A straightforward and Lagrangien proof of the mass as the internal energy of a system

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I propose a simple proof of the famous Einstein law: the mass system as its internal energy. The interest of this proof is to minimize the hypotheses. The essential point of the proof is to define more carefully the internal degree of freedom in order to distinguish it correctly from the center of mass.

1. Introduction

The Einstein Law is well established since its first publication in 1905, even if it was calculated in particular cases (which are almost always sufficient). The simpler way (that the author know), is to demonstrate that the momentum tensor is a 4 vector, but this is possible only with some restrictions.

In [Landau Lifchitz] we know that (in a general field theory):

- if the system is locally conserved : the momentum tensor has a null divergence $\partial_k T^{ik} = 0$;
- and if there is "nothing (other than gravitation field)" in infinite (in the sense of convergence to infinity).

⇨ It exists a conserved quantity $P^i(K) \equiv \frac{1}{c}\int\iiint_{space-time,K} T^{ik} dS_k = \frac{1}{c}\iiint_{space,K} T^{00} dV$, which is a 4-vector, so the proof is complete. Moreover, this vector doesn't depend on the hyperplane of integration (thanks to the conservation law).

[Janssen & Mecklenburg] explains to be careful because, in general, the 4 vector $P^i(K)$ which can be re-written like $P^i(K) = \frac{1}{c}\int\iiint_{space-time} T^{ik}\delta\left(n_{lm}x^l x^m\right). d\eta_k(K)\, d^4x$ depends on each reference frame K: $P^i(K) \neq P^i(K') \neq P^i(K^*)$. It is in the 2 particular conditions above-mentioned that we have the equality $P^i(K) = P^i(K') = P^i(K^*)$... via the fact that the choice of hyperplane doesn't matter.

Now, I propose another (much less sophisticated) proof which doesn't need:

- neither the locally conserved law;
- nor the demand of "nothing to infinity".

I give the proof for a material system (to present the method), a field (scalar to simplified) and finally a system where the two interact.

2. Material system free

We begin with the action principle for a set of particles:

$$S[\{\boldsymbol{r_a}(t)\}] = \int_{t_1}^{t_2} L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right) dt$$

In this expression, we are using coordinates in a Galilean reference frame K.
The degrees of freedom are the vectors $\{\boldsymbol{r_a}\}$, and we integrate the expression between the plan H_1 ($t_1 = cte$), and H_2 ($t_1 = cte'$) in this frame.
We want now separate:

- the internal degree of freedom $\{\boldsymbol{r}^*_{\boldsymbol{a}}\}$, defined in the frame K*of the center of mass ;
- from the external degree of freedom $\boldsymbol{R_c}$ defined in the Galilean frame K.

So the degrees of freedom $\{\boldsymbol{r_a}\}$, are equivalent to the degree o freedom $\{\boldsymbol{r}^*_{\boldsymbol{a}}, \boldsymbol{R_c}\}$.

Note:

Thanks to the relativist invariance we know that each terms of the action associated to a particle is invariant ($L.dt = \sum_a -m_a.cds_a$). However in the frame K*, the border plan H_1 and H_2 are associated to different time for each particle (in Einstein relativity the simultaneity is relative to a frame).

More explicitly, the Lorentz transformation said that a coordinate t' seen in the frame K is expressed like $t' = \gamma(t)\left(t^* + \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{r}^*_{\boldsymbol{a}}\right)$, with $\gamma(t) = \gamma(\boldsymbol{V_C}(t))$ *and* $\boldsymbol{V_C} \equiv \boldsymbol{V_{K*/K}}(t)$, in the frame K*(t) at the instant t (t'≠t, a priori, since t' is a generic coordinate of K but t define the time of K for which the center of mass has the speed $\boldsymbol{V_C}(t)$.

So a plane $t' = cte$ in K is view like a plane $\gamma(t)\left(t^* + \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{r}^*_{\boldsymbol{a}}\right) = cte$ in the frame K*(t) around t.

