*\left(x_a^{j*}\right)\frac{dx_a^{i*}}{dt^*}\frac{dx_a^{k*}}{dt^*}}\right]}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\right)$$

$$= \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})} \left(g_{i\alpha}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{\alpha*}}{dt^*}\frac{dx_a^{i*}}{ds}\right) + \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* \beta^2 - \left(\frac{\Sigma_a[-m_a.cds_a]}{\gamma(\boldsymbol{V}_{\mathbf{C}}) dt^*}\right)$$

$$= \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})} \left(g_{ik}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{k*}}{dt^*}\frac{dx_a^{i*}}{ds} - g_{i0}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{i*}}{ds}\frac{dx_a^{0*}}{dt^*}\right) + \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* \beta^2 - \left(\frac{\Sigma_a[-m_a.cds_a]}{\gamma(\boldsymbol{V}_{\mathbf{C}}) dt^*}\right)$$

$$= \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})} \left(\frac{ds}{dt^*} - u_0{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{0*}}{dt^*}\right) + \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* \beta^2 - \left(\frac{\Sigma_a[-m_a.cds]}{\gamma(\boldsymbol{V}_{\mathbf{C}}) dt^*}\right)$$

$$= \sum_a \frac{-m_a.c}{\gamma(\boldsymbol{V}_{\mathbf{C}})} \left(-u_0{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) \frac{dx_a^{0*}}{dt^*}\right) + \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* \beta^2$$

$$= \sum_a \frac{m_a.cu_0{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*)}{\gamma(\boldsymbol{V}_{\mathbf{C}})} + \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* \beta^2 = \frac{E^*}{\gamma(\boldsymbol{V}_{\mathbf{C}})} + \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* \beta^2 = \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* \left(\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})^2} + \beta^2\right) = \gamma(\boldsymbol{V}_{\mathbf{C}}) E^* (1 - \beta^2 + \beta^2) = \gamma(\boldsymbol{V}_{\mathbf{C}}) E^*$$

$$E = \gamma E^*$$

With

$$E^* = \sum_a E_a^* = \sum_a m_a.c^2 g_{0i}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*) u_a^{i*} = \left(\sum_a \gamma\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{ds_a}\right)^* m_a.c^2\sqrt{g_{00}{}^*(\boldsymbol{r}_{\boldsymbol{a}}^*)}\right)_{if\ stationary}$$

Thus we can also again write

$$\boxed{E = \gamma M c^2}$$

We can see here that :

- as the Gauge field theory, if we change the coordinate system, M is modified but not the role of the mass as again an origin of the Energy scale (accordance with Landau-Lifchitz [1]), which remain:
  - for every gravitational field;
  - for every coordinate space-time.
- But contrary to the Gauge field theory, the mass is not a sum of :
  - a free part;
  - plus an interacting part ;
  - plus an gauge part.

Indeed as the system is always and necessary in the space-time there is no way to separate it from the gravitational field: there is no free term relative to the gravitation, only interacting terms. This is in accordance with the fundamental and particular role of gravitation in physics. Otherwise any terms in the action would lose its signification due to the arbitrariness of the coordinate system: the truly role of the gravitation (via the metric) is the one of a filter which absorbs any effect of the coordinate transformation in the action.

Although the mass depend of the context (Einstein-Mach idea), in practice we work in a cosmological context where for quasi-every system, if we move sufficiently away from it, the space-time tends to be approximatively Minkowskian. Thus, in paragraph 96 of [1], Landau-Lifchitz have given the proof that the momentum-tensor, and so the mass, are independent of the frame of reference if the system studied is <u>free</u>. So the total mass in an internal volume surround by a Galilean domain are well defined. It is again a “cosmological” accident in the RG spirit.

## 7. Why the Einstein law (the mass as internal energy) does not appear in Newtonian mechanics? The crucial role of the Einstein non-universality of time law

If we put from the start of the theory the **Newtonian law** that time is universal, $dt = dt^*$, that is to say $\boxed{\gamma(\boldsymbol{V}_C) = 1}$, we have:

$$\boldsymbol{P}_c \equiv \frac{\partial L'}{\partial \boldsymbol{V}_C} = \frac{\partial L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)}{\partial \boldsymbol{V}_C} = \frac{\partial}{\partial \boldsymbol{V}_C}\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_C)} = \frac{\partial}{\partial \boldsymbol{V}_C}L^*\left(\{\boldsymbol{r}_a^*\},\frac{d\boldsymbol{r}_a^*}{dt}\right) = 0$$

Which is of course wrong, so why we cannot use the Lagrangian $L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_C,\boldsymbol{V}_C\right)$ ?

