# A straightforward and Lagrangien proof of the mass as the internal energy of a system

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I propose a Lagrangian proof of Einstein's well-known law that the mass system is its internal energy. The interest of this proof is to show how appears the distinction between internal degrees of freedom and the center of mass in the Lagrangian formalism.

## 1. Introduction

- **The law**

According the expression of the law of physics via the principle of least action [Landau-Lifchitz] and the relativistic invariance: the mass $m_a$ of a material point "a" is simply the multiplicative coefficient appearing in the Lagrangien of this material point, interacting or not with an external field.

$$\boxed{S[\boldsymbol{r_a}(t)] = -\int_{s_{a,1}}^{s_{a,2}} m_a.c.ds_a + \cdots = -\int_{t_1}^{t_2} \frac{m_a.c^2}{\gamma(\boldsymbol{v_a})} dt + \cdots}$$

In 1905, Einstein tells us that whatever the system: a set of material points (dynamically characterised with a Lagrangien $L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right)$) or a field (dynamically characterised with the Lagrangien $\Lambda\left(\varphi, \frac{\partial\varphi}{\partial r}, \frac{\partial\varphi}{\partial t}\right)$) we should have:

$$\boxed{S[\boldsymbol{R_c}(t), \dots] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma(\boldsymbol{V_c})} dt + \cdots}$$

- With $E^* = \sum_a \frac{d\boldsymbol{r_a^*}}{dt^*} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}} - L^*\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}\right)$ for a material point;
- Or $E^* \equiv \iiint \left(\frac{\partial\varphi^*}{\partial t^*} \frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)} \Lambda^* - \Lambda^*\right) dV^*$ for a scalar field (for example).

Where the quantities with a star * are relative to the reference frame associated to the mass center K*. So $E^*$ is the internal energy.

Thus, every system has a centre of mass which has a Lagrangian, analogous to a material point with a mass $M = \frac{E^*}{c^2}$**.** This is the famous law of Einstein.

- **The current proof**

This law is well established since its first publication in 1905 and was re-demonstrated more clearly after by other (Einstein himself, Von Laue ...). The simpler way (that the author know and read in [Landau Lifchitz]), is to demonstrate that the momentum is a 4 vector.

Indeed, tanks to the stress energy tensor $T^{ik}$ of the system, we can always associate to it a 4-vector

$P^i(K^*) \equiv \frac{1}{c}\int \iiint_{space-time,K} T^{ik} dS_k$ , where we choose the hyper-surface of integration as the hyperplane of the reference frame K* ($t^* = cte$).

In any frame ([Janssen & Mecklenburg]), $P^i(K^*)$ can be written equivalently

$P^i(K^*) = \frac{1}{c}\int \iiint_{space-tim} \; T^{ik}\delta\left(n_{lm}x^l\eta^m(K^*)\right).\eta_k(K^*)\, d^4x$

where $\eta_k(K^*)$ is an orthogonal vector of the hyperplane $t^* = cte$ of K* such that $\eta^*{}_k(K^*) = (1{,}0{,}0{,}0)$ in $K^*$.

Thus, the Lorentz transformations tell us:

$$P^i(K^*) = \frac{1}{c}\int \iiint_{space-time} L^i{}_r L^k{}_s T^{*rs}\delta(t^*).L^m_k.\eta^*{}_m(K^*)\, d^4x^* \; = L^i{}_r \frac{1}{c}\iiint_{x^{*\alpha}\epsilon V^*} T^{*r0}(0,x^{*\alpha})dV^*$$

So $P^i(K^*) = L^i{}_r P^{*r}(K^*)$ where $P^{*r}(K^*) = \frac{1}{c}\iiint_{Space} T^{*r0}(0,x^{*\alpha})dV^*$

But $E^* \equiv \iiint_{Space} T^{*00}(0,x^{*\alpha})dV^*$ and $P^{*\alpha}(K^*) \equiv 0$ by definition of K*

So we have $P^i(K^*) = \left(\gamma\frac{E^*}{c}, \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{K*/K}}\right)$, hence $\boldsymbol{P} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{K*/K}}$ **=>** $M = \frac{E^*}{c^2}$

That is to say, the 3-momentum of any system is the same as a material point:

- with a mass $M = \frac{E^*}{c^2}$;
- and a speed $\boldsymbol{v} = \boldsymbol{V}_{\mathbf{K*/K}}$.

2 remarks:

- $P^i(K^*)$ is here relative to the particular time $t^* = 0$ and is not a priori constant;
- $P^i(K^*)$ is not the only one 4-momentum since we can define a different one for each frame of reference, $P^i(K), P^i(K'), P^i(K^*)$ ..., all are associated to different hyperplane of simultaneity linked to each possible (an infinity) frame of reference K,K',K*...(see [Janssen & Mecklenburg]).