Thus a particle at the position $\boldsymbol{r}^*_{\boldsymbol{a}}$, see the plan $t' = cte$ at the instant $t^*_a = \frac{t'}{\gamma(t)} - \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{r}^*_{\boldsymbol{a}}$

Furthermore, we remark that, since in K*(t) *we observe simultaneous events at* $(t^*, \boldsymbol{C})$ we have $t^* \equiv t^*_c = t^*_a\ \forall a$. But $t^*_c = \left(\frac{t'}{\gamma(t)} - \frac{\boldsymbol{V_C}}{c^2}\boldsymbol{R}^*_{\boldsymbol{c}}\right)_{t'=t} = \left(\frac{t'}{\gamma(t)}\right)_{t'=t}$ around t.

So ∀ particle a: $t^*_a = t^* = \left(\frac{t'}{\gamma(t)}\right)_{t'=t}$ and $dt^*_a = dt^* = \left(\frac{dt}{\gamma(t)}\right)_{t'=t} = \frac{dt}{\gamma(t)}$.

Cautions:

In this note we use a reference frame K* for which the speed $\boldsymbol{V_{K*/K}}$ is constant from t_0=0 to t and from t_0*=0 to t*. That is to say the K* considered à the time t has a priori the right speed only at the instant t. So the above reasoning is valid only if at each time we change the origin of time.

$$\begin{pmatrix} t' - t'_0 \\ \boldsymbol{r_a} + \boldsymbol{R_c} - \boldsymbol{R_{c_0}} \end{pmatrix} = L.\begin{pmatrix} t^*_a - t^*_0 \\ \boldsymbol{r}^*_a + \boldsymbol{0}^* \end{pmatrix} \Leftrightarrow \begin{cases} t' - t'_0 = \gamma(t)\left((t^*_a - t^*_0) + \frac{\beta}{c}\boldsymbol{r}^*_{\boldsymbol{a}}\right) \\ \boldsymbol{r_a}(t') + \boldsymbol{R_c}(t') - \boldsymbol{R_{c_0}}(t'_0) = c(t^* - t^*_0)\gamma(t)\boldsymbol{\beta} + \boldsymbol{r}^*_{\boldsymbol{a}} + (\gamma - 1)\frac{\boldsymbol{\beta}}{\beta^2}.(\boldsymbol{\beta}\boldsymbol{r}^*_{\boldsymbol{a}}) \end{cases}$$

Or more simply, for a movement of K* along x : $\begin{cases} t' - t'_0 = \gamma(t)\left((t^*_a - t^*_0) + \frac{\beta}{c}.x^*_a\right) \\ x_a + X_c - X_{c_0} = \gamma(t)(c(t^*_a - t^*_0)\beta + x^*_a) \end{cases}$

Then we see $t' - t'_0 = \gamma(t)\left((t^*_a - t^*_0) + \frac{\beta}{c}\boldsymbol{r}^*_{\boldsymbol{a}}\right) \Leftrightarrow t^*_a = \frac{t' - t'_0}{\gamma(t)} - \frac{\beta}{c}\boldsymbol{r}^*_{\boldsymbol{a}} + t^*_0$

So by changing locally the origin of time, we have $t^*_a = \frac{t'}{\gamma(t)} - \frac{\beta}{c}\boldsymbol{r}^*_{\boldsymbol{a}}$

Now we express the action in the local frames K*(t):

$$S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*), \boldsymbol{R}_{\boldsymbol{c}}(t)\}] = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dt^*$$

Taking account $dt^* = \frac{dt}{\gamma(t)}$ and returnig to the Galilean frame K we have:

$$S = \int_{\{t_{a,1}^*\}}^{\{t_{a,2}^*\}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{dt}{dt^*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) \frac{dt^*}{dt} dt = \int_{t_1}^{t_2} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})} dt$$

So far, nothing new.

The important point to keep in mind is that we are not considering the variation of the internal degree of freedom $\boldsymbol{r}_{\boldsymbol{a}}^*$ **:**

- relative to the internal time of K* t^* ;
- **but instead relative to time of K, t**: $\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}$.