Actually this Lagrangian is correct...the problem is the passage to the limit:

- $Einsteinian$ non universality of time $dt \neq dt^*$,
- To the $Newtonian$ universality of time $dt = dt^*$,
- Before the derivation and not after.

Indeed, if keep the non universality of time during the derivation process we have necessary directly

$$\boldsymbol{P}_{c,Einstein} = \boldsymbol{V}_C.\gamma(\boldsymbol{V}_C{}^2)\frac{E^*}{c^2}$$

And then, the passage to the limit of universality of time gives

$$\boldsymbol{P}_{c,Newton} = \lim_{\gamma(\boldsymbol{V}_C{}^2)\to 1}\boldsymbol{V}_C.\gamma(\boldsymbol{V}_C{}^2)\frac{E^*}{c^2} = \boldsymbol{V}_C.\frac{E^*}{c^2}$$

Which keep the link between the mass and the energy in the Newtonian limit since the proportionality coefficient is again $\frac{E^*}{c^2}$.

Therefore, we are faced to a (famous) mathematical non equivalence (non-commutativity of derivative and the limit operation):

$$\left[\frac{\partial\left(\lim_{\gamma(\boldsymbol{V}_C)\to 1}L'\right)}{\partial \boldsymbol{V}_C} = \frac{\partial}{\partial \boldsymbol{V}_C}\left(\lim_{\gamma(\boldsymbol{V}_C)\to 1}\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_C)}\right)\right] \neq \left[\lim_{\gamma(\boldsymbol{V}_C)\to 1}\left(\frac{\partial L'}{\partial \boldsymbol{V}_C}\right) = \lim_{\gamma(\boldsymbol{V}_C)\to 1}\left(\frac{\partial}{\partial \boldsymbol{V}_C}\frac{L^*\left(\{\boldsymbol{r}_a^*\},\left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_C)}\right)\right]$$

So, what is the procedure used in the Newtonian theory, and why the procedure doesn't show the link between the mass and the internal energy?

Like explain in Landau-Lifchitz [2]:

- we start from a Lagrangian $L\left(\{\boldsymbol{r}_a\},\left\{\frac{d\boldsymbol{r}_a}{dt}\right\},t\right)$
- we pose the principle of the additivity of the Lagrangian for independent system (as in Einstein Special Relativity)

$$L\left(\{\boldsymbol{r}_a\},\left\{\frac{d\boldsymbol{r}_a}{dt}\right\},t\right) = \sum L\left(\boldsymbol{r}_a,\frac{d\boldsymbol{r}_a}{dt},t\right)$$

- we pose the principle of the homogeneity of space and for time

$$L\left(\boldsymbol{r}_a,\frac{d\boldsymbol{r}_a}{dt},t\right) = L\left(\frac{d\boldsymbol{r}_a}{dt}\right)$$

- we pose the principle of the isotropy of space

$$L\left(\frac{d\boldsymbol{r_a}}{dt}\right) = L\left(\left(\frac{dr_a}{dt}\right)^2\right)$$

- we pose the principle of Galileo-Newtonian kinematic between to Galilean frame K & K'

$$\begin{pmatrix} t = t' \ (universality\ of\ time) \\ x = x' + V_{K'/K}.t' \\ y = y' \\ z = z' \end{pmatrix} which\ implies\ the\ additivity\ formula\ \left(\frac{d\boldsymbol{r_a}}{dt} = \frac{d\boldsymbol{r_a}'}{dt} + \boldsymbol{V_{K'/K}}\right)$$

