A straightforward and Lagrangien proof of the mass as the internal energy of a system

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I propose a Lagrangian proof of Einstein's well-known law that the mass system is its internal energy. The interest of this proof is to show how appears the distinction between internal degrees of freedom and the center of mass in the Lagrangian formalism.

Sommaire

1. **The Einstein law**
   1.1. **The law**

According the expression of the law of physics via the principle of least action [Landau-Lifchitz] and the relativistic invariance: the mass m_a of a material point "a" is simply the multiplicative coefficient appearing in the Lagrangien of this material point, interacting or not with an external field.

$$\boxed{S[\boldsymbol{r_a}(t)] = -\int_{s_{a,1}}^{s_{a,2}} m_a.c.ds_a + \cdots = -\int_{t_1}^{t_2} \frac{m_a.c^2}{\gamma(\boldsymbol{v_a})} dt + \cdots}$$

In 1905, Einstein tells us that whatever the system: a set of material points (dynamically characterised with a Lagrangien $L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right)$) or a field (dynamically characterised with the Lagrangien $\Lambda\left(\varphi, \frac{\partial\varphi}{\partial r}, \frac{\partial\varphi}{\partial t}\right)$) we should have:

$$\boxed{S[\boldsymbol{R_c}(t), \dots] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma(\boldsymbol{V_c})} dt + \cdots}$$

- With $E^* = \sum_a \frac{d\boldsymbol{r_a^*}}{dt^*} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}} - L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}\right)$ for a material point;
- Or $E^* \equiv \iiint \left(\frac{\partial\varphi^*}{\partial t^*} \frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)} \Lambda^* - \Lambda^*\right) dV^*$ for a scalar field (for example).

Where the quantities with a star * are relative to the reference frame associated to the mass center K*. So E^* is the internal energy.

Thus, every system has a centre of mass which has a Lagrangian, analogous to a material point with a mass $M = \frac{E^*}{c^2}$**.** This is the famous law of Einstein.

   1.2. **The current proof**

This law is well established since its first publication in 1905 and was re-demonstrated more clearly after by other (Einstein himself, Von Laue ...). The simpler way (that the author know and read in [Landau Lifchitz]), is to demonstrate that the momentum is a 4 vector.

Indeed, tanks to the stress energy tensor T^{ik} of the system, we can always associate to it a 4-vector

$P^i(K^*) \equiv \frac{1}{c}\int\iiint_{space-time,K} T^{ik} dS_k$, where we choose the hyper-surface of integration as the hyperplane of the reference frame K* ($t^* = cte$).

In any frame ([Janssen & Mecklenburg]), $P^i(K^*)$ can be written equivalently

$$P^i(K^*) = \frac{1}{c}\int\iiint_{space-time} T^{ik} \delta\left(n_{lm} x^l \eta^m(K^*)\right).\eta_k(K^*)\, d^4x$$

where $\eta_k(K^*)$ is an orthogonal vector of the hyperplane $t^* = cte$ of K* such that ${\eta^*}_k(K^*) = (1{,}0{,}0{,}0)$ in K^*.

Thus, the Lorentz transformations tell us:

$$P^i(K^*) = \frac{1}{c}\int\iiint_{space-time} L^i_{\ r} L^k_{\ s} T^{*rs}\delta(t^*).L^m_k.\eta^*_{\ m}(K^*)\,d^4x^* \ = L^i_{\ r}\frac{1}{c}\iiint_{x^{*\alpha}\epsilon V^*} T^{*r0}(0,x^{*\alpha})dV^*$$

So $P^i(K^*) = L^i_{\ r}P^{*r}(K^*)$ where $P^{*r}(K^*) = \frac{1}{c}\iiint_{Space} T^{*r0}(0,x^{*\alpha})dV^*$

But $E^* \equiv \iiint_{Space} T^{*00}(0,x^{*\alpha})dV^*$ and $P^{*\alpha}(K^*) \equiv 0$ by definition of K*

So we have $P^i(K^*) = \left(\gamma\frac{E^*}{c}, \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{K*/K}}\right)$, hence $\boldsymbol{P} = \ \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{K*/K}}$ **=>** $M = \frac{E^*}{c^2}$

That is to say, the 3-momentum of any system is the same as a material point:

- with a mass $M = \frac{E^*}{c^2}$;
- and a speed $\boldsymbol{v} = \boldsymbol{V}_{\mathbf{K*/K}}$.

2 remarks:

- $P^i(K^*)$ is here relative to the particular time $t^* = 0$ and is not a priori constant;
- $P^i(K^*)$ is not the only one 4-momentum since we can define a different one for each frame of reference, $P^i(K), P^i(K'), P^i(K^*)$ ..., all are associated to different hyperplane of simultaneity linked to each possible (an infinity) frame of reference K,K',K*...(see [Janssen & Mecklenburg]).

It exists a particular case where there is only one 4-momentum P^i: $P^i(K) = P^i(K') = P^i(K^*)$...In [Landau Lifchitz] we know that (in a general field theory):

- if the system is locally conserved : the stress-energy tensor has a null divergence $\partial_k T^{ik} = 0$;
- and if there is "nothing (other than gravitation field)" in infinite (in the sense of convergence to infinity).
- ⇨ $P^i(K) \equiv \frac{1}{c}\int\iiint_{space-time,K} T^{ik}dS_k$ is conserved and doesn't depend on the hyperplane of integration (thanks to the conservation law).

In a less general theorem (but more old) from Von Laue (cf. [Wang]) we can also say that if $\partial_0 T^{ik} = 0$ (and nothing to infiny):

$$P^i = \frac{1}{c}\iiint_{Space} T^{i0}dV \text{ is a 4-momentum} \Leftrightarrow \frac{1}{c}\iiint_{Space} T^{\alpha\beta}dV{=}0$$

1.3. Why (I am) searching another proof ?

The proof above does not use the Lagrangian directly but indirectly via the stress energy tensor. However, the base of all dynamics in physics laws is (until now) always to start from the Lagrangien of a system with the appropriate variables (including degrees of freedom). We should be able to select the center of mass and the complementary degrees of freedom (which we called logically the internal degrees of freedom since they are seen in the "hidden" K*). Unfortunately (for myself at least...), I never found any proof using this point of view. With the current approach (even if it is sufficient for physics) it is not clear, for me, how the centre of mass appears in the Lagrangian, in parallel with the internal degrees of freedom. Indeed the Lagrangien is reconstructed only a posterior, after to demonstrate that $\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{C}}$ (using the stress-energy tensor) (see [Janssen & Mecklenburg]). So we don't clearly see the passage:

- From an initial Lagrangien $S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$ or $S[\{\varphi(x,t)\}] = \frac{1}{c}\int \iiint \Lambda\left(\varphi, \frac{\partial\varphi}{\partial \boldsymbol{r}}, \frac{\partial\varphi}{\partial t}\right) d\Omega$
- To a Lagrangien of an apparent material point $S[\boldsymbol{R_c}(t), \dots] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma(\boldsymbol{V_c})} dt + \cdots$

In this article, I propose, using directly the Lagrangien formalism, to give the proof, for a material system (to present the method), for a field (scalar in order to simplified) and finally a system where the two interact.

2. Material system free

2.1. The proof

We begin with the action principle for a set of particles:

$$S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$$

In this expression, we are using coordinates in a Galilean reference frame K.
The degrees of freedom are the vectors $\{\boldsymbol{r_a}\}$, and we integrate the expression between the plan H_1 ($t_1 = cte$), and H_2 ($t_2 = cte'$) in this frame.
We want now separate:

- the internal degree of freedom $\{\boldsymbol{r_a^*}\}$, defined in the frame K*of the center of mass ;
- from the external degree of freedom $\boldsymbol{R_c}$ defined in the Galilean frame K.

So the degrees of freedom $\{\boldsymbol{r_a}\}$, are equivalent to the degree o freedom $\{\boldsymbol{r_a^*}, \boldsymbol{R_c}\}$**.**

Note 1: a plane t=cte is seen differently for different internal particle in the frame of the center of mass K*

Thanks to the relativist invariance we know that each terms of the action associated to a particle is invariant ($L.dt = \sum_a -m_a.cds_a$). However in the frame K*, the border plan H_1 and H_2 are associated to different time for each particle (in Einstein relativity the simultaneity is relative to a frame).

More explicitly, the Lorentz transformation said that a coordinate t' seen in the frame K is expressed like

$$t' - t = \gamma(t)\left(\left(t'^* - t^*_{c(t)}\right) + \frac{\boldsymbol{\beta}(t)}{c}\boldsymbol{r}^*\right)$$

With:

- $\gamma(t) = \gamma(\boldsymbol{V_C}(t))$,
- $\boldsymbol{V_C} \equiv \boldsymbol{V_{K*/K}}(t)$,
- and t^*_c, the time measured by a clock ion C: $t^*_{c(t)} = \int_0^t \frac{dt'}{\gamma(t')}$

, in the frame $K^*(t)$ at the instant t (t'≠t, a priori, since t' is a generic coordinate of K but t defines the time of K for which the center of mass has the speed $\boldsymbol{V_C}(t)$).

So a plane $t' = cte$ in K is seen like a plane $\gamma(t)\left(\left(t'^* - t^*_{c(t)}\right) + \frac{\boldsymbol{\beta}}{c}\boldsymbol{r}^*\right) + t = cte$ in the frame $K^*(t)$ around t.

Thus a particle at the position $\boldsymbol{r_a^*}$, see the plane $t' = cte$ at the instant $t'^* = \frac{t'-t}{\gamma(t)} - \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{r_a^*} + t^*_{c(t)}$

This is the proof of the assertion in the title.

Note 2: measurement of a clock fixed on $K^*(t)$

Around t (t given and constant), a clock in $\boldsymbol{r}^*$ of $K^*(t)$**, and always in** $\boldsymbol{r}^*$ **,** measures the duration time

$t'^* - t^*_{c(t)} = \frac{t'-t}{\gamma(t)} - \frac{\boldsymbol{V_C}(t)}{c^2}\boldsymbol{r}^*$ between the event $\left(ct^*_{c(t)}, 0^*\right)_{K*(t)}$associated to C in $K^*(t)$ and a certain event $(ct'^*, \boldsymbol{r}^*)_{K*(t)}$ localised, by definition, in a **different position** than C: that is to say $\boldsymbol{r}^*$.

If we demand to this clock to measures now the duration between 2 events localised in its own position, the duration is now $\Delta\left(t'^* - t^*_{c(t)}\right) = \Delta\left(\frac{t'-t}{\gamma(t)} - \frac{\boldsymbol{V_C}(t)}{c^2}\boldsymbol{r}^*\right) <=> (\Delta t'^* - 0) = \left(\frac{\Delta t'-0}{\gamma(t)} - 0\right)$ since $\gamma(t), \boldsymbol{V_C}(t)$, t are constant since we work always in the **same reference frame** $K^*(t)$. More over $\boldsymbol{r}^* = \boldsymbol{cte}$ by definition of the 2 events considered.

So we have $\Delta t'^{*} = \frac{\Delta t'}{\gamma(t)}$ and $dt'^{*} = \frac{dt'}{\gamma(t)}$ for 2 infinitesimal events.

When we observe 2 events associated to a particle, we study the duration time between 2 hyperplanes $t'^{*} = ct$ of K*(t) where the 2 successive positions of the particle occurred. The duration is **always measured by a clock fix** in K*(t). So we can apply the relation above for the duration time associated to a particle:

$$\forall \text{ particle } a\text{: } dt_a^* = dt^* = \frac{dt}{\gamma(t)}$$

Note 3 : On the Lorentz transformation

A more general Lorentz transformation is:

$$\begin{pmatrix} t_a - t \\ \boldsymbol{r_a}(t_a) - \boldsymbol{R_c}(t) \end{pmatrix} = L(t).\begin{pmatrix} t_a^* - t_c^* \\ \boldsymbol{r_a^*} \end{pmatrix} <=> \begin{cases} t_a - t = \gamma(t)\left((t_a^* - t_c^*) + \frac{\boldsymbol{\beta}}{c}\boldsymbol{r_a^*}\right) \\ \boldsymbol{r_a}(t_a) - \boldsymbol{R_c}(t) = c(t_a^* - t_c^*)\gamma(t)\boldsymbol{\beta} + \boldsymbol{r_a^*} + (\gamma - 1)\frac{\boldsymbol{\beta}}{\beta^2}.(\boldsymbol{\beta r_a^*}) \end{cases}$$

For a movement of K* along x, we have the special Lorentz transformation principally used in this article:

$$\begin{cases} t_a - t = \gamma(t)\left(\left(t_a^* - t_{c(t)}^*\right) + \dfrac{\beta(t)}{c}.x_a^*\right) \\ x_a - X_c = \gamma(t)\left(c\left(t_a^* - t_{c(t)}^*\right)\beta(t) + x_a^*\right) \end{cases}$$

Now we express the action in the local frames K*(t):

$$S[\{\boldsymbol{r_a^*}(t^*), \boldsymbol{R_c}(t)\}] = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right) dt^*$$

Taking account $dt^* = \frac{dt}{\gamma(t)}$ and returnig to the Galilean frame K we have:

$$S = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{dt}{dt^*}\frac{d\boldsymbol{r_a^*}}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right)\frac{dt^*}{dt}dt = \int_{t_1}^{t_2} \frac{L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r_a^*}}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right)}{\gamma(\boldsymbol{V}_C)} dt$$

So far, nothing new.