That is to say, the Lagrangien considered is $L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) \equiv \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$, instead of using the most « natural » $L\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) \equiv \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$

So, we can now calculate the momentum of the center of mass, with $\boldsymbol{V}_{\mathrm{C}} \equiv \boldsymbol{V}_{\mathrm{K*/K}}$:

$$\boldsymbol{P}_{\boldsymbol{c}} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} = \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$

$$= L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)$$

- $\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} = \frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} \sqrt{1 - \frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{c^2}} = \frac{-\frac{1}{2}2\frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}}{\sqrt{1-\frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{c^2}}} = -\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}$
- $\frac{\partial}{\partial \boldsymbol{V}_{\mathrm{C}}} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) = \sum_a \frac{\partial \gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{\partial L^*}{\partial \gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} = \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \frac{\partial\left(1-\frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{c^2}\right)^{-1/2}}{\partial \boldsymbol{V}_{\mathrm{C}}} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$

$$= \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \left(\frac{\frac{1}{2} 2 \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}}{\left(1 - \frac{\boldsymbol{V}_{\mathrm{C}}{}^2}{\boldsymbol{c}^2}\right)^{3/2}} \right) \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} = \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \gamma^3(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$\boldsymbol{P}_{\boldsymbol{c}} = L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\left(-\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2}\right) + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \gamma^3(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}}$$

$$= \gamma(\boldsymbol{V}_{\mathrm{C}}) \frac{\boldsymbol{V}_{\mathrm{C}}}{\boldsymbol{c}^2} \left(-L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right) + \sum_a \gamma \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} \right)$$

$$= \gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}\left(\sum_a \gamma \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)\right)$$

$$= \gamma(\boldsymbol{V}_{\mathrm{C}})\frac{\boldsymbol{V}_{\mathrm{C}}}{c^2}\left(\sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right)\right) \text{ since } \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} = \gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}$$

$$\boldsymbol{P}_{\boldsymbol{c}} = \gamma \frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}$$

where $E^* \equiv \sum_a \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} - L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}\right)$ is the internal energy.

So we have our relation.

E^* is relative to the hyperplane t^*=cte, the mass $M = \frac{E^*}{c^2}$ is dealing with events (the spatio-temporal positions of the particles) simultaneous in the frame K* and not in the frame K. This is well defined since $t^* = \left(\frac{t' - t'_0(t)}{\gamma(t)}\right)_{t'=t} = \frac{t - t'_0(t)}{\gamma(t)}$. (Note that the center of mass is not a well defined space-time event according to [Landau-Lifchitz]).

$$M = M(t^*) = M\left(\frac{t - t'_0(t)}{\gamma(t)}\right) = M\left(\int_0^t \frac{dt'}{\gamma(t')}\right)$$

We see that we don't need to talk about closed system hypothesis or to have a 4 vector momentum to demonstrate it (we don't even use the expression $L.dt = \sum_a -m_a.cds_a$).

Finally, the Euler-Lagrange equations tell us that $\frac{d}{dt}\left(\gamma \frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right) = \frac{\partial}{\partial \boldsymbol{R}_{\mathrm{C}}} L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)$

We have to note, in the proof, <u>the importance to freeze the right variable $\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}$</u> (and not $\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}$) in order to have the good expression.

Below we give the proof for a field theory, but before we can also notice that

$\boldsymbol{P}_{\boldsymbol{a}} \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} = \gamma_a^* m_a.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}$ which is surprising but reassuring for the intelligibility of this quantity: this is the same as the one we would have in the frame of the centre of mass.

<u>Proof:</u>

Indeed $L.dt = -\sum_a m_a.cds_a$ => $L = -\sum_a m_a.c\frac{ds_a}{dt} = -\sum_a m_a.c\frac{ds_a}{dt^*}\frac{dt^*}{dt} = -\sum_a m_a.c^2\frac{1}{\gamma_a^*}\frac{1}{\gamma}$

But $\frac{1}{\gamma.\gamma_a^*} = \frac{1}{\gamma}\sqrt{1 - \frac{\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right)^2}{c^2}} = \sqrt{\frac{1}{\gamma^2} - \frac{\frac{1}{\gamma^2}\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right)^2}{c^2}} = \sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)^2}{c^2}}$ since $\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} = \gamma(\boldsymbol{V}_{\mathrm{C}})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}$