Therefore we can compute the momentum where K'=K* and $\boldsymbol{V_{K'/K}} = \boldsymbol{V_c}$

$$\boldsymbol{P_{c,Newton}} = \frac{\partial}{\partial \boldsymbol{V_C}} L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) = \frac{\partial}{\partial \boldsymbol{V_C}} \sum L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = \sum \frac{\partial}{\partial \boldsymbol{V_C}} L\left(\left(\frac{d\boldsymbol{r_a^*}}{dt} + \boldsymbol{V_C}\right)^2\right)$$

$$= \sum \frac{\partial \left(\frac{d\boldsymbol{r_a^*}}{dt} + \boldsymbol{V_C}\right)^2}{\partial \boldsymbol{V_C}} \frac{\partial}{\partial \frac{d\boldsymbol{r_a}}{dt}^2} L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = \sum 2\left(\frac{d\boldsymbol{r_a^*}}{dt} + \boldsymbol{V_C}\right) \frac{\partial}{\partial \frac{d\boldsymbol{r_a}}{dt}^2} L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right)$$

$$= \sum 2\left(\frac{d\boldsymbol{r_a^*}}{dt}\right) \frac{\partial}{\partial \frac{d\boldsymbol{r_a}}{dt}^2} L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) + 2\boldsymbol{V_C} \sum \frac{\partial}{\partial \frac{d\boldsymbol{r_a}}{dt}^2} L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = \sum \left(\frac{d\boldsymbol{r_a^*}}{dt}\right) \lambda_a + \boldsymbol{V_C} \sum \lambda_a$$

- if we define K* such that $\sum \left(\frac{d\boldsymbol{r_a^*}}{dt}\right) \lambda_a \equiv 0$
- and the quantities $\lambda_a \left(\frac{d\boldsymbol{r_a}}{dt}^2\right) \equiv 2 \frac{\partial}{\partial \frac{d\boldsymbol{r_a}}{dt}^2} L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right)$ a priori not constant (the future mass).

In conclusion with first Galileo-Newtonian principles we have, without any other hypothesis:

- Homogeneity & isotropy of space and homogeneity of time
- & Kinematic Galileo-Newtonian (Galilean transformation )
- & Additivity of the Lagrangian (for independent system)

$$=> \boldsymbol{P_{c,Newton}} = \frac{\partial}{\partial \boldsymbol{V_C}} L\left(\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) = \left(\sum \lambda_a\right) \boldsymbol{V_C}$$

With

- $\lambda_a \equiv 2 \frac{\partial}{\partial \frac{d\boldsymbol{r_a}}{dt}^2} L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right)$
- $\left(\sum \left(\frac{d\boldsymbol{r_a^*}}{dt}\right) \lambda_a \equiv 0\right)_{by\ defintion\ of\ a\ K^*}$

In order to complete the mechanical description, we have to express the Lagrangian a particule $L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right)$ more explicitly. For that we will call (following again [2]) another principle: the Galilean principle of relativity which affirm that the mechanical law have to be the same for any Galilean frame K, K',K*...

But, there is a problem, the only Galilean invariants are:

- The action $S[\{\boldsymbol{r_a^*}(t^*)\}, \boldsymbol{R_c}(t)]$ by construction (the quantity, not necessary the function!)
- The Newtonian time $dt = dt'$
- And....nothing else

We cannot construct an invariant with the basic kinematic quantities of a particle $\boldsymbol{r_a}, \frac{d\boldsymbol{r_a}}{dt}$ and so we cannot construct an invariant quantity $\left(\frac{d\boldsymbol{r_a}}{dt}^2\right).dt$ .

So we are a priori blocked. In fact, the only possibility is to use the "Gauge invariance" associated to the least action principle,

- $L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = L^*\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) + \frac{df(\boldsymbol{r_a}^*,\boldsymbol{t})}{dt}$
- that we complete by the Galilean principle of relativity $L^*\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) = L\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right)$

(Remark: Einstein tells us that this Galilean principles contains also "the principle of Galileo-Newtonian kinematic" defined above).