The important point to keep in mind is that we are not considering the variation of the internal degree of freedom $\boldsymbol{r_a^*}$ **:**

- relative to the internal time $t^*$of K*: $\frac{d\boldsymbol{r_a^*}}{dt^*}$;
- **but instead relative to time t of K:** $\frac{d\boldsymbol{r_a^*}}{dt}$.

That is to say, the Lagrangien considered is $L'\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right) \equiv \frac{L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r_a^*}}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_C)}$, instead of using the most « natural » $L\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right) \equiv \frac{L^*\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}\right)}{\gamma(\boldsymbol{V}_C)}$

So, we can now calculate the momentum of the center of mass, with $\boldsymbol{V}_C \equiv \boldsymbol{V}_{K*/K}$:

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} = \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)$$

- $\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} = \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \sqrt{1 - \frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{c^2}} = \frac{-\frac{1}{2} 2 \frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}}{\sqrt{1 - \frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{c^2}}} = -\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}$
- $\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) = \sum_a \frac{\partial \gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{\partial L^*}{\partial \gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} = \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \frac{\partial \left(1 - \frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{c^2}\right)^{-1/2}}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$

$$= \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \left(\frac{\frac{1}{2} 2 \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}}{\left(1 - \frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{\boldsymbol{c}^2}\right)^{3/2}} \right) \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} = \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \gamma^3(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$\boldsymbol{P_c} = L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) \left(-\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\right) + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \gamma^3(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$= \gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2} \left(-L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) + \sum_a \gamma \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} \right)$$

$$= \gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{c^2} \left(\sum_a \gamma \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) \right)$$

$$= \gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{c^2} \left(\sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right) \right) \text{since } \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} = \gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}$$

$$\boldsymbol{P_c} = \gamma \frac{E^*}{c^2} \boldsymbol{V}_{\mathbf{C}}$$

where $E^* \equiv \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right)$ is the internal energy.

So we have our relation.

E^* is relative to the hyperplane t^*=cte, the mass $M = \frac{E^*}{c^2}$ is dealing with events (the spatio-temporal positions of the particles) simultaneous in the frame K* and not in the frame K. This is well defined since $t^* = \int_0^t \frac{dt}{\gamma(t')}$.

$$M = M(t^*) = M\left(\int_0^t \frac{dt'}{\gamma(t')}\right)$$

We see that we don't need to talk about closed system hypothesis or to have a 4 vector momentum to demonstrate it (we don't even use the expression $L.dt = \sum_a -m_a.cds_a$).

We have to note, in the proof, the importance to freeze the right variable $\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}$ (and not $\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\}$) in order to have the good expression.

2.2. Momentum and energy

2.2.1. Momentum

We can also notice that $\boldsymbol{P_a} \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} = \gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*}$, so $\boldsymbol{P_a} = \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$ which is surprising but reassuring for the intelligibility of this quantity: this is the same as the one we would have in the frame of the centre of mass K*.

More over the total momentum $\boldsymbol{P_{total}}$ associated to the Lagrangien $L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)$ is

$\boldsymbol{P_{total}} = \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} + \frac{\partial L'}{\partial v_{\mathrm{C}}} = \sum_a \boldsymbol{P_a} + \boldsymbol{P_c} = \boldsymbol{P_c}$ since by definition of K*: $\sum_a \boldsymbol{P_a} \equiv 0$. This is interesting since despite considering the internal variables on the same level as the mass center, we obtain as it should the total momentum is the one associated to the mass center.

Proof:

Indeed $L.dt = -\sum_a m_a . c ds_a$ => $L = -\sum_a m_a . c \frac{ds_a}{dt} = -\sum_a m_a . c \frac{ds_a}{dt^*}\frac{dt^*}{dt} = -\sum_a m_a . c^2 \frac{1}{\gamma_a^*}\frac{1}{\gamma}$

But $\frac{1}{\gamma . \gamma_a^*} = \frac{1}{\gamma}\sqrt{1 - \frac{\left(\frac{d\boldsymbol{r}_a^*}{dt^*}\right)^2}{c^2}} = \sqrt{\frac{1}{\gamma^2} - \frac{\frac{1}{\gamma^2}\left(\frac{d\boldsymbol{r}_a^*}{dt^*}\right)^2}{c^2}} = \sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r}_a^*}{dt}\right)^2}{c^2}}$ since $\frac{d\boldsymbol{r}_a^*}{dt^*} = \gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_a^*}{dt}$

Moreover $\frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\left(\frac{1}{\gamma . \gamma_a^*}\right) = \frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r}_a^*}{dt}\right)^2}{c^2}} = -\frac{1}{2}\frac{2\frac{d\boldsymbol{r}_a^*}{dt}}{c^2}\frac{1}{\sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r}_a^*}{dt}\right)^2}{c^2}}} = -\frac{\frac{d\boldsymbol{r}_a^*}{dt}}{c^2}\gamma . \gamma_a^*$

So $\boldsymbol{P_a} = -\frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\sum_{a'} m_{a'} . c^2 \frac{1}{\gamma_{a'}^*}\frac{1}{\gamma} = m_a . c^2 \frac{\frac{d\boldsymbol{r}_a^*}{dt}}{c^2}\gamma . \gamma_a^* = m_a . \frac{d\boldsymbol{r}_a^*}{dt^*}\gamma_a^*$

2.2.2. Energy

By definition the energy is: $E \equiv \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}}\frac{d\boldsymbol{r}_a^*}{dt} + \frac{\partial L'}{\partial v_{\mathrm{C}}}\boldsymbol{V_c} - L'$

We can re-express it as:

$E = \sum_a \boldsymbol{P_a}\frac{d\boldsymbol{r}_a^*}{dt} + \boldsymbol{P_c}\boldsymbol{V_c} - \frac{L^{*\prime}}{\gamma}$ since $L' = \frac{L^{*\prime}}{\gamma}$

$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}\frac{d\boldsymbol{r}_a^*}{dt} + \left(\gamma\frac{E^*}{c^2} . \boldsymbol{V_c}\right)\boldsymbol{V_c} - \frac{L^{*\prime}}{\gamma}$ since $\boldsymbol{P_a} \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} = \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$

$$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt} - \frac{L^{*\prime}}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V_c}^2$$

$$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{\gamma dt^*} - \frac{L^{*\prime}}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V_c}^2 = \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L' \right) \frac{1}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V_c}^2$$

$$= \frac{E^*}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V_c}^2 = \frac{E^*}{\gamma} + \gamma \frac{E^*}{c^2} . \boldsymbol{V_c}^2 = \frac{E^* + \gamma^2 \frac{E^*}{c^2} . \boldsymbol{V_c}^2}{\gamma} = E^* \frac{1 + \gamma^2 . \beta^2}{\gamma} = E^* \frac{1 + \frac{\beta^2}{1 - \beta^2}}{\gamma}$$

$$= E^* \frac{\frac{1 - \beta^2 + \beta^2}{1 - \beta^2}}{\gamma} = E^* \frac{\frac{1}{1 - \beta^2}}{\gamma} = E^* \frac{\gamma^2}{\gamma} = \gamma E^*$$

So we have, as it should:

$$\boxed{E = \gamma E^*}$$

We can also conventionally note: $E = E^* + (\gamma - 1)E^*$ where we observe, for a closed system (E=cte), an exchange of Energy between the internal energy E^* and the kinetic energy $(\gamma - 1)E^*$, the one depending of the center of mass.

2.3. The Euler-Lagrange equation for the internal particles and the mass center

The Euler-Lagrange equations are :

$$\frac{d}{dt} \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} L' \left(\{\boldsymbol{r}_a^*\}, \left\{ \frac{d\boldsymbol{r}_a^*}{dt} \right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C} \right) = \frac{\partial}{\partial \boldsymbol{R}_\mathrm{C}} L' \left(\{\boldsymbol{r}_a^*\}, \left\{ \frac{d\boldsymbol{r}_a^*}{dt} \right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C} \right)$$

$$\forall a \quad \frac{d}{dt} \frac{\partial}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} L' \left(\{\boldsymbol{r}_a^*\}, \left\{ \frac{d\boldsymbol{r}_a^*}{dt} \right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C} \right) = \frac{\partial}{\partial \boldsymbol{r}_a^*} L' \left(\{\boldsymbol{r}_a^*\}, \left\{ \frac{d\boldsymbol{r}_a^*}{dt} \right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C} \right) = \frac{\partial}{\partial \boldsymbol{r}_a^*} \frac{L^* \left(\{\boldsymbol{r}_a^*\}, \left\{ \gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_a^*}{dt} \right\} \right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$= \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} \frac{\partial}{\partial \boldsymbol{r}_a^*} L^* \left(\{\boldsymbol{r}_a^*\}, \left\{ \gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_a^*}{dt} \right\} \right)$$

Taking account the momentum expression above we have therefore:

$$\frac{d}{dt} \left(\gamma \frac{E^*}{c^2} \boldsymbol{V}_\mathrm{C} \right) = \frac{\partial}{\partial \boldsymbol{R}_\mathrm{C}} L' \left(\{\boldsymbol{r}_a^*\}, \left\{ \frac{d\boldsymbol{r}_a^*}{dt} \right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C} \right)$$

$$\forall a \quad \frac{d}{dt} \left(\gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*} \right) = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} \frac{\partial}{\partial \boldsymbol{r}_a^*} L^* \left(\{\boldsymbol{r}_a^*\}, \left\{ \gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_a^*}{dt} \right\} \right)$$

As $dt^* = \frac{dt}{\gamma(V_\mathrm{C})}$, the second equation can be re-write:

$$\frac{d}{dt^*} \left(\gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*} \right) = \frac{\partial}{\partial \boldsymbol{r}_a^*} L^* \left(\{\boldsymbol{r}_a^*\}, \left\{ \frac{d\boldsymbol{r}_a^*}{dt^*} \right\} \right)$$

It is remarkable that we obtain the same equation that we should obtain for the dynamic in a K* frame. However, we should notice that, since the center of mass can a priori accelerate, this is not the equation for a material point in a truly Galilean frame. Indeed, dt^* is not constant as it is equal to $dt^* = \frac{dt}{\gamma(v_C)}$ where dt is the true constant differential element. $\boldsymbol{V}_C$ varies, so dt^* has to vary also.

This situation is totaly analog as the one encounter in Newtonian mechanic with Frenet-Serret frame. Indeed, the Frenet-Serret are not a frame of reference but only axis where we project the vectors of a particle (speed, acceleration). We do not derive the speed relative to this frame. The local Galilean frame used in this article is the relativistic analog of the Frenet-Serret frame. We use it to project the dynamical element of the material system (or field) but we do not derive these dynamical elements relative to this frame. This is why the inertial forces don't appear in the internal dynamic.

2.4. The material system seen as a material point: the reduced action

We can write:

$$S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}(t)] = \int_{t_1}^{t_2} L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right) dt$$

$$= \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P}_{\boldsymbol{a}} . \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \boldsymbol{P}_{\boldsymbol{c}} . \boldsymbol{V}_{\boldsymbol{c}} - E\right] dt$$

$$= \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P}_{\boldsymbol{a}} . \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \left(\gamma \frac{E^*}{c^2} \boldsymbol{V}_C\right) . \boldsymbol{V}_{\boldsymbol{c}} - \gamma E^*\right] dt$$

$$= \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P}_{\boldsymbol{a}} . \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \gamma E^*(\beta^2 - 1)\right] dt = \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P}_{\boldsymbol{a}} . \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} - \frac{E^*}{\gamma}\right] dt$$

So

$$\boxed{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}(t)] = \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P}_{\boldsymbol{a}} . \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} - \frac{E^*}{\gamma}\right] dt}$$

If we ignore the final position of the internal degree of freedom, we have like a "spatial Maupertuis principle" (instead of a temporal used in [Landau Lifchitz]):

$$\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}(t)] + \left(\sum_a \boldsymbol{P}_{\boldsymbol{a}} . \delta \boldsymbol{r}_{\boldsymbol{a}}^*\right)_{H_2} = 0$$

We can see that if all the internal momentum are constant, it exists a reduced action principle:

$$\boxed{S_0[\boldsymbol{R}_{\boldsymbol{c}}(t)] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma} dt}$$

We can surely generalize it for closed systems with internal separable variables where we've chosen well the variables with constant momentum. In this case, we see that for "stationary" system, in this restrict sense, the center of mass dynamic is the same as a material point.