It exists a particular case where there is only one 4-momentum $P^i$: $P^i(K) = P^i(K') = P^i(K^*)$...In [Landau Lifchitz] we know that (in a general field theory):

- if the system is locally conserved : the stress-energy tensor has a null divergence $\partial_k T^{ik} = 0$;
- and if there is "nothing (other than gravitation field)" in infinite (in the sense of convergence to infinity).
  - ⇨ $P^i(K) \equiv \frac{1}{c}\int \iiint_{space-time,K} T^{ik} dS_k$ is conserved and doesn't depend on the hyperplane of integration (thanks to the conservation law).

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In a less general theorem (but more old) from Von Laue (cf. [Wang]) we can also say that if $\partial_0 T^{ik} = 0$ (and nothing to infiny):

$P^i = \frac{1}{c}\iiint_{Space} T^{i0} dV$ is a 4-momentum ⇔ $\frac{1}{c}\iiint_{Space} T^{\alpha\beta} dV$=0

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- **Why (I am) searching another proof ?**

The proof above does not use the Lagrangian directly but indirectly via the stress energy tensor. However, the base of all dynamics in physics laws is (until now) always to start from the Lagrangien of a system with the appropriate variables (including degrees of freedom). We should be able to select the center of mass and the complementary degrees of freedom (which we called logically the internal degrees of freedom since they are seen in the “hidden” K*). Unfortunately (for myself at least...), I never found any proof using this point of view. With the current approach (even if it is sufficient for physics) it is not clear, for me, how the centre of mass appears in the Lagrangian, in parallel with the internal degrees of freedom. Indeed the Lagrangien is reconstructed only a posterior, after to demonstrate that $\boldsymbol{P_c} = \gamma \frac{E^*}{c^2} \boldsymbol{V_C}$ (using the stress-energy tensor) (see [Janssen & Mecklenburg]). So we don’t clearly see the passage:

- From an initial Lagrangien $S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$ or $S[\{\varphi(x,t)\}] = \frac{1}{c}\int \iiint \Lambda\left(\varphi, \frac{\partial\varphi}{\partial \boldsymbol{r}}, \frac{\partial\varphi}{\partial t}\right) d\Omega$
- To a Lagrangien of an apparent material point $S[\boldsymbol{R_c}(t), \dots] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma(\boldsymbol{V_c})} dt + \cdots$

In this article, I propose, using directly the Lagrangien formalism, to give the proof, for a material system (to present the method), for a field (scalar in order to simplified) and finally a system where the two interact.

2. **Material system free**

- **The proof**

We begin with the action principle for a set of particles:

$$S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\}, \left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$$

In this expression, we are using coordinates in a Galilean reference frame K.
The degrees of freedom are the vectors $\{\boldsymbol{r_a}\}$, and we integrate the expression between the plan $H_1$ $(t_1 = cte)$, and $H_2$ ( $t_2 = cte'$) in this frame.
We want now separate:

- the internal degree of freedom $\{\boldsymbol{r_a^*}\}$, defined in the frame K*of the center of mass ;
- from the external degree of freedom $\boldsymbol{R_c}$ defined in the Galilean frame K.

So the degrees of freedom $\{\boldsymbol{r_a}\}$, are equivalent to the degree o freedom $\{\boldsymbol{r_a^*}, \boldsymbol{R_c}\}$.

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**Note 1:**

Thanks to the relativist invariance we know that each terms of the action associated to a particle is invariant ($L.dt = \sum_a -m_a.cds_a$). However in the frame K*, the border plan $H_1$ and $H_2$ are associated to different time for each particle (in Einstein relativity the simultaneity is relative to a frame).

More explicitly, the Lorentz transformation said that a coordinate $t'$ seen in the frame K is expressed like

$t' = \gamma(t)\left(t^* + \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{r_a^*}\right)$, with $\gamma(t) = \gamma(\boldsymbol{V_C}(t))$ *and* $\boldsymbol{V_C} \equiv \boldsymbol{V_{K*/K}}(t)$ , in the frame K*$(t)$ at the instant t (t'≠t, a priori, since t' is a generic coordinate of K but t define the time of K for which the center of mass has the speed $\boldsymbol{V_C}(t)$ .

So a plane $t' = cte$ in K is view like a plane $\gamma(t)\left(t^* + \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{r_a^*}\right) = cte$ in the frame K*$(t)$ around t.

Thus a particle at the position $\boldsymbol{r_a^*}$, see the plan $t' = cte$ at the instant $t_a^* = \frac{t'}{\gamma(t)} - \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{r_a^*}$

Furthermore, we remark that, since in K*$(t)$ *we observe simultaneous events at* $(t^*, \boldsymbol{C})$ we have $t^* \equiv t_c^* = t_a^*\ \forall a$. But $t_c^* = \left(\frac{t'}{\gamma(t)} - \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{R_c^*}\right)_{t'=t} = \left(\frac{t'}{\gamma(t)}\right)_{t'=t}$ around t.

So ∀ particle $a$: $t_a^* = t^* = \left(\frac{t'}{\gamma(t)}\right)_{t'=t}$ and $dt_a^* = dt^* = \left(\frac{dt'}{\gamma(t)}\right)_{t'=t} = \frac{dt}{\gamma(t)}$.