Moreover $\frac{\partial}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\left(\frac{1}{\gamma.\gamma_a^*}\right) = \frac{\partial}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\sqrt{\frac{1}{\gamma^2} - \frac{\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)^2}{\boldsymbol{c}^2}} = -\frac{1}{2}\frac{2\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\boldsymbol{c}^2}\frac{1}{\sqrt{\frac{1}{\gamma^2}-\frac{\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)^2}{\boldsymbol{c}^2}}} = -\frac{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\boldsymbol{c}^2}\gamma.\gamma_a^*$

So $\boldsymbol{P}_{\boldsymbol{a}} = -\frac{\partial}{\partial\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\sum_{a'} m_{a'}.c^2\frac{1}{\gamma_{a'}^*}\frac{1}{\gamma} = m_a.c^2\frac{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}{\boldsymbol{c}^2}\gamma.\gamma_a^* = m_a.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\gamma_a^*$

- **The reduced action**

We can write:

$$S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\},\boldsymbol{R}_{\boldsymbol{c}}(t)] = \int_{t_1}^{t_2} L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right)dt$$

$$= \int_{t_1}^{t_2}\left[\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \boldsymbol{P}_{\boldsymbol{c}}.\boldsymbol{V}_{\boldsymbol{c}} - E\right]dt$$

$$= \int_{t_1}^{t_2}\left[\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \left(\gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right).\boldsymbol{V}_{\boldsymbol{c}} - \gamma E^*\right]dt$$

$$= \int_{t_1}^{t_2}\left[\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \gamma E^*(\beta^2 - 1)\right]dt = \int_{t_1}^{t_2}\left[\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} - \frac{E^*}{\gamma}\right]dt$$

So

$$\boxed{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\},\boldsymbol{R}_{\boldsymbol{c}}(t)] = \int_{t_1}^{t_2}\left[\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} - \frac{E^*}{\gamma}\right]dt}$$

If we ignore the final position of the internal degree of freedom, we have like a “spatial Maupertuis principle” (instead of a temporal used in [Landau Lifchitz]):

$$\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\},\boldsymbol{R}_{\boldsymbol{c}}(t)] + \left(\sum_a \boldsymbol{P}_{\boldsymbol{a}}.\delta\boldsymbol{r}_{\boldsymbol{a}}^*\right)_{H_2} = 0$$

We can see that if all the internal momentum are constant, it exists a reduced action principle:

$$\boxed{S_0[\boldsymbol{R}_{\boldsymbol{c}}(t)] = -\int_{t_1}^{t_2}\frac{E^*}{\gamma}dt}$$

We can surely generalize it for closed systems with internal separable variables where we’ve chosen well the variables with constant momentum. In this case, we see that for “stationary” system, in this restrict sense, the center of mass dynamic is the same as a material point.

Proof:

Indeed (do the same that [Landau lifchitz] but for space and not for time)):

$$\delta S[\{\boldsymbol{r}_a^*(t^*)\}, \boldsymbol{R}_c(t)] + \left(\sum_a \boldsymbol{P}_a . \delta \boldsymbol{r}_a^*\right)_{H_2} = 0$$

$$<=> \delta \int_{t_1}^{t_2} d \sum_a [\boldsymbol{P}_a . \boldsymbol{r}_a^*] + \delta \int_{t_1}^{t_2} \left[-\frac{E^*}{\gamma}\right] dt + \left(\sum_a \boldsymbol{P}_a . \delta \boldsymbol{r}_a^*\right)_{H_2} = 0$$

$$<=> \delta \left[\sum_a \boldsymbol{P}_a . \boldsymbol{r}_a^*\right]_{H_2} + \delta \int_{t_1}^{t_2} \left[-\frac{E^*}{\gamma}\right] dt + \left(\sum_a \boldsymbol{P}_a . \delta \boldsymbol{r}_a^*\right)_{H_2} = 0$$

$$<=> \delta \int_{t_1}^{t_2} \left[-\frac{E^*}{\gamma}\right] dt = 0$$

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3. Free field

Now, I will repeat the same method for a field theory (a scalar field φ for simplify), and again:

The important point to keep in mind is that we are not considering the variation of the internal degree of freedom φ^* **:**

- relative to the internal time of K* t^* ;
- **but instead relative to time of K, t**: $\frac{\partial \varphi^*}{\partial t}$.