Note: my idea to consider the quantity $\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt}\right\}$ comes initially from the willingness to make appear the Lagrangien of the apparent material point with this reduced action (in the same manner we make appear the virtual work theorem: $\delta\int_{t_1}^{t_2}[\sum_a \boldsymbol{P_a}.\boldsymbol{dr} - H[\boldsymbol{P_a},\boldsymbol{r_a}]dt] + (\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r_a})_{H_2} = 0$ and $\boldsymbol{P_a} = \boldsymbol{cte}$=> $\delta\int_{t_1}^{t_2} H_{\boldsymbol{P_a}=\boldsymbol{cte},}(\{\boldsymbol{r_a}\})\,dt = 0$), cf .[Landau Lifchitz]).

Proof:

Indeed (do the same that [Landau lifchitz] but for space and not for time)):

$$\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^{*}(t^*)\},\boldsymbol{R}_c(t)] + \left(\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r}_{\boldsymbol{a}}^{*}\right)_{H_2} = 0$$

$$<=> \delta\int_{t_1}^{t_2} d\sum_a[\boldsymbol{P_a}.\boldsymbol{r}_{\boldsymbol{a}}^{*}] + \delta\int_{t_1}^{t_2}\left[-\frac{E^*}{\gamma}\right]dt + \left(\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r}_{\boldsymbol{a}}^{*}\right)_{H_2} = 0$$

$$<=> \delta\left[\sum_a \boldsymbol{P_a}.\boldsymbol{r}_{\boldsymbol{a}}^{*}\right]_{H_2} + \delta\int_{t_1}^{t_2}\left[-\frac{E^*}{\gamma}\right]dt + \left(\sum_a \boldsymbol{P_a}.\delta\boldsymbol{r}_{\boldsymbol{a}}^{*}\right)_{H_2} = 0$$

$$<=> \delta\int_{t_1}^{t_2}\left[-\frac{E^*}{\gamma}\right]dt = 0$$

2.5. The material system seen as a material point: the internal dynamic is known

As already written:

$$S[\{\boldsymbol{r}_{\boldsymbol{a}}^{*}(t^*)\},\boldsymbol{R_c}(t)] = \int_{t_i}^{t_f} L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)dt$$

$$= \int_{t_i}^{t_f}\left[\sum_a \boldsymbol{P_a}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} - \frac{E^*}{\gamma}\right]dt = \int_{t_i}^{t_f}\left[\sum_a \gamma_a^* m_a.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} - \frac{E^*}{\gamma}\right]dt$$

We **decide** to say that **we know** the internal dynamic of the system.

That is to say we know the maps:

- $\{\boldsymbol{r}_{\boldsymbol{a}}^{*}(t^*)\}$
- $\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^*}(t^*)\right\}$

So, it results that the mass center is **in the field** (in the [Landau Lifchitz] terms) of the internal degree of freedom$\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\}$. We can inject this information $\{\boldsymbol{r}_{\boldsymbol{a}}^{*}(t^*)\}$, $\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^*}(t^*)\right\}$ in the Action :

$$S[\{\boldsymbol{r}_{\boldsymbol{a}}^{*}(t^*)\},\boldsymbol{R_c}(t)] = \int_{t_i}^{t_f} L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)dt$$

$$= \int_{t_i}^{t_f} \left[\sum_a \gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*} d\boldsymbol{r}_a^* - \frac{E^*(t^*)}{\gamma} dt \right]$$

$$= \int_{\{\boldsymbol{t}_{a,i}^*\}}^{\{\boldsymbol{t}_{a,f}^*\}} \left[\sum_a \gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*} d\boldsymbol{r}_a^* \right] + \int_{t_i}^{t_f} -\frac{E^*(t^*)}{\gamma} dt$$

$$= \int_{\{\boldsymbol{t}_{a,i}^*\}}^{\{\boldsymbol{t}_{a,f}^*\}} df\{t_a^*\} + \int_{t_i}^{t_f} -\frac{E^*(t^*)}{\gamma} dt$$

The least action principle can therefore be express with the following action:

$$S''[\boldsymbol{R_c}(t), t] = \int_{t_i}^{t_f} L''\,(t, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}) dt = \int_{t_i}^{t_f} -\frac{E^*(t^*)}{\gamma(\boldsymbol{V}_\mathrm{C})} dt$$

With $t^* = t^*(t) = \int_{t_i}^{t} \frac{dt\prime}{\gamma(t\prime)}$

It is important to not that <u>we a priori don't know the expression of t^*</u> although we know the internal dynamic express relative to it. Indeed, knowing t^* required to know the map $\boldsymbol{V}_\mathrm{C}(t)$ (part of the solution we are looking for) since $t^* = \int_{t_i}^{t} \frac{dt\prime}{\gamma(t\prime)}$,which is absurd. Another proof: knowing t^*, implies the undesirable consequence that $\frac{E^*(t^*(t))}{\gamma} dt = E^*\big(t^*(t)\big) dt^*(t) = df\big(t^*(t)\big) = dg(t).$. This would suppress (according to the least action principle) the only one term of the action that we want to maintain in order to find the trajectory. We see therefore that the center of mass is again <u>in the field</u> of a variable : his own proper time t^*, as for a material point.

It seems difficult to find any relevant way in order to take account the constraint $t^* = \int_{t_i}^{t} \frac{dt\prime}{\gamma(t\prime)}$ in the Lagrangien.

Despite this problem, we can make a stronger supposition that we know, in addition to the internal dynamic, the behaviour of the energy relative to t (and not): $E^*\big(t^*(t)\big)$ noted abusively $E^*(t)$.

Indeed even if we don't know $t^*(t)$ we can pretend to know $E^*(t)$. More precisely

$E^*\big(t^*(t)\big) = (E^* o\ t^*)(t)$. Knowing the map $(E^* o\ t^*)$ is not sufficient to know the map t^* since the inverse map E^{*-1} could eventually not exist.

Knowing $E^*\big(t^*(t)\big)$ and inject it in the Lagrangien, is equivalent to say that the center of mass is now in the field of the energy.

This situation is automatically realized in the classical case where we put $t^* \approx t$ in the Energy. However, we do not make the same approximation for dt^*, indeed we put $dt^* \approx dt\left(1 - \frac{Vc^2}{c^2}\right)$. Otherwise, all the information would be lost:

we do $\frac{E^*(t^*(t))}{\gamma} dt \approx E^*(t) dt \left(1 - \frac{Vc^2}{c^2}\right)$ but not $\frac{E^*(t^*(t))}{\gamma} dt \approx E^*(t) dt$

2.6. A strong link between the Einstein law and the dilatation of time

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$= L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)$$

Since in special relativity, the space is isotropic (≡the **laws of a material system** in a homogeneous & isotropic gravitational field are isotropic) $\gamma(\boldsymbol{V}_\mathrm{C})$ depends only on the norm of $\boldsymbol{V}_\mathrm{C}$ or equivalently on $\boldsymbol{V}_\mathrm{C}{}^2$.

- $\frac{\partial}{\partial V_{\mathrm{C,x}}} \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} = -\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C})}{\partial V_{\mathrm{C,x}}} = -\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial V_{\mathrm{C,x}}} = -\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2} \frac{\partial \boldsymbol{V}_\mathrm{C}{}^2}{\partial V_{\mathrm{C,x}}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2} = -\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2} 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2}$
- $\frac{\partial}{\partial V_{\mathrm{C,x}}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) = \sum_a \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\partial V_{\mathrm{C,x}}} \frac{\partial L^*}{\partial \gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} = \sum_a \left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C})}{\partial V_{\mathrm{C,x}}}\right) \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} =$
  $\sum_a \left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2}\right) \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} = \sum_a \left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2}\right) \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$

$$P_{c,x} = \frac{\partial L'}{\partial V_{\mathrm{C,x}}} = L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\left(-\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2} 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} \sum_a \left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2}\right) \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$= 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2} \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2}\left[\gamma(\boldsymbol{V}_\mathrm{C}) \sum_a \left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right) \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\right]$$

$$= 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2} \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2}\left[\sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C}) \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\right]$$

$$= 2V_{\mathrm{C,x}} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2} \frac{E^*}{\gamma(\boldsymbol{V}_\mathrm{C})^2} = V_{\mathrm{C,x}}\left(\frac{2c^2}{\gamma(\boldsymbol{V}_\mathrm{C})^2} \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{\partial \boldsymbol{V}_\mathrm{C}{}^2}\right) \frac{E^*}{c^2}$$

Starting from $\boldsymbol{P_c} \equiv \frac{\partial L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)}{\partial \boldsymbol{V}_\mathrm{C}}$, the fact that the space is isotropic in special relativity

and without express explicitly $\gamma(\boldsymbol{V}_\mathrm{C})$, we have:

$$\boldsymbol{P_c} = \boldsymbol{V}_\mathrm{C} . \gamma^{eff}(\boldsymbol{V}_\mathrm{C}{}^2) \frac{E^*}{c^2}$$

With $\gamma^{eff}(\boldsymbol{V}_\mathrm{C}) \equiv \frac{2c^2}{\gamma(\boldsymbol{V}_\mathrm{C}{}^2)^2} \frac{d\gamma(\boldsymbol{V}_\mathrm{C}{}^2)}{d\boldsymbol{V}_\mathrm{C}{}^2}$

And of course $\gamma(\boldsymbol{V}_\mathrm{C}{}^2) \equiv \frac{dt}{dt^*}$ is dilatation of time

This is the expression of the 3-momentum of a material system without knowing explicitly the relation between the dilatation of time and the speed of the mass center $\boldsymbol{V}_\mathrm{C}$.

Now using this general result, we want to know if the Einstein law is sufficient to obtain the right expression of the dilatation of time γ relative to $\boldsymbol{V}_\mathrm{C}$, that is to say the expression $\gamma({\boldsymbol{V}_\mathrm{C}}^2)$

We start from $\frac{E^*}{c^2} = M$. This expression **means** that the form of the impulsion of a system, with internal energy E^*, is the same of a material point of mass M verifying $\frac{E^*}{c^2} = M$.

But for a material point we have $\boldsymbol{P_c} = \boldsymbol{V}_\mathrm{C}.\gamma({\boldsymbol{V}_\mathrm{C}}^2)M$, so the Einstein law implies

$$\frac{E^*}{c^2} = M \Rightarrow \gamma^{eff}({\boldsymbol{V}_\mathrm{C}}^2) = \gamma(\boldsymbol{V}_\mathrm{C})$$

$$\gamma(\boldsymbol{V}_\mathrm{C}) = \left(\frac{2c^2}{\gamma({V_\mathrm{C}}^2)^2}\frac{d\gamma({V_\mathrm{C}}^2)}{d{V_\mathrm{C}}^2}\right) \Rightarrow \frac{1}{2c^2} = \frac{1}{\gamma({V_\mathrm{C}}^2)^3}\frac{d\gamma({V_\mathrm{C}}^2)}{d{V_\mathrm{C}}^2} \Rightarrow \frac{d\gamma({V_\mathrm{C}}^2)}{\gamma({V_\mathrm{C}}^2)^3} = \frac{d{V_\mathrm{C}}^2}{2c^2} \Rightarrow -\frac{1}{2}d[\gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2}] = \frac{d{V_\mathrm{C}}^2}{2c^2}$$

$$\Rightarrow -\frac{1}{2}d[\gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2}] = \frac{d{V_\mathrm{C}}^2}{2c^2} \Rightarrow -\frac{1}{2}[\gamma({\boldsymbol{V'}_\mathrm{C}}^2)^{-2}]_0^{{V_\mathrm{C}}^2} = \frac{[{V'_\mathrm{C}}^2]_0^{{V_\mathrm{C}}^2}}{2c^2} \Rightarrow -[\gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2} - \gamma(0)^{-2}] = \frac{{V_\mathrm{C}}^2}{c^2}$$

$$\Rightarrow \gamma(0)^{-2} - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2} = \gamma({\boldsymbol{V}_\mathrm{C}}^2)^{-2} \Rightarrow \gamma({\boldsymbol{V}_\mathrm{C}}^2) = \frac{1}{\sqrt{\gamma(0)^{-2} - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2}}}$$

But $\gamma^{eff}(\boldsymbol{0}) = \gamma(\boldsymbol{0})$=1 => $1 = \frac{2c^2}{\gamma(\boldsymbol{0})^2}\left(\frac{d\gamma({V_\mathrm{C}}^2)}{d{V_\mathrm{C}}^2}\right)_{{V_\mathrm{C}}^2=0} = \frac{2c^2}{\gamma(\boldsymbol{0})^2}\left(-\frac{1}{c^2}\frac{-1}{2}\frac{1}{\left(\gamma(0)^{-2}-\frac{{V_\mathrm{C}}^2}{c^2}\right)^{3/2}}\right)_{{V_\mathrm{C}}^2=0} =$

$$\frac{c^2}{\gamma(\boldsymbol{0})^2}\frac{\boldsymbol{1}}{c^2}\frac{1}{(\gamma(0)^{-2})^{3/2}} = \frac{c^2}{\gamma(\boldsymbol{0})^2}\frac{\boldsymbol{1}}{c^2}\gamma(0)^3 = \gamma(0)$$