**Note 2 :**

In this note we use a reference frame K* for which the speed $\boldsymbol{V_{K*/K}}$ is constant from $t_0$=0 to t and from $t_0$*=0 to t*. That is to say the K* considered à the time t has a priori the right speed only at the instant t. So the above reasoning is valid only if at each time we change the origin of time.

$$\begin{pmatrix} t' - t'_0 \\ \boldsymbol{r_a} + \boldsymbol{R_c} - \boldsymbol{R_{c_0}} \end{pmatrix} = L.\begin{pmatrix} t_a^* - t_0^* \\ \boldsymbol{r_a^*} + \boldsymbol{0^*} \end{pmatrix} \Leftrightarrow \begin{cases} t' - t'_0 = \gamma(t)\left((t_a^* - t_0^*) + \frac{\boldsymbol{\beta}}{c}\boldsymbol{r_a^*}\right) \\ \boldsymbol{r_a}(t') + \boldsymbol{R_c}(t') - \boldsymbol{R_{c_0}}(t'_0) = c(t^* - t_0^*)\gamma(t)\boldsymbol{\beta} + \boldsymbol{r_a^*} + (\gamma - 1)\frac{\boldsymbol{\beta}}{\beta^2}.(\boldsymbol{\beta}\boldsymbol{r_a^*}) \end{cases}$$

Or more simply, for a movement of K* along x : $\begin{cases} t' - t'_0 = \gamma(t)\left((t_a^* - t_0^*) + \frac{\beta}{c}.x_a^*\right) \\ x_a + X_c - X_{c_0} = \gamma(t)(c(t_a^* - t_0^*)\beta + x_a^*) \end{cases}$

Then we see $t' - t'_0 = \gamma(t)\left((t_a^* - t_0^*) + \frac{\boldsymbol{\beta}}{c}\boldsymbol{r_a^*}\right) \Leftrightarrow t_a^* = \frac{t' - t'_0}{\gamma(t)} - \frac{\boldsymbol{\beta}}{c}\boldsymbol{r_a^*} + t_0^*$

So by changing locally the origin of time, we have $t_a^* = \frac{t'}{\gamma(t)} - \frac{\beta}{c}\boldsymbol{r}_a^*$

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Now we express the action in the local frames K*(t):

$$S[\{\boldsymbol{r}_a^*(t^*), \boldsymbol{R}_c(t)\}] = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dt^*$$

Taking account $dt^* = \frac{dt}{\gamma(t)}$ and returnig to the Galilean frame K we have:

$$S = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{dt}{dt^*}\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)\frac{dt^*}{dt}dt = \int_{t_1}^{t_2} \frac{L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})} dt$$

So far, nothing new.

The important point to keep in mind is that we are not considering the variation of the internal degree of freedom $\boldsymbol{r}_a^*$ :

- relative to the internal time of K*, $t^*$ $\frac{d\boldsymbol{r}_a^*}{dt^*}$;
- **but instead relative to time of K, t**: $\frac{d\boldsymbol{r}_a^*}{dt}$.

That is to say, the Lagrangien considered is $L'\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) \equiv \frac{L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$, instead of using the most « natural » $L\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) \equiv \frac{L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$

So, we can now calculate the momentum of the center of mass, with $\boldsymbol{V}_\mathrm{C} \equiv \boldsymbol{V}_{\mathrm{K}*/\mathrm{K}}$:

$$\boldsymbol{P}_c \equiv \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} \frac{L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$$

$$= L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)$$

- $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\sqrt{1 - \frac{\boldsymbol{V}_\mathrm{C}^2}{c^2}} = \frac{-\frac{1}{2}2\frac{\boldsymbol{V}_\mathrm{C}}{c^2}}{\sqrt{1-\frac{\boldsymbol{V}_\mathrm{C}^2}{c^2}}} = -\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{c^2}$
- $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right) = \sum_a \frac{\partial \gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}}{\partial \boldsymbol{V}_\mathrm{C}} \frac{\partial L^*}{\partial \gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_a^*}{dt}} = \sum_a \frac{d\boldsymbol{r}_a^*}{dt}\frac{\partial\left(1-\frac{\boldsymbol{V}_\mathrm{C}^2}{c^2}\right)^{-1/2}}{\partial \boldsymbol{V}_\mathrm{C}}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$

$$= \sum_a \frac{d\boldsymbol{r}_a^*}{dt}\left(\frac{\frac{1}{2}2\frac{\boldsymbol{V}_\mathrm{C}}{c^2}}{\left(1-\frac{\boldsymbol{V}_\mathrm{C}^2}{c^2}\right)^{3/2}}\right)\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} = \sum_a \frac{d\boldsymbol{r}_a^*}{dt}\gamma^3(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{c^2}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$$

$$\boldsymbol{P_c} = L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_{\rm C})\frac{d\boldsymbol{r_a^*}}{dt}\right\}\right)\left(-\gamma(\boldsymbol{V}_{\rm C})\frac{\boldsymbol{V}_{\rm C}}{\boldsymbol{c}^2}\right) + \frac{1}{\gamma(\boldsymbol{V}_{\rm C})}\sum_a \frac{d\boldsymbol{r_a^*}}{dt}\gamma^3(\boldsymbol{V}_{\rm C})\frac{\boldsymbol{V}_{\rm C}}{\boldsymbol{c}^2}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}}$$