So without comments, we have successively:

$$S[\{\varphi(x,t)\}] = \frac{1}{c}\int \iiint \Lambda\left(\varphi, \frac{\partial \varphi}{\partial \boldsymbol{r}}, \frac{\partial \varphi}{\partial t}\right) d\Omega$$

$$= \frac{1}{c}\int \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) d\Omega^* = \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*\right] dt^*$$

$$= \int \left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*\right] \frac{dt}{\gamma}$$

=>

$$S[\{\varphi^*(x^*,t^*)\}, \boldsymbol{R}_{\mathrm{C}}(\boldsymbol{t})] = \int L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right] dt$$

With $L'\left[\{\varphi^*\}, \left\{\frac{\partial \varphi^*}{\partial r^*}\right\}, \left\{\frac{\partial \varphi^*}{\partial t}\right\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right] = \frac{1}{\gamma} \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial r^*}, \gamma \frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}\right) dV^*$

So we can calculate the 3-momentum as:

$$\boldsymbol{P_c} \equiv \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\left[\frac{1}{\gamma}\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right]$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma} + \frac{1}{\gamma}\iiint \frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

$$= \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \left(-\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)$$
$$+ \frac{1}{\gamma}\iiint \frac{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} \frac{\partial}{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

But $\frac{\partial}{\partial \boldsymbol{V}_\mathrm{C}}\frac{1}{\gamma} = -\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}$; $\gamma\frac{\partial \varphi^*}{\partial t} = \frac{\partial \varphi^*}{\partial t^*}$

And $\frac{\partial\left(\gamma\frac{\partial \varphi^*}{\partial t}\right)}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{\partial \gamma}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{\partial\frac{1}{\sqrt{1-\frac{\boldsymbol{V}_\mathrm{C}^2}{c^2}}}}{\partial \boldsymbol{V}_\mathrm{C}} = \frac{\partial \varphi^*}{\partial t}\frac{-1}{2}\left(-2\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)\frac{1}{\left(1-\frac{\gamma^2\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right)^{2}}{\boldsymbol{c}^2}\right)^{3/2}} = \frac{\partial \varphi^*}{\partial t}\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\gamma^3$

$$\boldsymbol{P_c} = \iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^* \left(-\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\right)$$
$$+ \frac{1}{\gamma}\iiint \left(\frac{\partial \varphi^*}{\partial t}\frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\gamma^3\right)\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

$$= \frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\gamma\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*(-1)$$
$$+ \iiint \left(\frac{\partial \varphi^*}{\partial t}\gamma\right)\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*$$

$$= \frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\gamma\left[\iiint \Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \gamma\frac{\partial \varphi^*}{\partial t}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*(-1) + \iiint \frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^*\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}, \frac{\partial \varphi^*}{\partial t^*}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}\right) dV^*\right]$$

$$= \frac{\boldsymbol{V}_\mathrm{C}}{\boldsymbol{c}^2}\gamma\iiint\left[\frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right] dV^*$$

So we have again:

$$\boldsymbol{P_c} = \gamma\frac{E^*}{c^2}\boldsymbol{V}_\mathbf{C}$$

where $E^* \equiv \iiint\left(\frac{\partial \varphi^*}{\partial t^*}\frac{\partial}{\partial\left(\frac{\partial \varphi^*}{\partial t^*}\right)}\Lambda^* - \Lambda^*\right) dV^*$ is the internal energy (associated to the hyperplane $t^* = cte$)

We see that we don't need to talk about closed system hypothesis or to have a 4 vector to demonstrate it (we don't even use the expression of any density Lagrangien).

The Euler-Lagrange equations tell us that $\frac{d}{dt}\left(\gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right)=\frac{\partial}{\partial \boldsymbol{R}_{\mathrm{C}}}L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial r^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}}\right]$

We have to note, in the proof, the importance to freeze the right variable $\frac{\partial\varphi^*}{\partial t}$ (and not $\frac{\partial\varphi^*}{\partial t^*}$) in order to have the good expression.