So $\frac{E^*}{c^2} = M$ with $\boldsymbol{P_c} = \boldsymbol{V}_\mathrm{C}.\gamma({\boldsymbol{V}_\mathrm{C}}^2)M \Rightarrow \gamma({\boldsymbol{V}_\mathrm{C}}^2) = \frac{1}{\sqrt{1-\frac{{V_\mathrm{C}}^2}{c^2}}}$

We have the final result:

Starting from $\boldsymbol{P_c} \equiv \frac{\partial L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)}{\partial V_\mathrm{C}}$, the fact that the space is isotropic in special relativity and without express explicitly $\gamma(\boldsymbol{V}_\mathrm{C})$, we have the equivalence:

$$E^* = Mc^2 \Leftrightarrow \gamma({\boldsymbol{V}_\mathrm{C}}^2) = \frac{1}{\sqrt{1 - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2}}}$$

With the definition

$$\left\{\frac{E^*}{c^2} = M\right\} \equiv$$

$\equiv$ {the form of the **impulsion** of a system, with internal energy E^*,
is the same of a material point of mass M verifying $\frac{E^*}{c^2} = M$.}

Hence the Einstein law is **not only a necessary condition** of special relativity (via kinematic and least action principle), but also **a sufficient condition** for the dilatation factor expression $\gamma(\boldsymbol{V}_c{}^2)$.
In this sense, this theorem shows that the dilatation of time and the Einstein law are strongly related. So any proof of the dilatation of time, is a proof of the Einstein Law and inversely.
This can also be illustrated by showing that any empirical deviation of the Einstein law $\Delta \equiv \frac{E^*}{c^2} - M$ is linked to a deviation of the Special Relativity relation $\frac{1}{\gamma(\boldsymbol{V}_c{}^2)^2} = 1 - \frac{\boldsymbol{V}_c{}^2}{c^2}$.

$$\Delta \equiv \frac{E^*}{c^2} - M = \frac{\gamma(\boldsymbol{V}_c{}^2)}{\gamma^{eff}(\boldsymbol{V}_c{}^2)} M - M = M\left(\frac{\gamma(\boldsymbol{V}_c{}^2)}{\gamma^{eff}(\boldsymbol{V}_c{}^2)} - 1\right) = M\left(\frac{\gamma(\boldsymbol{V}_c{}^2)}{\frac{2c^2}{\gamma(\boldsymbol{V}_c{}^2)^2}\frac{d\gamma(\boldsymbol{V}_c{}^2)}{d\boldsymbol{V}_c{}^2}} - 1\right)$$

$$= M\left(\frac{\gamma(\boldsymbol{V}_c{}^2)^3}{2c^2\frac{d\gamma(\boldsymbol{V}_c{}^2)}{d\boldsymbol{V}_c{}^2}} - 1\right) = M\left(\frac{1}{\frac{2c^2}{-2}\frac{d[\gamma(\boldsymbol{V}_c{}^2)^{-2}]}{d\boldsymbol{V}_c{}^2}} - 1\right) = M\left(1 - \frac{1}{c^2\frac{d[\gamma(\boldsymbol{V}_c{}^2)^{-2}]}{d\boldsymbol{V}_c{}^2}}\right)$$

So we have

$$\Delta \equiv \frac{E^*}{c^2} - M = M\left(1 - \frac{1}{c^2\frac{d[\gamma(V_c{}^2)^{-2}]}{dV_c{}^2}}\right) \text{ or } c^2\frac{d[\gamma(V_c{}^2)^{-2}]}{dV_c{}^2} = \left(\frac{1}{\frac{\Delta}{M}-1}\right) = \left(\frac{1}{1-\frac{\Delta}{M}}\right) \approx 1 + \frac{\Delta}{M}$$

If we measures $\frac{1}{\gamma(V_c{}^2)^2}$ in function of $\boldsymbol{V}_c{}^2$, we can obtain an empiric law like

$\frac{1}{\gamma(V_c{}^2)^2} = \sum_{n=0}^{\infty}(a_n^{SR} + \varepsilon_n).\|\boldsymbol{V}_c\|^{2n}$ with $a_n^{SR} = (1, -1, 0, 0, 0, \dots)$

<=> $\frac{1}{\gamma(V_c{}^2)^2} = (1 + \varepsilon_0) + (-1 + \varepsilon_1).\boldsymbol{V}_c{}^2 + \sum_{n=2}^{\infty}\varepsilon_n.\|\boldsymbol{V}_c\|^{2n}$

$$=> \frac{d[\gamma(\boldsymbol{V}_c{}^2)^{-2}]}{d\boldsymbol{V}_c{}^2} = \varepsilon_1 - 1 + 2\sum_{n=2}^{n}\varepsilon_n.\|\boldsymbol{V}_c\|^{2n-1}$$

Then we have the following relation between the empiric deviation of the 2 law:

$$c^2\left(\varepsilon_1 - 1 + 2\sum_{n=2}^{n}\varepsilon_n.\|\boldsymbol{V}_c\|^{2n-1}\right) = \left(1 + \frac{\Delta}{M}\right)$$

Any deviation of the Einstein law is linked to a deviation of the expression of the dilatation of time :

$$c^2\left(\varepsilon_1 - 1 + 2\sum_{n=2}^{n}\varepsilon_n.\|\boldsymbol{V}_c\|^{2n-1}\right) \approx \left(1+\frac{\Delta}{M}\right)$$

with

- $\Delta \equiv \frac{E^*}{c^2} - M$
- $\frac{1}{\gamma({V_c}^2)^2} = (1+\varepsilon_0) + (-1+\varepsilon_1).{\boldsymbol{V}_c}^2 + \sum_{n=2}^{\infty}\varepsilon_n.\|\boldsymbol{V}_c\|^{2n}$

This is another way to express the link between the 2 laws.

2.7. Questions about the meaning of events and physical quantities used in the proof

2.7.1. The meaning of a speed

There is a priori a problem with the speed $\frac{d\boldsymbol{r}_a^*}{dt}$ since it combines 2 quantities that each relies to 2 different references frames: K* for $d\boldsymbol{r}_{\boldsymbol{a}}^*$ and K for dt. It may be thought to be ill-defined, which would break the demonstration.

In many textbook like in [Yvan Simon] we can "traditionally" write $\frac{d\boldsymbol{r}_a^*}{dt} = \frac{d\boldsymbol{r}_a^*}{dt^*}\frac{dt^*}{dt}$, and according to the Lorentz Transformation $\frac{dt^*}{dt} = \frac{1}{\gamma\left(dt^*+\frac{\beta}{c}dx_a^*\right)} = \frac{1}{dt_1+dt_2}$ with $dt_1 \equiv \gamma.dt^*$ and $dt_2 \equiv \gamma\frac{\beta}{c}dx_a^*$.

$$\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt_1+dt_2} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{\gamma\left(dt^*+\frac{\beta}{c}dx_a^*\right)}$$

However **we don't use this textbook (or traditional) formula** above in this article but another instead (consequently $d\boldsymbol{r}_{\boldsymbol{a}}^*$ has also another meaning):

$$\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt_1} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{\gamma dt^*}$$

So what the 2 expressions really mean, why are we using the second whereas the first ? and is there any sense to use the second ? The latter question is important since my proof is totally based on it.

a. In the first expression $\frac{d\boldsymbol{r}_a^*}{dt} = \frac{d\boldsymbol{r}_a^*}{\gamma\left(dt^*+\frac{\beta}{c}dx_a^*\right)}$, we are actually using the Lorentz transformation about **the 2 same two events** seen in 2 different Galilean Frames K and K*:

- $a_1 = \left(ct_{K_1^*}, \boldsymbol{r}_{\boldsymbol{a},K_1^*}(t_{K_1^*})\right)_{K_1^*} = \left(ct_1, \boldsymbol{r}_{\boldsymbol{a}}(t_1)\right)_K$
- $a_2 = \left(c\left(t_{K_1^*}+dt_{K_1^*}\right), \boldsymbol{r}_{\boldsymbol{a},K_1^*}\left(t_{K_1^*}+dt_{K_1^*}\right)\right)_{K_1^*} = \left(c\left(t_1+(t_2-t_1)\right), \boldsymbol{r}_{\boldsymbol{a}}(t_1)+d\boldsymbol{r}_{\boldsymbol{a}}(t_1)\right)_K$

Indeed, at the time t_1 of K we associate to the center of mass C, at the position $x_c(t_1)$. Any coordinate $\left(ct_{K_1^*}, x_{K_1^*}\right)_{K_1^*}$ of the local (current Galiean) reference frame K* is related to that of K $(ct,x)_K$ with the Lorentz transformation:

$$\begin{cases} c.t - c.t_1 = \gamma_{t_1}.\left(c\left(t_{K_1^*} - \int_0^{t_1}\frac{dt}{\gamma_t}\right) + \beta_{t_1}.\left(x_{K_1^*} - x_{C,K_1^*}\right)\right) \\ x - x_c(t_1) = \gamma_{t_1}.\left(\left(x_{K_1^*} - x_{C,K_1^*}\right) + \beta_{t_1}.\left(t_{K_1^*} - \int_0^{t_1}\frac{dt}{\gamma_t}\right)\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} c.t - c.t_1 = \gamma_{t_1}.\left(c\left(t_{K_1^*} - t_{C(t_1)}^*\right) + \beta_{t_1}.x_{K_1^*}\right) \\ x - x_c(t_1) = \gamma_{t_1}.\left(x_{K_1^*} + \beta_{t_1}.\left(t_{K_1^*} - t_{C(t_1)}^*\right)\right) \end{cases}$$

- $t^*_{C(t_1)} \equiv \int_0^{t_1} \frac{dt}{\gamma_t}$ the time seen from the clock in C ;
- $x_{C,K_1^*} \equiv 0$ by deciding that C is the spatial origin of the current K*.
- $\gamma_{t_1}, t^*_{C(t_1)}, \beta_{t_1}$ are **constants** associated to the Lorentz transformation at the time t_1.

So we simply apply this transformation for the 2 events:

- On one hand:
  $d(c.t - c.t_1) \equiv (c.t - c.t_1)_{a_2} - (c.t - c.t_1)_{a_1} = (c.t)_{a_2} - c.t_1 - (c.t)_{a_1} + c.t_1 = c.dt$
- On the other hand: $(c.t - c.t_1)_{a_2} - (c.t - c.t_1)_{a_1} =$
  $= \left[\gamma_{t_1}.\left(c\left(t_{K_1^*} - t^*_{C(t_1)}\right) + \beta_{t_1}.x_{K_1^*}\right)\right]_{a_2} - \left[\gamma_{t_1}.\left(c\left(t_{K_1^*} - t^*_{C(t_1)}\right) + \beta_{t_1}.x_{K_1^*}\right)\right]_{a_1}$

$= \gamma_{t_1}\left[\left(c\left(t_{K^*_{1,a_2}} - t_{K^*_{1,a_1}}\right) + \beta_{t_1}.\left(x_{K^*_{1,a_2}} - x_{K^*_{1,a_1}}\right)\right)\right]$ since γ_{t_1}, t^*_C & β_{t_1} are constant

$= \gamma_{t_1}\left(cdt_{K_1^*} + \beta_{t_1}.dx_{K_1^*}\right)$

So we got what we expected $\boxed{c.dt = \gamma_{t_1}\left(c.dt_{K_1^*} + \beta_{t_1}.dx_{K_1^*}\right)}$

b. <u>Now what is the meaning of</u> <u>the second expression</u> $\frac{d\boldsymbol{r}^*_{\boldsymbol{a}}}{dt} = \frac{d\boldsymbol{r}^*_{\boldsymbol{a}}}{dt_1} = \frac{d\boldsymbol{r}^*_{\boldsymbol{a}}}{\gamma dt^*}$?

The answer of the question need to clarify what we are actually doing in the reasoning of this article. First, we start to suppose the **knowledge of** the movement of the center of mass C, for each time t of K. This knowledge **imposes** the movement of the reference frame K* since we **choose to define** it such that, around each time t, it coincides with the family of Galilean reference frame $\left(\mathrm{K}^*(\mathrm{t})\right)_{\mathrm{t}\in\mathrm{R}}$

- in a uniform rectilinear translation relative to K (with the speed of C: $\boldsymbol{V}_{\boldsymbol{C}/\mathbf{K}^*}$);
- and having for spatial origin the position of C.