$$= \gamma(\boldsymbol{V}_{\rm C})\frac{\boldsymbol{V}_{\rm C}}{\boldsymbol{c}^2}\left(-L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_{\rm C})\frac{d\boldsymbol{r_a^*}}{dt}\right\}\right) + \sum_a \gamma\frac{d\boldsymbol{r_a^*}}{dt}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}}\right)$$

$$= \gamma(\boldsymbol{V}_{\rm C})\frac{\boldsymbol{V}_{\rm C}}{c^2}\left(\sum_a \gamma\frac{d\boldsymbol{r_a^*}}{dt}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}} - L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_{\rm C})\frac{d\boldsymbol{r_a^*}}{dt}\right\}\right)\right)$$

$$= \gamma(\boldsymbol{V}_{\rm C})\frac{\boldsymbol{V}_{\rm C}}{c^2}\left(\sum_a \frac{d\boldsymbol{r_a^*}}{dt^*}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r_a^*}}{dt^*}} - L^*\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}\right)\right) \text{since } \frac{d\boldsymbol{r_a^*}}{dt^*} = \gamma(\boldsymbol{V}_{\rm C})\frac{d\boldsymbol{r_a^*}}{dt}$$

$$\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{C}}$$

where $E^* \equiv \sum_a \frac{d\boldsymbol{r_a^*}}{dt^*}\frac{\partial L^*}{\partial\frac{d\boldsymbol{r_a^*}}{dt^*}} - L^*\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}\right)$ is the internal energy.

So we have our relation.

$E^*$ is relative to the hyperplane $t^*$=cte, the mass $M = \frac{E^*}{c^2}$ is dealing with events ( the spatio-temporal positions of the particles) simultaneous in the frame K* and not in the frame K. This is well defined since $t^* = \left(\frac{t'-t'_0(t)}{\gamma(t)}\right)_{t'=t} = \frac{t-t'_0(t)}{\gamma(t)}$.

$$M = M(t^*) = M\left(\frac{t-t'_0(t)}{\gamma(t)}\right) = M\left(\int_0^t \frac{dt'}{\gamma(t')}\right)$$

We see that we don't need to talk about closed system hypothesis or to have a 4 vector momentum to demonstrate it (we don't even use the expression $L.dt = \sum_a -m_a.cds_a$).

Finally, the Euler-Lagrange equations tell us that $\frac{d}{dt}\left(\gamma\frac{E^*}{\boldsymbol{c}^2}\boldsymbol{V}_{\mathbf{C}}\right) = \frac{\partial}{\partial \boldsymbol{R}_{\rm C}}L'\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_{\rm C},\boldsymbol{V}_{\rm C}\right)$

We have to note, in the proof, <u>the importance to freeze the right variable</u> $\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\}$ (and not $\left\{\frac{d\boldsymbol{r_a^*}}{dt^*}\right\}$ ) in order to have the good expression.

- **But what $\frac{dr_a^*}{dt}$ means ?**

Indeed the speed $\frac{d\boldsymbol{r_a^*}}{dt}$ combines 2 quantities that each rely to 2 different reference frames: K* for $d\boldsymbol{r_a^*}$ and K for $dt$. We can think that this ill-defined which would break the proof.

We can write $\frac{dr_a^*}{dt} = \frac{dr_a^*}{dt^*}\frac{dt^*}{dt}$,and according to according to [Yvan Simon] via Lorentz Transformation $\frac{dt^*}{dt} = \frac{1}{dt_1+dt_2} = \frac{1}{\gamma\left(dt^*+\frac{\beta}{c}dx_a^*\right)}$ . So we have $\frac{dr_a^*}{dt} = \frac{dr_a^*}{dt^*}\frac{1}{dt_1+dt_2}$.