Thanks to this "Gauge invariance", we can make the following calculation:

$$L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = L\left(\left(\frac{d\boldsymbol{r_a}}{dt}^* + \boldsymbol{v}_{\mathrm{C}}\right)^2\right) = L\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2 + (\boldsymbol{v}_{\mathrm{C}})^2 + 2\frac{d\boldsymbol{r_a}}{dt}^*\boldsymbol{v}_{\mathrm{C}}\right)$$

The expression, should be valid for any $\boldsymbol{v}_{\mathrm{C}}$, and so even for infinitesimal value $\boldsymbol{\varepsilon}$:

$$L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = L\left(\left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2 + \boldsymbol{\varepsilon}^2 + 2\frac{d\boldsymbol{r_a}}{dt}^*\boldsymbol{\varepsilon}\right)$$

$$\approx L\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) + \frac{d\boldsymbol{r_a}}{dt}^*\boldsymbol{\varepsilon}2\frac{\partial L}{\partial\left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2}\left(\left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2\right)$$

$$= L\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) + \frac{d\boldsymbol{\varepsilon}.\boldsymbol{r_a}^*}{dt}2\frac{\partial L}{\partial\left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2}\left(\left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2\right)$$

$$= L\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) + \lambda_a\frac{d\boldsymbol{\varepsilon}.\boldsymbol{r_a}^*}{dt} \text{ with } \lambda_a = \lambda_a\left(\left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2\right)$$

Galilean relativity principle => $L\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) = L^*\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right)$ => $L^*\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = L^*\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) + \lambda_a{}^*\frac{d\boldsymbol{\varepsilon}.\boldsymbol{r_a}^*}{dt}$

And the "Gauge invariance"=> $L^*\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) + \lambda_a{}^*\frac{d\boldsymbol{\varepsilon}.\boldsymbol{r_a}^*}{dt} = L^*\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) + \frac{df(\boldsymbol{r_a}^*)}{dt}$

=> $\frac{df(\boldsymbol{r_a}^*)}{dt} = \lambda_a{}^*\frac{d\boldsymbol{\varepsilon}.\boldsymbol{r_a}^*}{dt} = \lambda_a{}^*\frac{dg(\boldsymbol{r_a}^*)}{dt}$ => $\frac{\partial f(\boldsymbol{r_a}^*)}{\partial \boldsymbol{r_a}^*} = \lambda_a{}^*\frac{\partial\boldsymbol{\varepsilon}.\boldsymbol{r_a}^*}{\partial\boldsymbol{r_a}^*}$ =>$\lambda_a{}^* = f(\boldsymbol{r_a}^*)$

But at the same time $\lambda_a{}^* = \lambda_a{}^*\left(\left(\frac{d\boldsymbol{r_a}}{dt}^*\right)^2\right)$, $then\ \lambda_a{}^* = cte,\ \forall\boldsymbol{r_a}^*\ \forall\frac{d\boldsymbol{r_a}}{dt}^*\ \forall t$

The constant Lagrangian characteristic coefficents $\lambda_a{}^* \equiv 2\frac{\partial}{\partial\frac{d\boldsymbol{r_a}}{dt}^*}L^*\left(\frac{d\boldsymbol{r_a}}{dt}^{*2}\right) = 2\frac{\partial}{\partial\frac{d\boldsymbol{r_a}}{dt}^2}L\left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = \lambda_a$

are what we call the mass $m_a$ of a particule.

It results from that, the expression of the momentum of the center of mass.

$$\boldsymbol{P}_{c,Newton} = \left(\sum \lambda_a\right) \boldsymbol{v}_{\mathrm{C}} = \left(\sum m_a\right) \boldsymbol{V}_{\mathrm{C}} = M.\boldsymbol{V}_{\mathrm{C}}$$

This result à la Landau [2] shows us:

- the crucial role playing by the indeterminacy of the Lagrangian (the "Gauge invariance");
- And no more the existence of an invariant L.dt in the action.

This is the complete opposite of the Einsteinian case where:

- the invariant of the action was used in the start of the reasoning;
- and the indeterminacy of the Lagrangian was kept away.