So we have parameterized the reference frame K* with the time t_i of K with a map, say g:

$$\boldsymbol{g}: t_i \rightarrow K^*(t_i) \text{ also noted } K_i^*$$

Secondly, <u>what are the events involved in the two frames ?</u> We are studying a particle “a” of a material system with C as its mass center. We can a priori think that, at the instant t_1 of K, since we study an event $\left(ct_1, x_a(t_1)\right)_K$, we have to study in $K^*(t_1)$ the **same event** seen with the different coordinate due to the direct application of the Lorentz transformation to $\left(ct_1, x_a(t_1)\right)_K$ ...But it is actually **not the case.**

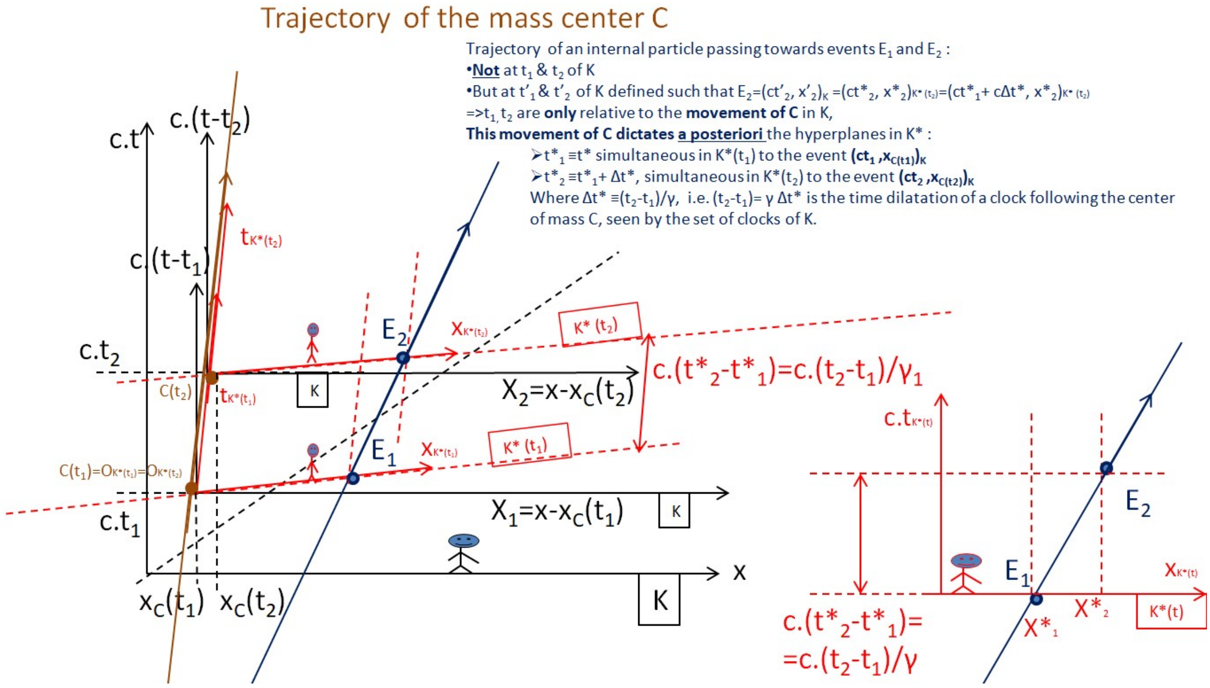
Indeed, at the instant t_1 of K we apply the map **g** defined above and we observe in $K^*(t_1)$ **all the elements which are <u>simultaneous</u> with the event associated to the spatio-temporal position of C**: $(ct_1, x_c)_K$.

So contrary to the case 1), in the case 2): we are not studying the same event (the same spatio-temporal position of the partcicle “a”) in two different frame but :

- An event $\left(ct_1, x_a(t_1)\right)_K$ in K;
- And an event $\mathrm{E}_1 = \left(c.t^*_{C(t_1)}, x_{a,K_1^*}\right)_{K_1^*}$ in $K^*(t_1)$ defined by its **simultaneity** with $(ct_1, x_c)_K$.

By the relativity of the simultaneity, this event E_1 in $K^*(t_1)$ cannot be associated to the instant t_1 of K. In fact, only the event $(ct_1, x_c)_K$ is analysed with the two reference frame K & $K^*(t_1)$. So we understand why we cannot use the expression of the case a).

In order to visualize the situation, we show below the schematic view of what we are truly doing.



This schematic view use the 2 following expressions calculated in ANNEX:

- $c.t_{(x_{K_i^*}=cte)}(x) = c.t_i + \frac{x-x_c(t_i)}{\beta_{t_i}} - \frac{cte}{\gamma_{t_i}.\beta_{t_i}}$
- $c.t_{(ct_{K_i^*}=cte)}(x) = c.t_i + \beta_{t_i}.(x - x_c(t_i)) + \frac{c\left(cte - t^*_{C(t_i)}\right)}{\gamma_{t_i}}$

We also use the fact that, according to the definition of the reference frame of the centre of mass, the orientation all the hyperplane of simultaneity of $K^*(t_1)$ are (around t_1):

- the hyperplanes $t_{K_1^*} = t^*_{C(t_1)}$
- and all the other separated by $dt_{K_1^*} = \frac{dt}{\gamma_{t_1}}$

Indeed, thanks to the Lorentz transformation between the reference frame K and $K_i^* \equiv K^*(t_i)$

$$\begin{cases} c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) \\ x - x_c(t_i) = \gamma_{t_i}.\left(x_{K_i^*} + \beta_{t_i}.\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right) \end{cases}, \text{ we have}$$

$$c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) \Rightarrow c.t = c.t_i + \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right)$$

$$\Rightarrow \boxed{c.t_{(x_{K_i^*}=cte)}(ct_{K_i^*}) = c.t_i + \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.cte\right)}$$

Its results that relative to K, events situated, at rest, at the origin of K_1^* (that is to say C) and having the time $t_{K_i^*}$ are observed at the time $t_{(x_{K_1^*}=0)}(ct_{K_i^*}) = t_1 + \gamma_{t_1}.\left(t_{K_1^*} - t^*_{C(t_1)}\right)$.

This situation is of course relevant for the centre of mass C between the instant t_1 and t_2:

$$\boxed{t_2 - t_1 = \gamma_{t_1}.\left(t_{K_2^*} - t_{1,K_1^*}\right) \Leftrightarrow t_{K_1^*} - t^*_{C(t_1)} = \frac{t_2 - t_1}{\gamma_{t_1}}}$$

This relation also relevant to all couples of events having the same position (at rest) in $K^*(t_1)$. So, we have the relation affirmed in 2) and showed in the picture above.

The particle event of the reference frame K* are also parameterized by the time t of K
Indeed, we can define for a particle "a" a map:

$$\boldsymbol{g_a}: t_i \to \mathrm{E}_{t_i} = \left(c.\,t^*_{C(t_i)}, x_{a,K_i^*}\right)_{K_1^*}$$

That is to say, at each time t_i of K, we associate a frame $K^*(t_i)$, then the event E_{t_i} associated to the particle is the one localized in the hyperplane of $K^*(t_i)$ which **contain also C** at the instant t_i.

We are not saying that the particle "a" is seen at the instant t_i in $K^*(t_i)$ (a non-sense in relativity) but instead it is associated to the instant t_i in the map $\boldsymbol{g_a}$ sense: indeed, the hyperplane of simultaneity of $K^*(t_i)$ is parameterized by t_i.

In order to more untangle these relation, we give just below the explicit expression of $\mathrm{E}_i = \mathrm{E}_{t_i}$ in K. To insist in the fact that E_i is parameterized by the time t_i, I will always write it E_{t_i}.

2.7.2. What is the coordinates of E_{t_1} in K?

We suppose the knowledge of the trajectory of C and the internal particle "a" relative to K $x_a(t)$.

At t_1, E_{t_1} has the same plane $c.t^* = ct_{K_1^*}$ than C which has the coordinate $\left(ct^*_{C(t_1)}, 0^*\right)_{K_1^*} = \left(c.\int_0^{t_1}\frac{dt}{\gamma_t}, 0^*\right)_{K_1^*}$ in $K^*(t_1)$.

Moreover at a given coordinate x of "a" in K we have:

$$c.\,t_{(c.t^*=ct^*_{C(t_1)})}(x) = c.\,t_1 + \beta_{t_1}.\,(x - x_c(t_1))$$

What can we choose for x ?

The expression was calculated for a particle "a" on the x-axis of K at a time of K where the function $x_a(t)$ is the x-coordinate associate to $c.\,t_{(c.t^*=ct^*_{C(t_1)})}$ which different from t_1 with a certain duration Δt_1. The time of K where E_{t_1} took place is :

$$c.\,t_{(c.t^*=ct^*_{C(t_1)})}\big(x_a(t_1+\Delta t_1)\big) = c.\,t_1 + \beta_{t_1}.\,(x_a(t_1+\Delta t_1) - x_c(t_1))$$

We can notice that, knowing the trajectories $x_a(t)$, $x_c(t)$, Δt_1 is a solution of the equation:

$$\boxed{\Delta t_1 = \frac{\beta_{t_1}}{c}.\,(x_a(t_1+\Delta t_1) - x_c(t_1))}$$

- In a particular case where $x_a(t_1+\Delta t_1)$ can be developed at the first order, the latter equation is reduced to:

$$\boxed{\Delta {t_1}^{(1)} \approx \frac{\beta_{t_1}}{c}.\left(x_a(t_1) + \frac{dx_a}{dt}(t_1)\Delta {t_1}^{(1)} - x_c(t_1)\right)}$$

$$<=> \Delta {t_1}^{(1)}\left(1 - \frac{\beta_{t_1}}{c}\frac{dx_a}{dt}(t_1)\right) \approx \frac{\beta_{t_1}}{c}.\,(x_a(t_1) - x_c(t_1))$$

$$<=> \boxed{\Delta t_1 \approx \Delta {t_1}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1) - x_c(t_1)}{1 - \beta_{t_1}\frac{V_a}{c}(t_1)}}$$

- In a particular case where $x_a(t_1+\Delta t_1)$ can be developed at the second order, the latter equation is reduced to:

$$\Delta t_1^{(2)} \approx \frac{\beta_{t_1}}{c}.\left(x_a(t_1)+\frac{dx_a}{dt}(t_1).\Delta t_1^{(2)}+\frac{d^2x_a}{dt^2}(t_1).\frac{{\Delta t_1^{(2)}}^2}{2}-x_c(t_1)\right)$$

$$<=> 0 \approx \left[\frac{\beta_{t_1}}{2c}\frac{d^2x_a}{dt^2}(t_1)\right]{\Delta t_1^{(2)}}^2+\left[\frac{\beta_{t_1}}{c}\frac{dx_a}{dt}(t_1)-1\right]\Delta t_1^{(2)}+\frac{\beta_{t_1}}{c}[x_a(t_1)-x_c(t_1)]$$

$$<=> \boxed{0 \approx \left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]{\Delta t_1^{(2)}}^2-\left[1-\beta_{t_1}\frac{V_a}{c}(t_1)\right]\Delta t_1^{(2)}+\Delta t_1^{(1)}\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}$$

We can try to solve it directly, using the standard solution of the second order equation, but it should be not useful since the solution will not be applicable in the usual case where there is no acceleration...However, there is another way to solve it with the perturbation ε of the first order solution $\Delta t_1^{(1)}$: $\Delta t_1^{(2)}=\Delta t_1^{(1)}+\varepsilon$

$$0 \approx \left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]{\Delta t_1^{(2)}}^2-\left(\Delta t_1^{(2)}-\Delta t_1^{(1)}\right)\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)$$

$$<=> \Delta t_1^{(2)}-\Delta t_1^{(1)} \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}{\Delta t_1^{(2)}}^2$$

Using $\Delta t_1^{(2)}=\Delta t_1^{(1)}+\varepsilon$, we have:

$$\varepsilon \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left(\Delta t_1^{(1)}+\varepsilon\right)^2=\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left({\Delta t_1^{(1)}}^2+\varepsilon^2+2\varepsilon\Delta t_1^{(1)}\right)$$

$$<=> \varepsilon = \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left({\Delta t_1^{(1)}}^2+\varepsilon^2+2\varepsilon\Delta t_1^{(1)}\right)$$

$$<=> \varepsilon \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\left({\Delta t_1^{(1)}}^2+2\varepsilon\Delta t_1^{(1)}\right) \text{ with } \Delta t_1^{(1)} \gg \varepsilon$$

$$<=> \varepsilon\left(1-2\Delta t_1^{(1)}\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\right) \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}{\Delta t_1^{(1)}}^2$$

$$<=> \varepsilon \approx \frac{\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}}{1-2\Delta t_1^{(1)}\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}}{\Delta t_1^{(1)}}^2$$

$$=> \varepsilon \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.\Delta t_1{}^{(1)^2}\left(1+2\Delta t_1{}^{(1)}\frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\right) \approx \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.\Delta t_1{}^{(1)^2}$$

$$\Delta t_1{}^{(2)} = \Delta t_1{}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\Delta t_1{}^{(1)^2}$$

With $\Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$

The traditional calculation gives:

$$\Delta = \left[1-\beta_{t_1}\frac{V_a}{c}(t_1)\right]^2 - 4.\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1).\Delta t_1{}^{(1)}\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)$$

$$<=> \boxed{\Delta = \left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)\left(1-\beta_{t_1}\left(\frac{V_a}{c}(t_1)+\frac{a_a}{c}(t_1).2\Delta t_1{}^{(1)}\right)\right)}$$

$$<=> \boxed{\Delta = \left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)\left(1-\beta_{t_1}\frac{V_a}{c}\left(t_1+2\Delta t_1{}^{(1)}\right)\right)}$$