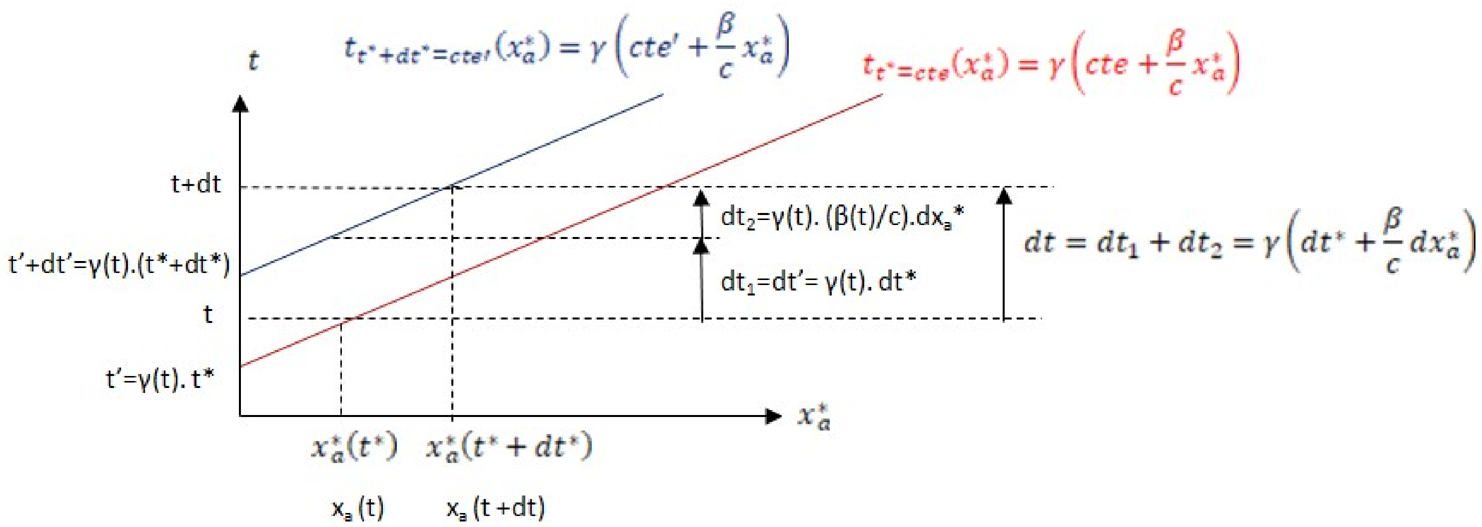
However *we don't use this formula* above in this article but $\frac{dr_a^*}{dt_1} = \frac{dr_a^*}{\gamma dt^*}$ instead.

The difference between the first $\frac{d\boldsymbol{r}_a^*}{dt}$ and the second $\frac{d\boldsymbol{r}_a^*}{dt_1}$ is (see graph below) the fact that :

- the former $\frac{d\boldsymbol{r}_a^*}{dt}$ use the differential time $dt$ between two events $\left(t^*, \boldsymbol{r}_a^*(t^*)\right)$ and $\left(t^* + dt^*, \boldsymbol{r}_a^*(t^* + dt^*)\right)$ seen in the frame K;
- the latter $\frac{d\boldsymbol{r}_a^*}{dt_1}$ use the differential time $dt_1 = \gamma dt^*$ which is the duration between two hyperplane $t^* = cte$ and $t^* + dt^* = cte'$ of K* measured in the reference frame K.

In short, $dt$ is about the 2 positions of a material point seen in K and $dt_1$ is the temporal distance between the 2 hyperplane of K* where the 2 events are contained.

During the proof we only use the different time element$dt_1 = \gamma dt^*$: in the integration time element, in the expression of the speed $\frac{d\boldsymbol{r}_a^*}{dt_1}$ and in using the time $t^*$ of the reference frame which is actually




$t^* = \int_0^t \frac{dt}{\gamma(t\prime)}$. This is coherent with viewing the entire internal dynamic (like internal energy) relative to the hyperplanes of K*. Moreover and interestingly, $dt_1 = \gamma dt^*$ is also the duration of the proper time associated to the apparent material point represented by the mass center C of the system.

- **Some remarks**

We can also notice that $\boldsymbol{P_a} \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} = \gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*}$ , so $\boldsymbol{P_a} = \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}}$which is surprising but reassuring for the intelligibility of this quantity: this is the same as the one we would have in the frame of the centre of mass K*.

More over the total momentum $\boldsymbol{P_{total}}$ associated to the Lagrangien $L'\left(\{\boldsymbol{r}_a^*\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C\right)$ is

$\boldsymbol{P_{total}} = \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r_a^*}}{dt}} + \frac{\partial L'}{\partial \boldsymbol{v}_c} = \sum_a \boldsymbol{P_a} + \boldsymbol{P_c} = \boldsymbol{P_c}$ since by definition of K*: $\sum_a \boldsymbol{P_a} \equiv 0$. This is interesting since despite considering the internal variables on the same level as the mass center, we obtain as it should the total momentum is the one associated to the mass center.

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**Proof:**

Indeed $L.dt = -\sum_a m_a.c ds_a$ => $L = -\sum_a m_a.c\frac{ds_a}{dt} = -\sum_a m_a.c\frac{ds_a}{dt^*}\frac{dt^*}{dt} = -\sum_a m_a.c^2\frac{1}{\gamma_a^*}\frac{1}{\gamma}$

But $\frac{1}{\gamma.\gamma_a^*} = \frac{1}{\gamma}\sqrt{1 - \frac{\left(\frac{d\boldsymbol{r_a^*}}{dt^*}\right)^2}{\boldsymbol{c}^2}} = \sqrt{\frac{1}{\gamma^2} - \frac{\frac{1}{\gamma^2}\left(\frac{d\boldsymbol{r_a^*}}{dt^*}\right)^2}{\boldsymbol{c}^2}} = \sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r_a^*}}{dt}\right)^2}{\boldsymbol{c}^2}}$ since $\frac{d\boldsymbol{r_a^*}}{dt^*} = \gamma(\boldsymbol{V}_\mathbf{C})\frac{d\boldsymbol{r_a^*}}{dt}$