Moreover the latter "Gauge invariance", when taking account after in relativity, was responsible of the undesirable change of the expression of the mass (although the link between the mass and the origin of the energy scale still remains). The consequence of this Newtonian inversion of the role between the "Gauge invariance" and the invariance of the action has 2 impacts in the description of the Newtonian mechanic.

a. The loss of connexion of the mass and the energy scale

Ineed, from above, we calcultate the Lagrangian and we find

$$L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right)$$

$$= \sum \frac{1}{2} m_a \left(\frac{d\boldsymbol{r_a}}{dt}^2\right) = \sum \frac{1}{2} m_a \left(\frac{d\boldsymbol{r_a}^*}{dt} + \boldsymbol{V}_{\mathrm{C}}\right)^2$$

$$= \sum \frac{1}{2} m_a \left(\left(\frac{d\boldsymbol{r_a}^*}{dt}\right)^2 + \boldsymbol{V}_{\mathrm{C}}^2 + 2\frac{d\boldsymbol{r_a}^*}{dt}\boldsymbol{V}_{\mathrm{C}}\right)$$

$$= \sum \frac{1}{2} m_a \left(\frac{d\boldsymbol{r_a}^*}{dt}\right)^2 + \frac{1}{2} M \boldsymbol{V}_{\mathrm{C}}^2$$

$$\boxed{L'\left(\{\boldsymbol{r_a}^*\},\left\{\frac{d\boldsymbol{r_a}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right) = \sum \frac{1}{2} m_a \left(\frac{d\boldsymbol{r_a}^*}{dt}\right)^2 + \frac{1}{2} M \boldsymbol{V}_{\mathrm{C}}^2}$$

The resulting energy expression is:

$$E' \equiv \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} \frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} \boldsymbol{V}_c - L' = \sum_a m_a \left(\frac{d\boldsymbol{r_a}^*}{dt}\right)^2 + M.\boldsymbol{V}_{\mathrm{C}}\boldsymbol{V}_c - \sum \frac{1}{2} m_a \left(\frac{d\boldsymbol{r_a}^*}{dt}\right)^2 - \frac{1}{2} M \boldsymbol{V}_{\mathrm{C}}^2$$

$$=> \boxed{E' = \sum \frac{1}{2} m_a \left(\frac{d\boldsymbol{r_a}^*}{dt}\right)^2 + \frac{1}{2} M \boldsymbol{V}_{\mathrm{C}}^2}$$

The mass no longer defines the origin of the energy scale.

b. the non natural fixation of the origin of the energy scale by the relativistic invariance

Since, we do not used relativistic invariance quantities, in order to express the Lagrangian, we cannot of course use it to the fixation of the Lagrangian expression with the invariance relativist.

Thus the Lagrangian is only relative to a gauge, therefore also the origin of the energy scale.

$$\boxed{L'_{modif}\left(\{\boldsymbol{r_a}^*\},\left\{\frac{d\boldsymbol{r_a}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)=L'\left(\{\boldsymbol{r_a}^*\},\left\{\frac{d\boldsymbol{r_a}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)+\sum_a\frac{df(\boldsymbol{r_a}^*,t)}{dt}}$$

This indeterminacy contaminates the one of the origin of the energy scale:

$$\Rightarrow \quad E'_{modif}\equiv\sum_a\frac{\partial L'_{modif}}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt}+\frac{\partial L'_{modif}}{\partial\boldsymbol{V}_{\mathrm{C}}}\boldsymbol{V}_c-L'_{modif}$$

$$=\sum_a\frac{\partial\left(L'\left(\{\boldsymbol{r_a}^*\},\left\{\frac{d\boldsymbol{r_a}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)+\frac{df(\boldsymbol{r_a}^*,t)}{dt}\right)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt}+M.\boldsymbol{V}_{\mathrm{C}}\boldsymbol{V}_c-L'\left(\{\boldsymbol{r_a}^*\},\left\{\frac{d\boldsymbol{r_a}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)-\frac{df(\boldsymbol{r_a}^*,t)}{dt}$$

$$=\sum_a m_a\left(\frac{d\boldsymbol{r_a}^*}{dt}\right)^2+\sum_a\frac{\partial f(\boldsymbol{r_a}^*,t)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt}+M.\boldsymbol{V}_{\mathrm{C}}\boldsymbol{V}_c^2-L'\left(\{\boldsymbol{r_a}^*\},\left\{\frac{d\boldsymbol{r_a}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)-\sum_a\frac{df(\boldsymbol{r_a}^*,t)}{dt}$$

$$=E'+\sum_a\frac{\partial f(\boldsymbol{r_a}^*,t)}{\partial\frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt}-\sum_a\frac{df(\boldsymbol{r_a}^*,t)}{dt}=E'-\sum_a\frac{\partial f(\boldsymbol{r_a}^*,t)}{\partial t}$$

$$\boxed{E'_{modif}=E'-\sum_a\frac{\partial f(\boldsymbol{r_a}^*,t)}{\partial t}}$$

**Summary**:

The Einstein energy-mass equivalence law comes from Einsteinian non-universality of time law by the fact it gives the existence of invariants in the action, this has also 2 others consequences, the fixation of the energy scale by the mass and the fixation of the value of the mass (in free system) by the natural demand of the Lagrangian L.dt invariant expression in the action (by saying that the "Gauge invariance" although permit, is "not natural").