$\Delta > 0 <=> 1 > \beta_{t_1}\frac{V_a}{c}\left(t_1 + .\Delta t_1{}^{(1)}\right)$ which is always true

$$=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right) \pm \sqrt{\left(1-\beta_{t_1}V_a(t_1)\right)\left(1-\beta_{t_1}\frac{V_a}{c}\left(t_1+2\Delta t_1{}^{(1)}\right)\right)}}{\beta_{t_1}\frac{a_a}{c}(t_1)}$$

$$<=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right) \pm \sqrt{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)\left(1-\beta_{t_1}\left(\frac{V_a}{c}(t_1)+\frac{a_a}{c}(t_1).2\Delta t_1{}^{(1)}\right)\right)}}{\beta_{t_1}\frac{a_a}{c}(t_1)}$$

$$<=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}\left[1 \pm \sqrt{\frac{1-\beta_{t_1}\left(\frac{V_a}{c}(t_1)+\frac{a_a}{c}(t_1).2\Delta t_1{}^{(1)}\right)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}}\right]$$

$$<=> \Delta t_1{}^{(2)} = \frac{\left(1-\beta_{t_1}V_a(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}\left[1 \pm \sqrt{1-\frac{\beta_{t_1}\frac{a_a}{c}(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.2\Delta t_1{}^{(1)}}\right]$$

$$<=> \Delta t_1{}^{(2)} \approx \frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}\left[1 \pm 1 \mp \frac{\beta_{t_1}\frac{a_a}{c}(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}.\Delta t_1{}^{(1)}\right]$$

$$<=> \Delta t_1{}^{(2)} \approx \mp\Delta t_1{}^{(1)} + (1 \pm 1)\frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}\frac{a_a}{c}(t_1)}$$

$$\Delta t_1{}^{(2)} \approx \Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$$

$$or \ \approx -\Delta t_1{}^{(1)} + 2\frac{\left(1-\beta_{t_1}\frac{V_a}{c}(t_1)\right)}{\beta_{t_1}a_a(t_1)}$$

As explained, this solution relevant only when $a_a(t_1) \neq 0$

I will not use this one, I will use the first showed above.

The position where E_{t_1} takes place in K is therefore $x_a(t_1 + \Delta t_1)$:

$$\text{With } \Delta t_1 \approx \Delta {t_1}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)} {\Delta {t_1}^{(1)}}^2, \text{ and } \Delta {t_1}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$$

We have finally:

$$\boxed{\mathrm{E}_{t_1} = \left(c.t^*_{C(t_1)}, x_{a,K_1^*}\right)_{K_1^*} = \left(c(t_1 + \Delta\mathrm{t}), x_a(t_1 + \Delta t_1)\right)_K}$$

with $\Delta t_1 = \frac{\beta_{t_1}}{c}.(x_a(t_1 + \Delta t_1) - x_c(t_1))$, that we can call it **the shift time** : the time to wait after t_1 in order to have the event "the particle "a" arrives on the hyper plane of $K^*(t_1)$ ".

We can notice that:

- $\mathrm{E}_{t_1} \neq (c.t_1, \ldots)_K$
- $\boldsymbol{g_a}: t_i \to \mathrm{E}_{t_i} = \left(c.(t_i + \Delta t_i), x_a(t_i + \Delta t_i)\right)_K$

We clearly see that E_{t_1} is parameterized by t_1 although it is not seen at this instant in K but at the instant $t = t_1 + \Delta t_1$.

Another interesting point is that, at the t_1 , the internal events that take place in $K^*(t_1)$ are not of the kind $\left(c(t_1), x_a(t_1)\right)_K$ but the "shifted" version$\left(c(t_1 + \Delta t_1), x_a(t_1 + \Delta t_1)\right)_K$. That is to say the internal events considered (spatio-temporal position of particle) will happen in the future (or the past, depending the position compared to the mass centre). The weird consequence (another one of relativity...) is that **the internal energy and so the mass, is relative to the future and the past of the material system** (and also field as we will see below).

2.7.3. What is the difference of coordinates of the particle for infinitesimal interval dt, seen in K ?

With the same reasoning, we have at the instant t_2 just after t_1:

$$\mathrm{E}_{t_2} = \left(c.t^*_{C(t_2)}, x_{a,K_2^*}\right)_{K_2^*} = \left(c(t_2 + \Delta t_2), x_a(t_2 + \Delta t_2)\right)_K$$

$$\text{With } \Delta t_2 \approx \Delta {t_2}^{(1)} + \frac{\left[\frac{\beta_{t_2}}{2}\frac{a_a}{c}(t_2)\right]}{1-\beta_{t_2}\frac{V_a}{c}(t_2)} {\Delta {t_2}^{(1)}}^2, \text{ and } \Delta {t_1}^{(1)} \equiv \frac{\beta_{t_2}}{c}.\frac{x_a(t_2)-x_c(t_2)}{1-\beta_{t_2}\frac{V_a}{c}(t_2)}$$

So by doing the simple algebraic difference in K, we have:

$$\mathrm{E}_{t_2} - \mathrm{E}_{t_1} = \left(c.(t_2 + \Delta t_2), x_a(t_2 + \Delta t_2)\right)_K - \left(c.(t_1 + \Delta t_1), x_a(t_1 + \Delta t_1)\right)_K$$

$$= \left(c.(t_2 - t_1) + c(\Delta t_2 - \Delta t_1), [x_a]_{t_1 + \Delta t_1}^{t_2 + \Delta t_2}\right)_K$$

With $[x_a]_{t_1 + \Delta t_1}^{t_2 + \Delta t_2} \equiv x_a(t_2 + \Delta t_2) - x_a(t_1 + \Delta t_1)$

When $(t_2 - t_1)$ tends to dt (no 2nd degree), we have:

- $\beta_{t_2} = \beta_{t_1} + \left(\frac{d\beta_t}{dt}\right)_{t_1} (t_2 - t_1)$
- $\Delta t_2 = \Delta t + (t_2 - t_1)\left(\frac{d}{dt}\Delta t\right)_{t_1}$

With:

- $\Delta t_1 \approx {\Delta t_1}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)} {{\Delta t_1}^{(1)}}^2$
- ${\Delta t_1}^{(1)} \equiv \frac{\beta_{t_1}}{c} \cdot \frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$
- $\left(\frac{d}{dt}\Delta t_1\right)_{t_1} = \frac{d}{dt}\left({\Delta t_1}^{(1)} + {\Delta t_1}^{(1)2}\,\frac{1}{2c}\frac{\beta_{t_1} a_a}{1-\frac{1}{c}\beta_{t_1} V}\right)$

Moreover $[x_a]_{t_1+\Delta t_1}^{t_2+\Delta t_2} = x_a(t_2 + \Delta t_2) - x_a(t_1 + \Delta t_1) = x_a(t_1 + (t_2 - t_1) + \Delta t_2) - x_a(t_1 + \Delta t_1)$

$= x_a(t_1 + (t_2 - t_1) + \Delta t_2) - x_a(t_1 + \Delta t_1)$

$= x_a\left(t_1 + (t_2 - t_1) + \Delta t_1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}(t_2 - t_1)\right) - x_a(t_1 + \Delta t_1)$

$= x_a\left(t_1 + \Delta t_1 + (t_2 - t_1)\left[1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right]\right) - x_a(t_1 + \Delta t_1)$

$= x_a(t_1 + \Delta t_1) + \left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}(t_2 - t_1)\left[1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right] - x_a(t_1 + \Delta t_1)$

$$\boxed{[x_a]_{t_1+\Delta t_1}^{t_2+\Delta t_2} = (t_2 - t_1).\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}.\left[1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right]}$$

$\Rightarrow \mathrm{E}_{t_2} - \mathrm{E}_{t_1} = \left(c.(t_2 - t_1) + c(\Delta t_2 - \Delta t_1), [x_a]_{t_1+\Delta t_1}^{t_2+\Delta t_2}\right)_K$

$= \left(c.(t_2 - t_1) + c\left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}(t_2 - t_1), (t_2 - t_1).\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}.\left(1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right)\right)_K$

$= c.(t_2 - t_1)\left(1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right).\left(1, \frac{1}{c}\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}\right)_K$

$$\Rightarrow \boxed{\mathrm{E}_{t_2} - \mathrm{E}_{t_1} = c.(t_2 - t_1)\left(1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right).\left(1, \frac{1}{c}\left(\frac{\mathrm{d}x_a}{dt}\right)_{t_1+\Delta t_1}\right)_K}$$

With:

- $\Delta t_1 \approx {\Delta t_1}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)} {{\Delta t_1}^{(1)}}^2$
- ${\Delta t_1}^{(1)} \equiv \frac{\beta_{t_1}}{c} \cdot \frac{x_a(t_1)-x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$

- $\left(\frac{d}{dt}\Delta t_1\right)_{t_1} = \frac{d}{dt}\left(\Delta t_1{}^{(1)} + \Delta t_1{}^{(1)2}\frac{1}{2c}\frac{\beta_{t_1}a_a}{1-\frac{1}{c}\beta_{t_1}V_a}\right)$

We can use this difference of events in order to calculate the speed of a particle "a" with these 2 events, we have:

$$\boxed{\left(\frac{x_{\mathrm{E}_{t_2}} - x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}} - t_{\mathrm{E}_{t_1}}}\right)_{t_1,K} = \frac{V_a(t_1 + \Delta t_1)}{1 + \left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}}}$$

The speed associated to the 2 events E_{t_2}& E_{t_1} is actually different than the one associated to the speed measured by K in the standard way. It is of course different to study in K 2 events observed at the instant t_1 & $t_1 + dt$ than the 2 others at $t_1 + \Delta t_1$ & $t_2 + \Delta t_2$.

We recover the standard speed at a given time t when:

- The particle is sufficiently close to the mass centre C =>$\Delta t_1{}^{(1)} \approx 0$
- the relative position of the particle and C is constant $\frac{d}{dt}\left(\Delta t_1{}^{(1)2}\frac{1}{2c}\frac{\beta_{t_1}a_a}{1-\frac{1}{c}\beta_{t_1}V_a}\right) = 0$;
- the speed of C, the speed and the acceleration of "a" a are constant $\frac{d}{dt}\left(\frac{1}{2c}\frac{\beta_{t_1}a_a}{1-\frac{1}{c}\beta_{t_1}V_a}\right) = 0$.

2.7.4. What is the difference of coordinates of the particle for infinitesimal interval dt, seen in K*

The first event is:

$$\mathrm{E}_{t_1} = \left(ct_{a_{(t_1)},K_1^*}, x_{a_{(t_1)},K_1^*}\right)_{K_1^*} = \left(ct^*_{C(t_1)}, x_{a_{(t_1)},K_1^*}\right)_{K_1^*} = (c.t_1 + c.\Delta t_1, x_a(t_1 + \Delta t_1))_K$$

Remark: we use the expression $x_{a_{(t_1)},K_1^*}$ as we have explained above that the events in $K^*(t_1)$ are parameterized via the map **g_a**.