Moreover $\frac{\partial}{\partial\frac{d\boldsymbol{r_a^*}}{dt}}\left(\frac{1}{\gamma.\gamma_a^*}\right) = \frac{\partial}{\partial\frac{d\boldsymbol{r_a^*}}{dt}}\sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r_a^*}}{dt}\right)^2}{\boldsymbol{c}^2}} = -\frac{1}{2}\frac{\mathbf{2}\frac{d\boldsymbol{r_a^*}}{dt}}{\boldsymbol{c}^2}\frac{\mathbf{1}}{\sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r_a^*}}{dt}\right)^2}{\boldsymbol{c}^2}}} = -\frac{\frac{d\boldsymbol{r_a^*}}{dt}}{\boldsymbol{c}^2}\gamma.\gamma_a^*$

So $\boldsymbol{P_a} = -\frac{\partial}{\partial\frac{d\boldsymbol{r_a^*}}{dt}}\sum_{a'} m_{a'}.c^2\frac{1}{\gamma_{a'}^*}\frac{1}{\gamma} = m_a.c^2\frac{\frac{d\boldsymbol{r_a^*}}{dt}}{\boldsymbol{c}^2}\gamma.\gamma_a^* = m_a.\frac{d\boldsymbol{r_a^*}}{dt^*}\gamma_a^*$

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- **The reduced action**

We can write:

$$S[\{\boldsymbol{r_a^*}(t^*)\}, \boldsymbol{R_c}(t)] = \int_{t_1}^{t_2} L'\left(\{\boldsymbol{r_a^*}\}, \left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\}, \boldsymbol{R}_\mathbf{C}, \boldsymbol{V}_\mathbf{C}\right) dt$$

$$= \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P_a}.\frac{d\boldsymbol{r_a^*}}{dt} + \boldsymbol{P_c}.\boldsymbol{V_c} - E\right] dt$$

$$= \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P_a}.\frac{d\boldsymbol{r_a^*}}{dt} + \left(\gamma\frac{E^*}{c^2}\boldsymbol{V}_\mathbf{C}\right).\boldsymbol{V_c} - \gamma E^*\right] dt$$

$$= \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P_a}.\frac{d\boldsymbol{r_a^*}}{dt} + \gamma E^*(\beta^2 - 1)\right] dt = \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P_a}.\frac{d\boldsymbol{r_a^*}}{dt} - \frac{E^*}{\gamma}\right] dt$$

So

$$\boxed{S[\{\boldsymbol{r_a^*}(t^*)\}, \boldsymbol{R_c}(t)] = \int_{t_1}^{t_2} \left[\sum_a \boldsymbol{P_a}.\frac{d\boldsymbol{r_a^*}}{dt} - \frac{E^*}{\gamma}\right] dt}$$

If we ignore the final position of the internal degree of freedom, we have like a “spatial Maupertuis principle” (instead of a temporal used in [Landau Lifchitz]):

$$\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}(t)] + \left(\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\delta \boldsymbol{r}_{\boldsymbol{a}}^*\right)_{H_2} = 0$$

We can see that if all the internal momentum are constant, it exists a reduced action principle:

$$\boxed{S_0[\boldsymbol{R}_{\boldsymbol{c}}(t)] = -\int_{t_1}^{t_2} \frac{E^*}{\gamma} dt}$$

We can surely generalize it for closed systems with internal separable variables where we've chosen well the variables with constant momentum. In this case, we see that for "stationary" system, in this restrict sense, the center of mass dynamic is the same as a material point.

Note: my idea to consider the quantity $\left\{\frac{dr_a^*}{dt}\right\}$ comes initially from the willingness to make appear the Lagrangien of the apparent material point with this reduced action (in the same manner we make appear the virtual work theorem: $\delta \int_{t_1}^{t_2} [\sum_a \boldsymbol{P}_{\boldsymbol{a}}.d\boldsymbol{r} - H[\boldsymbol{P}_{\boldsymbol{a}}, \boldsymbol{r}_{\boldsymbol{a}}]dt] + (\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\delta \boldsymbol{r}_{\boldsymbol{a}})_{H_2} = 0$ and $\boldsymbol{P}_{\boldsymbol{a}} = \boldsymbol{cte} \Rightarrow \delta \int_{t_1}^{t_2} H_{\boldsymbol{P}_{\boldsymbol{a}}=\boldsymbol{cte},}(\{\boldsymbol{r}_{\boldsymbol{a}}\})\, dt = 0$ ).