The Newonian universality of time reverses completely the situation. There is no more sufficiently invariant quantities, this oblige us to use the "not natural Gauge invariance" of the action, which hides the Einstein law and suppress the role of the mass as the origin of the energy scale. Moreover, this necessary use of "gauge invariance" does not permit us to talk about (and even think of) a natural invariant L.dt expression which was so necessary to set the origin of the energy scale in a "natural" way.

Worst, this frequent use of "gauge invariance" in Newtonian mechanic accustomed us to consider that energy scale has "no natural" fixed value (in free system). Therefore when Einstein discovered the Special Relativity in 1905 and the mass-energy equivalence, he was again conditioned by this habit and so hesitated to set a fixed value of the energy scale origin by the mass. He waits several years before to fix it (cf. [7] or the original Einstein articles where he talked about difference of energy, instead of the "absolute" energy). But he set the origin of the energy scale via an intuition of the naturalness than a Lagrangian explanation. The latter was not the only formal expression of physics law, it was surely not as mature as today (in electromagnetism, gravitation...etc, and even in the future quantum mechanics as Feynman showed us) and so a priori not the unique convincing road to physics. I suppose that, if my proof of his law was not derived by himself is surely due to the lack of confidence of this way of thinking even if he used it after many time in General Relativity and his others modifications of his theory.

## 8. The momentum tensor and the mass as a scalar

A simple Lorentz transformation, shows that the 3-momentum is actually the one associated to the 4-vector defined above $P^i(K^*) = \frac{1}{c}\int \iiint_{space-time} T^{ik}\delta\left(n_{lm}x^l x^m\right).\,d\eta_k(K^*)\,d^4x$.

Thus, among all the 4-momentum $P^i(K)$, $P^i(K')$, $P^i(K^*)$... the Lagrangian method selects $P^i(K^*)$.

Moreover, thanks to this association we can naturally affirm that the mass, and so the internal energy, is a scalar: this is the well known norm of the 4-momentum.

## 9. Conclusion

We have a way to demonstrate the famous Einstein formula $E^*=Mc^2$ directly from an appropriate Lagrangian function selecting the correct variable.

Instead of $L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right)$, we use $L'\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R_\mathrm{C}},\boldsymbol{V_\mathrm{C}}\right)=\frac{L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V_\mathrm{C}})\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R_\mathrm{C}},\boldsymbol{V_\mathrm{C}}\right)}{\gamma(\boldsymbol{V_\mathrm{C}})}$.

Instead of $L'\left[\{\varphi\},\left\{\frac{\partial\varphi}{\partial r}\right\},\left\{\frac{\partial\varphi}{\partial t}\right\}\right]$, we use $L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial r^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t}\right\},\boldsymbol{R_\mathrm{C}},\boldsymbol{V_\mathrm{C}}\right]=\frac{\iiint \Lambda^*\left(\varphi^*,\frac{\partial\varphi^*}{\partial r^*},\gamma\frac{\partial\varphi^*}{\partial t},\boldsymbol{R_\mathrm{C}},\boldsymbol{V_\mathrm{C}}\right)dV^*}{\gamma}$.

In the two cases we've calculated directly that $\boldsymbol{P_c}\equiv\frac{\partial L'}{\partial \boldsymbol{V_\mathrm{C}}}=\gamma\frac{E^*}{c^2}\boldsymbol{V_\mathrm{C}}$

In this article, we have also showed:

- The strong link with this law and the dilatation of time formula which highlight the crucial role of the Einstein demand of non universality of time;
- A discussion on the meaning of the new set of variable chosen with an amusing modified speed addition formula which do not contradict the one of Einstein-Poincaré;
- A discussion on the origin of the energy scale and the link with the mass as stated by Landau-Lifchitz.
- Why in Newtonian mechanic the Einstein law is hidden.