According to Lorentz $\begin{cases}(c.t_1 + c.\Delta t_1) - c.t_1 = \gamma_{t_i}.\left(\mathrm{c}\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) + \beta_{t_1}.x_{a_{(t_1)},K_1^*}\right)\\ x_a(t_1 + \Delta t_1) - x_c(t_1) = \gamma_{t_1}.\left(x_{a_{(t_1)},K_1^*} + \beta_{t_1}.\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right)\right)\end{cases}$

<=>

$$\begin{cases}c\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) = \gamma_{t_1}.\left((c.t_1 + c.\Delta t_1 - c.t_1) - \beta_{t_1}.(x_a(t_1 + \Delta t_1) - x_c(t_1))\right)\\ x_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1) - \beta_{t_1}.(c.t_1 + c.\Delta t_1 - c.t_1)\right)\end{cases}$$

$$\text{<=>}\begin{cases}t_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(\beta_{t_1}.(x_a(t_1 + \Delta t_1) - x_c(t_1)) - \beta_{t_1}.(x_a(t_1 + \Delta t_1) - x_c(t_1))\right)\\ x_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1) - \beta_{t_1}.c.\Delta t_1\right)\end{cases}$$

$$\text{<=>}\begin{cases}c\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) = 0 \Rightarrow \textit{as it should}\\ x_{a_{(t_1)},K_1^*} = \gamma_{t_1}.\left(x_a(t_1 + \Delta t_1) - x_c(t_1) - \beta_{t_1}.\beta_{t_1}.x_a(t_1 + \Delta t_1) - x_c(t_1)\right)\end{cases}$$

$$\boxed{\text{<=>}\begin{cases}\left(t_{a_{(t_1)},K_1^*} - t^*_{C(t_1)}\right) = 0\\ x_{a_{(t_1)},K_1^*} = \frac{x_a(t_1 + \Delta t_1) - x_c(t_1)}{\gamma_{t_1}} = \frac{c\Delta t_1}{\beta_{t_1}.\gamma_{t_1}}\end{cases}}$$

We use $\Delta t_1 = \frac{\beta_{t_1}}{c}.(x_a(t_1 + \Delta t_1) - x_c(t_1))$

The second event is:

$$\mathrm{E}_2 = \left(ct_{a_{(t_2)},K_2^*}, x_{a_{(t_2)},K_2^*}\right)_{K_1^*} = \left(c.t_{C(t_2)}^*, x_{a_{(t_2)},K_2^*}\right)_{K_2^*} = \left(c.(t_2+\Delta t_2), x_a(t_2+\Delta t_2)\right)_K$$

But, in point of view of K_1^* we have also

$$\mathrm{E}_2 = \left(c.t_{a_{(t_2)},K_1^*}, x_{a_{(t_2)},K_1^*}\right)_{K_1^*} = \left(c.t_2 + \beta_{t_2}.(x_a(t_2) - x_c(t_2)), x_a(t_2)\right)_K$$

Remark:

- In the notation $t_{a_{(t_2)},K_1^*}$ we have to note the small change: this is the event in the hyperperplane of $K^*(t_2)$ parametrized at t_2 but seen by an observatory in the frame$K^*(t_1)$.
- $c.t_{a_{(t_2)},K_1^*} \neq t_{C(t_1)}^*$ a priori

$$\begin{cases} c\left(t_{a_{(t_2)},K_1^*} - t_{C(t_1)}^*\right) = \gamma_{t_1}.\left(c.(t_2+\Delta t_2) - c.t_1 - \beta_{t_1}.(x_a(t_2+\Delta t_2) - x_c(t_1))\right) \\ x_{a_{\mathrm{E}_2},K_1^*} = \gamma_{t_1}.\left(x_a(t_2+\Delta t_2) - x_c(t_1) - \beta_{t_1}.(c.(t_2+\Delta t_2) - c.t_1)\right) \end{cases}$$

$$c\left(t_{a_{(t_2)},K_1^*} - t_{C(t_1)}^*\right) = \gamma_{t_1}.\left(c.(t_2+\Delta t_2) - c.t_1 - \beta_{t_1}.(x_a(t_2+\Delta t_2) - x_c(t_2) - x_c(t_1) + x_c(t_2))\right)$$

$$= \gamma_{t_1}.\left(c.(t_2+\Delta t_2) - c.t_1 - c\frac{\Delta t_2}{\beta_{t_2}}\beta_{t_1} - \beta_{t_1}(-x_c(t_1) + x_c(t_2))\right)$$

$$= \gamma_{t_1}.\left(c.t_2 + c.\Delta t_2 - c.t_1 - c\frac{\Delta t_2}{\beta_{t_2}}\beta_{t_1} - (t_2 - t_1)\frac{\beta_{t_1}}{c}V_c(t_1)\right)$$

$$= \gamma_{t_1}.\left(c.(t_2 - t_1) + c.\Delta t_2 - c\frac{\Delta t_2}{\beta_{t_2}}\beta_{t_1} - (t_2 - t_1){\beta_{t_1}}^2\right)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(1 - \frac{\beta_{t_1}}{\beta_{t_2}}\right) + (t_2 - t_1)\left(1 - \frac{{\beta_{t_1}}^2}{c}\right)\right)$$

because $\Delta t_2 = \Delta t_1 + (t_2 - t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}$

and $\Delta t_2 = \frac{\beta_{t_2}}{c}.(x_a(t_2+\Delta t_2) - x_c(t_2))$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(1 - \frac{\beta_{t_1}}{\beta_{t_1} + \left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)}\right) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$\frac{1}{\beta_{t_1} + \left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)} = \frac{1}{\beta_{t_1}} + \left(\frac{d}{dX}\left(\frac{1}{X}\right)\right)_{X=\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1) = \frac{1}{\beta_{t_1}} + \left(\frac{-1}{X^2}\right)_{X=\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$= \frac{1}{\beta_{t_1}} - \frac{1}{{\beta_{t_1}}^2}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(1 - \beta_{t_1}\left(\frac{1}{\beta_{t_1}} - \frac{1}{{\beta_{t_1}}^2}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)\right)\right) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\left(\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)\right) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$= \gamma_{t_1}.c.\left(\Delta t_2\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1) + \frac{(t_2 - t_1)}{{\gamma_{t_1}}^2}\right)$$

$$= c.\frac{(t_2 - t_1)}{\gamma_{t_1}} + \gamma_{t_1}.c.\Delta t_2\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$= c.\frac{(t_2 - t_1)}{\gamma_{t_1}} + \gamma_{t_1}.c.\left(\Delta t_1 + (t_2 - t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\frac{1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)$$

$$\boxed{c\left(t_{a_{(t_2)},K_1^*} - t^*_{C(t_1)}\right)_{K*not\ Galilean} = c.\frac{(t_2 - t_1)}{\gamma_{t_1}} + \gamma_{t_1}.\frac{c.\Delta t_1}{\beta_{t_1}}\left(\frac{d\beta_t}{dt}\right)_{t_1}(t_2 - t_1)}$$

But, since we use at each time a local Galilean frame, there are non acceleration for this frame (the condition for the use of Lorentz transformation): $\left(\frac{d\beta_t}{dt}\right)_{t_1,Galilean} \equiv 0$

$$\boxed{c\left(t_{a_{(t_2)},K_1^*} - t^*_{C(t_1)}\right) = c.\frac{(t_2 - t_1)}{\gamma_{t_1}}}$$

$$x_{a_{(t_2)},K_1^*} = \gamma_{t_1}.\left(x_a(t_2 + \Delta t_2) - x_c(t_1) - \beta_{t_1}.(c.(t_2 + \Delta t_2) - c.t_1)\right)$$
$$= \gamma_{t_1}.\left(x_a(t_1 + (t_2 - t_1) + \Delta t_2) - x_c(t_1) - \beta_{t_1}.(c.\Delta t_2 + c.(t_2 - t_1))\right)$$
$$= \gamma_{t_1}.\left(x_a\left(t_1 + (t_2 - t_1) + \Delta t_1 + (t_2 - t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right) - x_c(t_1) - \beta_{t_1}.c\left(\Delta t_1 + (t_2 - t_1)\left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right) - \beta_{t_1}.(c.(t_2 - t_1))\right)$$
$$= \gamma_{t_1}.\left[x_a\left(t_1 + \Delta t_1 + (t_2 - t_1)\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right) - x_c(t_1) - \beta_{t_1}.c\left(\Delta t_1 + (t_2 - t_1)\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right)\right]$$
$$= \gamma_{t_1}.\left[x_a(t_1 + \Delta t_1) + \left(\frac{d}{dt}x_a\right)_{t_1+\Delta t_1}.(t_2 - t_1)\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right) - x_c(t_1) - \beta_{t_1}.c\left(\Delta t_1 + (t_2 - t_1)\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right)\right]$$
$$= \gamma_{t_1}.\left[x_a(t_1 + \Delta t_1) - x_c(t_1) + V_a(t_1 + \Delta t_1).(t_2 - t_1)\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right) - \beta_{t_1}.c\Delta t_1 - \beta_{t_1}.c(t_2 - t_1)\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\right]$$
$$= \gamma_{t_1}.\left(c\frac{\Delta t_1}{\beta_{t_1}} - \beta_{t_1}.c\Delta t_1 + (t_2 - t_1).\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\left[V_a(t_1 + \Delta t_1) - \beta_{t_1}.c\right]\right)$$
$$= \gamma_{t_1}.\left(c\Delta t_1\left(\frac{1 - {\beta_{t_1}}^2}{\beta_{t_1}}\right) + (t_2 - t_1).\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\left[V_a(t_1 + \Delta t_1) - \beta_{t_1}.c\right]\right)$$
$$= \gamma_{t_1}.\left(\frac{c\Delta t_1}{{\gamma_{t_1}}^2}\frac{1}{\beta_{t_1}} + (t_2 - t_1).\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)\left[V_a(t_1 + \Delta t_1) - \beta_{t_1}.c\right]\right)$$
$$= \gamma_{t_1}(t_2 - t_1).\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).c\left(\frac{V_a}{c}(t_1 + \Delta t_1) - \beta_{t_1}\right)$$

Because $x_{a_{(t_1)},K_1^*} = \frac{x_a(t_1 + \Delta t_1) - x_c(t_1)}{\gamma_{t_1}} = \frac{c\Delta t_1}{\beta_{t_1}.\gamma_{t_1}}$

The expression of the

$$\boxed{\begin{gathered} x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*} = \gamma_{t_1}(t_2 - t_1).\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).c\left(\frac{V_a}{c}(t_1 + \Delta t_1) - \beta_{t_1}\right) \\ c\left(t_{a_{(t_2)},K_1^*} - t^*_{C(t_1)}\right) = c.\frac{(t_2 - t_1)}{\gamma_{t_1}} \end{gathered}}$$

2.7.5. What is the expression of the speed in K and K*and what are their relation (velocity addition formula)?

Using the expression above, we calculate different speed for different frame.

- Relative to the internal frame $K^*(t_1)$

$$\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} = \frac{\gamma_{t_1}(t_2 - t_1).\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).c\left[\frac{V_a}{c}(t_1 + \Delta t_1) - \beta_{t_1}\right]}{c.\frac{(t_2 - t_1)}{\gamma_{t_1}}}$$

- <=> $\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} = \gamma_{t_1}{}^2\left(1 + \left(\frac{d}{dt}\Delta t\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)$

- => $\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_2 - t_1} = \gamma_{t_1}.\left(1 + \left(\frac{d}{dt}\Delta t\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)$

- A new velocity addition formula

Since $\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K} = \frac{V_a(t_1 + \Delta t_1)}{1 + \left(\frac{d\Delta t}{dt}\right)_{t_1}}$, we have

$$\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} = \gamma_{t_1}{}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)$$

$$= \gamma_{t_1}{}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).\left(\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K}\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right) - V_C(t_1)\right)$$

$$= \gamma_{t_1}{}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right).\left(\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K}\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right) - V_C(t_1)\right)$$

<=> $\frac{\frac{x_{a_{E_2},K_1^*} - x_{a,K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}}}{\gamma_{t_1}{}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)} + V_C(t_1) = \left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K}\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)$

<=> $\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K} = \frac{\frac{x_{a_{E_2},K_1^*} - x_{a,K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}} + V_C(t_1)\gamma_{t_1}{}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)}{\gamma_{t_1}{}^2\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)^2}$

<=> $\left(\frac{x_{E_{t_2}} - x_{E_{t_1}}}{t_{E_{t_2}} - t_{E_{t_1}}}\right)_{t_1,K} = \frac{\left(\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_{a_{(t_2)},K_1^*} - t_{a_{(t_1)},K_1^*}}\right)\frac{1}{\gamma_{t_1}{}^2} + V_C(t_1)\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)}{\left(1 + \left(\frac{d}{dt}\Delta t_1\right)_{t_1}\right)^2}$

- A second new velocity addition formula

Since $\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_2 - t_1} = \gamma_{t_1}.\left(1 + \left(\frac{d}{dt}\Delta t\right)_{t_1}\right).\left(V_a(t_1 + \Delta t_1) - V_C(t_1)\right)$

<=> $\frac{\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_2 - t_1}}{\gamma_{t_1}\left(1+\left(\frac{d}{dt}\Delta t\right)_{t_1}\right)} + V_C(t_1) = V_a(t_1 + \Delta t_1)$

We use now: $\left(\frac{x_{\mathrm{E}_{t_2}} - x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}} - t_{\mathrm{E}_{t_2}}}\right)_{t_1,K} = \frac{V_a(t_1 + \Delta t_1)}{1+\left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}}$

$$=> \left(\frac{x_{\mathrm{E}_{t_2}} - x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}} - t_{\mathrm{E}_{t_2}}}\right)_{t_1,K} = \frac{1}{\left(1+\left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right)}\left[\frac{\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_2 - t_1}}{\gamma_{t_1}\left(1+\left(\frac{d}{dt}\Delta t\right)_{t_1}\right)} + V_C(t_1)\right]$$

$$<=> \boxed{\left(\frac{x_{\mathrm{E}_{t_2}} - x_{\mathrm{E}_{t_1}}}{t_{\mathrm{E}_{t_2}} - t_{\mathrm{E}_{t_2}}}\right)_{t_1,K} = \frac{\left(\frac{x_{a_{(t_2)},K_1^*} - x_{a_{(t_1)},K_1^*}}{t_2 - t_1}\right)\frac{1}{\gamma_{t_1}} + V_C(t_1)\left(1+\left(\frac{d}{dt}\Delta t\right)_{t_1}\right)}{\left(1+\left(\frac{\mathrm{d}\Delta t}{dt}\right)_{t_1}\right)^2}}$$

With

- $\Delta t_1 \approx \Delta t_1{}^{(1)} + \frac{\left[\frac{\beta_{t_1}}{2}\frac{a_a}{c}(t_1)\right]}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}\Delta t_1{}^{(1)^2}$
- $\Delta t_1{}^{(1)} \equiv \frac{\beta_{t_1}}{c}.\frac{x_a(t_1) - x_c(t_1)}{1-\beta_{t_1}\frac{V_a}{c}(t_1)}$
- $\left(\frac{d}{dt}\Delta t_1\right)_{t_1} = \frac{d}{dt}\left(\Delta t_1{}^{(1)} + \Delta t_1{}^{(1)^2}\frac{1}{2c}\frac{\beta_{t_1}a_a}{1-\frac{1}{c}\beta_{t_1}V_a}\right)$

2.7.6. Conclusion about the proof

We can conclude that although during the proof we use a particular duration of time $t_1 = \gamma dt^*$, it is well defined as I try to convince the reader in this paragraph 2.6. We should carefully take care to the events implied by this way of reasoning.