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**Proof:**

Indeed (do the same that [Landau lifchitz] but for space and not for time)):

$$\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}(t)] + \left(\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\delta \boldsymbol{r}_{\boldsymbol{a}}^*\right)_{H_2} = 0$$

$$\Leftrightarrow \delta \int_{t_1}^{t_2} d \sum_a [\boldsymbol{P}_{\boldsymbol{a}}.\boldsymbol{r}_{\boldsymbol{a}}^*] + \delta \int_{t_1}^{t_2} \left[-\frac{E^*}{\gamma}\right] dt + \left(\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\delta \boldsymbol{r}_{\boldsymbol{a}}^*\right)_{H_2} = 0$$

$$\Leftrightarrow \delta \left[\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\boldsymbol{r}_{\boldsymbol{a}}^*\right]_{H_2} + \delta \int_{t_1}^{t_2} \left[-\frac{E^*}{\gamma}\right] dt + \left(\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\delta \boldsymbol{r}_{\boldsymbol{a}}^*\right)_{H_2} = 0$$

$$\Leftrightarrow \delta \int_{t_1}^{t_2} \left[-\frac{E^*}{\gamma}\right] dt = 0$$

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### 3. Free field

Now, I will repeat the same method for a field theory (a scalar field $\varphi$ for simplify), and again:

The important point to keep in mind is that we are not considering the variation of the internal degree of freedom $\varphi^*$ **:**

- relative to the internal time of K*, $t^*$ : $\frac{\partial \varphi^*}{\partial t^*}$;
- **but instead relative to time of K, t**: $\frac{\partial \varphi^*}{\partial t}$.

So without comments, we have successively:

$$S[\{\varphi(x,t)\}] = \frac{1}{c}\int \iiint \Lambda\left(\varphi, \frac{\partial \varphi}{\partial \boldsymbol{r}}, \frac{\partial \varphi}{\partial t}\right) d\Omega$$

$$= \frac{1}{c}\int \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) d\Omega^* = \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right] dt^*$$

$$= \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right] \frac{dt}{\gamma}$$

=>

$$S[\{\varphi^*(x^*,t^*)\}, \boldsymbol{R}_\mathrm{C}(\boldsymbol{t})] = \int L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right] dt$$

With $L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right] = \frac{1}{\gamma}\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$

So we can calculate the 3-momentum as:

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\left[\frac{1}{\gamma}\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right]$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} \frac{1}{\gamma} + \frac{1}{\gamma}\iiint \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \left(-\gamma(\boldsymbol{V}_\mathrm{C}) \frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)$$

$$+ \frac{1}{\gamma}\iiint \frac{\partial\left(\gamma \frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} \frac{\partial}{\partial\left(\gamma \frac{\partial \varphi^*}{\partial t}\right)} \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

But $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}} \frac{1}{\gamma} = -\gamma(\boldsymbol{V}_\mathrm{C}) \frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}$ ; $\gamma \frac{\partial \varphi^*}{\partial t} = \frac{\partial \varphi^*}{\partial t^*}$

And $\frac{\partial\left(\gamma \frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t} \frac{\partial \gamma}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t} \frac{\partial \frac{1}{\sqrt{1-\frac{\boldsymbol{V}_\mathrm{C}^2}{c^2}}}}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t} \frac{-1}{2}\left(-2\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right) \frac{1}{\left(1-\frac{\gamma^2\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)^2}{\boldsymbol{c}^2}\right)^{3/2}} = \frac{\partial \varphi^*}{\partial t} \frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2} \gamma^3$

$$\boldsymbol{P_c} = \iiint \Lambda^*\left(\varphi^*, \frac{\partial\varphi^*}{\partial\boldsymbol{r}^*}, \gamma\frac{\partial\varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* \left(-\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\right)$$
$$+\frac{1}{\gamma}\iiint \left(\frac{\partial\varphi^*}{\partial t}\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma^3\right)\frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial\varphi^*}{\partial\boldsymbol{r}^*}, \frac{\partial\varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma \iiint \Lambda^*\left(\varphi^*, \frac{\partial\varphi^*}{\partial\boldsymbol{r}^*}, \gamma\frac{\partial\varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* \,(-1)$$
$$+\iiint \left(\frac{\partial\varphi^*}{\partial t}\gamma\right)\frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial\varphi^*}{\partial\boldsymbol{r}^*}, \frac{\partial\varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma\left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial\varphi^*}{\partial\boldsymbol{r}^*}, \gamma\frac{\partial\varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^* \,(-1) + \iiint \frac{\partial\varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial\varphi^*}{\partial\boldsymbol{r}^*}, \frac{\partial\varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*\right]$$

$$= \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\gamma \iiint \left[\frac{\partial\varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right] dV^*$$

So we have again:

$$\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathbf{C}}$$

where $E^* \equiv \iiint \left(\frac{\partial\varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial\varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right) dV^*$ is the internal energy (associated to the hyperplane $t^* = cte$)

We see that we don't need to talk about closed system hypothesis or to have a 4 vector to demonstrate it (we don't even use the expression of any density Lagrangien).

The Euler-Lagrange equations tell us that $\frac{d}{dt}\left(\gamma\frac{E^*}{\boldsymbol{c}^2}\boldsymbol{V}_{\mathbf{C}}\right) = \frac{\partial}{\partial\boldsymbol{R}_{\mathrm{C}}}L'\left[\{\varphi^*\}, \left\{\frac{\partial\varphi^*}{\partial r^*}\right\}, \left\{\frac{\partial\varphi^*}{\partial t}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right]$

We have to note, in the proof, the importance to freeze the right variable $\frac{\partial\varphi^*}{\partial t}$ (and not $\frac{\partial\varphi^*}{\partial t^*}$ ) in order to have the good expression.