## 10. Annex

### 10.1. Annex calculation

We want to draw the K* axis seen by K, that is to say the different axis in function of the x axis.

$$\begin{cases} c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) \\ x - x_c(t_i) = \gamma_{t_i}.\left(x_{K_i^*} + \beta_{t_i}.\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right) \end{cases}$$

- In K, the equation of a static point in K* ($x_{K_i^*}$=cte) in function of x , that is to say $c.t_{(x_{K_i^*}=cte)}(x)$ is

$$c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) = \gamma_{t_i}.\left(\frac{x - x_c(t_i)}{\beta_{t_i}.\gamma_{t_i}} - \frac{x_{K_i^*}}{\beta_{t_i}} + \beta_{t_i}.x_{K_i^*}\right)$$

$$= \frac{x - x_c(t_i)}{\beta_{t_i}} - x_{K_i^*}.\gamma_{t_i}\frac{1}{\beta_{t_i}}\left(1 - \beta_{t_i}{}^2.\right) = \frac{x - x_c(t_i)}{\beta_{t_i}} - \frac{x_{K_i^*}}{\gamma_{t_i}.\beta_{t_i}}$$

$$\boxed{c.t = c.t_i + \frac{x - x_c(t_i)}{\beta_{t_i}} - \frac{x_{K_i^*}}{\gamma_{t_i}.\beta_{t_i}}}$$

=> $\boxed{c.t_{(x_{K_i^*}=K)}(x) = c.t_i + \frac{x - x_c(t_i)}{\beta_{t_i}} - \frac{K}{\gamma_{t_i}.\beta_{t_i}} \text{ at time } t=t_i}$

So the equation of $x_{K_i^*}$=0 is $\boxed{c.t_{\left(x_{K_i^*}=0\right)}(x) = c.t_i + \frac{x - x_c(t_i)}{\beta_{t_i}} \text{ at time } t=t_i}$

Between $x_c(t_1)$ and $x_c(t_2)$, the variation is at should:

$$c.t_{\left(x_{K_1^*}=0\right)}(x_c(t_2)) - c.t_{\left(x_{K_i^*}=0\right)}(x_c(t_1)) = \frac{x_c(t_2) - x_c(t_1)}{\beta_{t_1}} = \frac{V_c(t_1).(t_2 - t_1)}{\beta_{t_1}} = c.(t_2 - t_1)$$

- In K, the equation of (t*=cte) in function of x , that is to say $c.t_{(t^*=cte)}(x)$ is

$$\begin{cases} c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) \\ x - x_c(t_i) = \gamma_{t_i}.\left(x_{K_i^*} + \beta_{t_i}.\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right) \end{cases}$$

$$c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right)$$

$$= \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.\left(\frac{x - x_c(t_i)}{\gamma_{t_i}} - \beta_{t_i}.c\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right)\right)$$

$$c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.\frac{x - x_c(t_i)}{\gamma_{t_i}} - \beta_{t_i}{}^2.c\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right)$$

$$= \gamma_{t_i}.\left(\left(1 - \beta_{t_i}{}^2\right)c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.\frac{x - x_c(t_i)}{\gamma_{t_i}}\right) = \frac{c\left(t_{K_i^*} - t^*_{C(t_i)}\right)}{\gamma_{t_i}} + \beta_{t_i}.(x - x_c(t_i))$$

$$\boxed{c.t = c.t_i + \beta_{t_i}.(x - x_c(t_i)) + \frac{c\left(t_{K_i^*} - t^*_{C(t_i)}\right)}{\gamma_{t_i}}}$$

$$=> \boxed{c.t_{(ct_{K_i^*}=K)}(x) = c.t_i + \beta_{t_i}.(x - x_c(t_i)) + \frac{c\left(K - t^*_{C(t_i)}\right)}{\gamma_{t_i}}}$$

And in particular

$$\boxed{c.t_{(ct_{K_i^*}=t^*_{C(t_i)})}(x) = c.t_i + \beta_{t_i}.(x - x_c(t_i))}$$

### 10.2. Application: the toy model of the electron

### 10.3. Application: the effective description of a particle in an external electromagnetic field

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