3. Free field

Now, I will repeat the same method for a field theory (a scalar field φ for simplify), and again:

> The important point to keep in mind is that we are not considering the variation of the internal degree of freedom φ^* **:**
>
> - relative to the internal time t^* of K* : $\frac{\partial \varphi^*}{\partial t^*}$;
> - **but instead relative to time t of K** : $\frac{\partial \varphi^*}{\partial t}$.

So without comments, we have successively:

$$S[\{\varphi(x,t)\}] = \frac{1}{c}\int \iiint \Lambda\left(\varphi, \frac{\partial \varphi}{\partial \boldsymbol{r}}, \frac{\partial \varphi}{\partial t}\right) d\Omega$$

$$= \frac{1}{c}\int \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) d\Omega^* = \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right] dt^*$$

$$= \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right] \frac{dt}{\gamma}$$

=>

> $$S[\{\varphi^*(x^*,t^*)\}, \boldsymbol{R}_\mathrm{C}(\boldsymbol{t})] = \int L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right] dt$$
>
> With $L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right] = \frac{1}{\gamma}\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$

So we can calculate the 3-momentum as:

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\left[\frac{1}{\gamma}\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right]$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma} + \frac{1}{\gamma}\iiint \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \left(-\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)$$

$$+ \frac{1}{\gamma}\iiint \frac{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} \frac{\partial}{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

But $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma} = -\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}$; $\gamma\frac{\partial \varphi^*}{\partial t} = \frac{\partial \varphi^*}{\partial t^*}$

And $\frac{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{\partial \gamma}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{\partial\frac{1}{\sqrt{1-\frac{\boldsymbol{V}_\mathrm{C}^2}{c^2}}}}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{-1}{2}\left(-2\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)\frac{1}{\left(1-\frac{\gamma^2\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)^2}{\boldsymbol{c}^2}\right)^{3/2}} = \frac{\partial \varphi^*}{\partial t}\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\gamma^3$

$$\boldsymbol{P_c} = \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* \left(-\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\right)$$
$$+\frac{1}{\gamma}\iiint \left(\frac{\partial \varphi^*}{\partial t}\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma^3\right)\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* \,(-1)$$
$$+ \iiint \left(\frac{\partial \varphi^*}{\partial t}\gamma\right)\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma\left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* \,(-1) + \iiint \frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*\right]$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma \iiint \left[\frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right] dV^*$$

So we have again:

$$\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{C}}$$

where $E^* \equiv \iiint \left(\frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right) dV^*$ is the internal energy (associated to the hyperplane $t^* = cte$)

We see that we don't need to talk about closed system hypothesis or to have a 4 vector to demonstrate it (we don't even use the expression of any density Lagrangien).

The Euler-Lagrange equations tell us that $\frac{d}{dt}\left(\gamma\frac{E^*}{\boldsymbol{c}^2}\boldsymbol{V}_{\mathbf{C}}\right) = \frac{\partial}{\partial \boldsymbol{R}_{\mathrm{C}}} L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial r^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right]$

We have to note, in the proof, the importance to freeze the right variable $\frac{\partial \varphi^*}{\partial t}$ (and not $\frac{\partial \varphi^*}{\partial t^*}$) in order to have the good expression.

4. Interaction between a field and a particle

We consider the simplified action:

$$S[\boldsymbol{r_a}(t),\{\varphi(x,t)\}] = \int_{t_1}^{t_2}\left(\sum_a\left[-m_a.c\frac{ds_a}{dt} - \frac{e_a}{c}.\frac{ds_a}{dt}\varphi(\boldsymbol{r_a},t)\right]\right)dt + \frac{1}{c}\int\iiint \Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega$$

So we have also:

$$S = \int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a+\frac{e_a}{c^2}\varphi(\boldsymbol{r_a},t)\right).c.\frac{ds_a}{dt}\right]\right)dt + \frac{1}{c}\int\iiint \Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega$$

$$= \int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a+\frac{e_a}{c^2}\varphi^{K_m}\right).c^2\frac{1}{\gamma_a^{K_m}}\right]\right)\frac{dt}{\gamma_m} + \int\left[\iiint \Lambda^{K_\varphi}\left(\varphi^{K_\varphi},\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}},\gamma\frac{\partial\varphi^{K_\varphi}}{\partial t},\boldsymbol{R}_{\mathrm{C}_\varphi},\boldsymbol{V}_{\mathrm{C}_\varphi}\right)dV^{K_\varphi}\right]\frac{dt}{\gamma_\varphi}$$

Where we have specified the quantities relative to:

- the frame K_φ of the center of mass C_φ of the field φ ;
- the frame K_m of the center of mass C_m of the material points.

$$S\left[\left\{\boldsymbol{r_a}^{K_m}(t^{K_m}),\boldsymbol{R}_{\mathrm{C}_m}(t)\right\},\{\varphi^{K_\varphi}(x^{K_\varphi},t^{K_\varphi})\},\boldsymbol{R}_{\mathrm{C}_\varphi}(\boldsymbol{t})\right] = \int_{t_1}^{t_2} L'\left(\left\{\boldsymbol{r_a}^{K_m}\right\},\left\{\frac{d\boldsymbol{r_a}^{K_m}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}_m},\boldsymbol{V}_{\mathrm{C}_m},t\right)dt + \int L'\left[\{\varphi^{K_\varphi}\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}}\right\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial t}\right\},\boldsymbol{R}_{\mathrm{C}_\varphi},\boldsymbol{V}_{\mathrm{C}_\varphi}\right]dt$$

So in this form, we can calculate the dynamic of the center of mass of one system and the other.
We can see that each system is not free at all, but we have again:

$$\boldsymbol{P}_{\mathrm{C}_m} = \gamma(\boldsymbol{V}_{\mathrm{C}_m})\frac{E^{K_m}}{c^2}\boldsymbol{V}_{\mathrm{C}_m}$$
$$\boldsymbol{P}_{\mathrm{C}_\varphi} = \gamma\left(\boldsymbol{V}_{\mathrm{C}_\varphi}\right)\frac{E^{K_\varphi}}{c^2}\boldsymbol{V}_{\mathrm{C}_\varphi}$$

So $M_m = \frac{E^{K_m}}{c^2}$, $M_\varphi = \frac{E^{K_\varphi}}{c^2}$

With the same method we can consider any set of systems.

5. Conclusion

We have a way to demonstrate the famous Einstein formula $E^*=Mc^2$ directly from an appropriate Lagrangien function selecting the correct variable.

Instead of $L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right)$, we use $L'\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)=\frac{L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$.

Instead of $L'\left[\{\varphi\},\left\{\frac{\partial\varphi}{\partial r}\right\},\left\{\frac{\partial\varphi}{\partial t}\right\}\right]$, we use $L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial t^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right]=\frac{\iiint \Lambda^*\left(\varphi^*,\frac{\partial\varphi^*}{\partial r^*},\gamma\frac{\partial\varphi^*}{\partial t},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)dV^*}{\gamma}$.

In the two cases we've calculated directly that $\boldsymbol{P_c}\equiv\frac{\partial L'}{\partial v_{\mathrm{C}}}=\gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}$

Some remark:

1) A simple Lorentz transformation, shows that the 3-momentum is actually the one associated to $P^i(K^*)=\frac{1}{c}\int\iiint_{space-time}T^{ik}\delta\left(n_{lm}x^lx^m\right).d\eta_k(K^*)\,d^4x$, so it is a part of a 4-vector.Thus, among all the 4-momentum $P^i(K)$, $P^i(K')$, $P^i(K^*)$... the Lagrangien method selects $P^i(K^*)$.
2) Since we have defined the mass center in K^*, it allows us to associated to it a true event (the center of the frame) which doesn't change from a frame K to another K', by the relativity of simultaneity. In consequence, we can show (not here) that the internal energy, so the mass, is an invariant in our case (like for a material point).

References

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6. Annex

6.1. Annex calculation

We want to draw the K* axis seen by K, that is to say the different axis in function of the x axis.

$$\begin{cases} c.t - c.t_i = \gamma_{t_i}.\left(\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) \\ x - x_c(t_i) = \gamma_{t_i}.\left(x_{K_i^*} + \beta_{t_i}.\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right) \end{cases}$$

- In K, the equation of a static point in K* ($x_{K_i^*}$=cte) in function of x , that is to say $c.t_{(x_{K_i^*}=cte)}(x)$ is

$$c.t - c.t_i = \gamma_{t_i}.\left(c\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) = \gamma_{t_i}.\left(\frac{x - x_c(t_i)}{\beta_{t_i}.\gamma_{t_i}} - \frac{x_{K_i^*}}{\beta_{t_i}} + \beta_{t_i}.x_{K_i^*}\right)$$

$$= \frac{x - x_c(t_i)}{\beta_{t_i}} - x_{K_i^*}.\gamma_{t_i}\frac{1}{\beta_{t_i}}\left(1 - \beta_{t_i}{}^2.\right) = \frac{x - x_c(t_i)}{\beta_{t_i}} - \frac{x_{K_i^*}}{\gamma_{t_i}.\beta_{t_i}}$$

$$\boxed{c.t = c.t_i + \frac{x - x_c(t_i)}{\beta_{t_i}} - \frac{x_{K_i^*}}{\gamma_{t_i}.\beta_{t_i}}}$$

=> $\boxed{c.t_{(x_{K_i^*}=K)}(x) = c.t_i + \frac{x - x_c(t_i)}{\beta_{t_i}} - \frac{K}{\gamma_{t_i}.\beta_{t_i}} \text{ at time } t=t_i}$

So the equation of $x_{K_i^*}$=0 is $\boxed{c.t_{\left(x_{K_i^*}=0\right)}(x) = c.t_i + \frac{x - x_c(t_i)}{\beta_{t_i}} \text{ at time } t=t_i}$

Between $x_c(t_1)$ and $x_c(t_2)$, the variation is at should:

$$c.t_{\left(x_{K_1^*}=0\right)}(x_c(t_2)) - c.t_{\left(x_{K_i^*}=0\right)}(x_c(t_1)) = \frac{x_c(t_2) - x_c(t_1)}{\beta_{t_1}} = \frac{V_c(t_1).(t_2 - t_1)}{\beta_{t_1}} = c.(t_2 - t_1)$$

- In K, the equation of (t*=cte) in function of x , that is to say $c.t_{(t^*=cte)}(x)$ is

$$\begin{cases} c.t - c.t_i = \gamma_{t_i}.\left(\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right) \\ x - x_c(t_i) = \gamma_{t_i}.\left(x_{K_i^*} + \beta_{t_i}.\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right) \end{cases}$$

$$c.t - c.t_i = \gamma_{t_i}.\left(\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.x_{K_i^*}\right)$$

$$= \gamma_{t_i}.\left(\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.\left(\frac{x - x_c(t_i)}{\gamma_{t_i}} - \beta_{t_i}.\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right)\right)$$

$$c.t - c.t_i = \gamma_{t_i}.\left(\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.\frac{x - x_c(t_i)}{\gamma_{t_i}} - \beta_{t_i}{}^2.\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right)\right)$$

$$= \gamma_{t_i}.\left(\left(1 - \beta_{t_i}{}^2\right)\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right) + \beta_{t_i}.\frac{x - x_c(t_i)}{\gamma_{t_i}}\right) = \frac{\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right)}{\gamma_{t_i}} + \beta_{t_i}.(x - x_c(t_i))$$

$$\boxed{c.t = c.t_i + \beta_{t_i}.(x - x_c(t_i)) + \frac{\mathrm{c}\left(t_{K_i^*} - t^*_{C(t_i)}\right)}{\gamma_{t_i}}}$$

$$=> \boxed{c.t_{(ct_{K_i^*}=K)}(x) = c.t_i + \beta_{t_i}.(x - x_c(t_i)) + \frac{c\left(K - t^*_{C(t_i)}\right)}{\gamma_{t_i}}}$$

And in particular

$$\boxed{c.t_{(ct_{K_i^*}=t^*_{C(t_i)})}(x) = c.t_i + \beta_{t_i}.(x - x_c(t_i))}$$