## 4. Interaction between a field and a particle

We consider the simplified action:

$$S[\boldsymbol{r_a}(t),\{\varphi(x,t)\}] = \int_{t_1}^{t_2}\left(\sum_a\left[-m_a.c\frac{ds_a}{dt} - \frac{e_a}{c}.\frac{ds_a}{dt}\varphi(\boldsymbol{r_a},t)\right]\right)dt + \frac{1}{c}\int\iiint \Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega$$

So we have also:

$$S = \int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a + \frac{e_a}{c^2}\varphi(\boldsymbol{r_a},t)\right).c.\frac{ds_a}{dt}\right]\right)dt + \frac{1}{c}\int\iiint \Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega$$

$$= \int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a + \frac{e_a}{c^2}\varphi^{K_m}\right).c^2\frac{1}{\gamma_a^{K_m}}\right]\right)\frac{dt}{\gamma_m} + \int\left[\iiint \Lambda^{K_\varphi}\left(\varphi^{K_\varphi},\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}},\gamma\frac{\partial\varphi^{K_\varphi}}{\partial t},\boldsymbol{R}_{\mathrm{C}_\varphi},\boldsymbol{V}_{\mathrm{C}_\varphi}\right)dV^{K_\varphi}\right]\frac{dt}{\gamma_\varphi}$$

Where we have specified the quantities relative to:

- the frame $K_\varphi$ of the center of mass $\mathrm{C}_\varphi$ of the field $\varphi$ ;
- the frame $K_m$ of the center of mass $\mathrm{C}_m$ of the material points.

$$S\left[\left\{\boldsymbol{r_a}^{K_m}(t^{K_m}),\boldsymbol{R}_{\mathrm{C}_m}(t)\right\},\{\varphi^{K_\varphi}(x^{K_\varphi},t^{K_\varphi})\},\boldsymbol{R}_{\mathrm{C}_\varphi}(\boldsymbol{t})\right] = \int_{t_1}^{t_2} L'\left(\{\boldsymbol{r_a}^{K_m}\},\left\{\frac{d\boldsymbol{r_a}^{K_m}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}_m},\boldsymbol{V}_{\mathrm{C}_m},t\right)dt + \int L'\left[\{\varphi^{K_\varphi}\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}}\right\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial t}\right\},\boldsymbol{R}_{\mathrm{C}_\varphi},\boldsymbol{V}_{\mathrm{C}_\varphi}\right]dt$$

So in this form, we can calculate the dynamic of the center of mass of one system and the other.
We can see that each system is not free at all, but we have again:

$$\boldsymbol{P}_{\mathrm{C}_m} = \gamma(\boldsymbol{V}_{\mathrm{C}_m})\frac{E^{K_m}}{c^2}\boldsymbol{V}_{\mathrm{C}_m}$$
$$\boldsymbol{P}_{\mathrm{C}_\varphi} = \gamma\left(\boldsymbol{V}_{\mathrm{C}_\varphi}\right)\frac{E^{K_\varphi}}{c^2}\boldsymbol{V}_{\mathrm{C}_\varphi}$$

So $M_m = \frac{E^{K_m}}{c^2}$, $M_\varphi = \frac{E^{K_\varphi}}{c^2}$

With the same method we can consider any set of systems.

## 5. Conclusion

We have a way to demonstrate the famous Einstein formula $E^*=Mc^2$ directly from an appropriate Lagrangien function selecting the correct variable.

Instead of $L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right)$, we use $L'\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)=\frac{L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$.

Instead of $L'\left[\{\varphi\},\left\{\frac{\partial\varphi}{\partial r}\right\},\left\{\frac{\partial\varphi}{\partial t}\right\}\right]$, we use $L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial r^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right]=\frac{\iiint \Lambda^*\left(\varphi^*,\frac{\partial\varphi^*}{\partial r^*},\gamma\frac{\partial\varphi^*}{\partial t},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)dV^*}{\gamma}$.

In the two cases we've calculated directly that $\boldsymbol{P_c}\equiv\frac{\partial L'}{\partial v_\mathrm{C}}=\gamma\frac{E^*}{c^2}\boldsymbol{V}_\mathrm{C}$

**Some remark:**

1) A simple Lorentz transformation, shows that the 3-momentum is actually the one associated to $P^i(K^*)=\frac{1}{c}\int\iiint_{space-time}T^{ik}\delta\left(n_{lm}x^l x^m\right).d\eta_k(K^*)\,d^4x$, so it is a part of a 4-vector.Thus, among all the 4-momentum $P^i(K)$, $P^i(K')$, $P^i(K^*)$... the Lagrangien method selects $P^i(K^*)$.
2) Since we have defined the mass center in $K^*$, it allows us to associated to it a true event (the center of the frame) which doesn't change from a frame K to another K', by the relativity of simultaneity. In consequence, we can show (not here) that the internal energy, so the mass, is an invariant in our case (like for a material point).

It will be interesting to derive the stress-energy tensor with our method, in order to show how the internal degrees of freedom generate the pressure like the formula (35,2) of [Landau-Lifchitz].

**References**


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