4. Interaction between a field and a particle

We consider the simplified action:

$$S[\boldsymbol{r_a}(t),\{\varphi(x,t)\}]=\int_{t_1}^{t_2}\left(\sum_a\left[-m_a.c\frac{ds_a}{dt}-\frac{e_a}{c}.\frac{ds_a}{dt}\varphi(\boldsymbol{r_a},t)\right]\right)dt+\frac{1}{c}\int\iiint\Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega$$

So we have also:

$$\begin{aligned}S&=\int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a+\frac{e_a}{c^2}\varphi(\boldsymbol{r_a},t)\right).c.\frac{ds_a}{dt}\right]\right)dt+\frac{1}{c}\int\iiint\Lambda\left(\varphi,\frac{\partial\varphi}{\partial\boldsymbol{r}},\frac{\partial\varphi}{\partial t}\right)d\Omega\\&=\int_{t_1}^{t_2}\left(\sum_a\left[-\left(m_a+\frac{e_a}{c^2}\varphi^{K_m}\right).c^2\frac{1}{\gamma_a^{K_m}}\right]\right)\frac{dt}{\gamma_m}\\&\quad+\int\left[\iiint\Lambda^{K_\varphi}\left(\varphi^{K_\varphi},\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}},\gamma\frac{\partial\varphi^{K_\varphi}}{\partial t},\boldsymbol{R}_{\mathrm{C}_\varphi},\boldsymbol{V}_{\mathrm{C}_\varphi}\right)dV^{K_\varphi}\right]\frac{dt}{\gamma_\varphi}\end{aligned}$$

Where we have specified the quantities relative to:

- the frame K_φ of the center of mass C_φ of the field φ ;
- the frame K_m of the center of mass C_m of the material points.

$$\begin{aligned}&S\left[\left\{\boldsymbol{r_a}^{K_m}(t^{K_m}),\boldsymbol{R}_{\mathrm{C}_m}(t)\right\},\{\varphi^{K_\varphi}(x^{K_\varphi},t^{K_\varphi})\},\boldsymbol{R}_{\mathrm{C}_\varphi}(\boldsymbol{t})\right]\\&=\int_{t_1}^{t_2}L'\left(\left\{\boldsymbol{r_a}^{K_m}\right\},\left\{\frac{d\boldsymbol{r_a}^{K_m}}{dt}\right\},\boldsymbol{R}_{\mathrm{C}_m},\boldsymbol{V}_{\mathrm{C}_m},t\right)dt\\&+\int L'\left[\{\varphi^{K_\varphi}\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial\boldsymbol{r}^{K_\varphi}}\right\},\left\{\frac{\partial\varphi^{K_\varphi}}{\partial t}\right\},\boldsymbol{R}_{\mathrm{C}_\varphi},\boldsymbol{V}_{\mathrm{C}_\varphi}\right]dt\end{aligned}$$

So in this form, we can calculate the dynamic of the center of mass of one system and the other. We can see that each system is not free at all, but we have again:

$$\boldsymbol{P}_{\mathrm{C}_m}=\gamma\frac{E^{K_m}}{c^2}\boldsymbol{V}_{\mathrm{C}_m}$$
$$\boldsymbol{P}_{\mathrm{C}_\varphi}=\gamma\frac{E^{K_\varphi}}{c^2}\boldsymbol{V}_{\mathrm{C}_\varphi}$$

So $M_m=\frac{E^{K_m}}{c^2}$, $M_\varphi=\frac{E^{K_\varphi}}{c^2}$

With the same method we can consider any set of systems.

5. Conclusion

We have a way to demonstrate the famous Einstein formula $E^*=Mc^2$ directly from an appropriate Lagrangien function selecting the correct variable.

Instead of starting from $L\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt^*}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)$, we've started the reasoning with $L'\left(\{\boldsymbol{r}_a^*\},\left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)$.

Again, instead of starting from $L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial r^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t^*}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right]$, we've started with $L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial r^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right]$.

In the two cases we've calculated directly that $\boldsymbol{P_c}\equiv\frac{\partial L'}{\partial v_\mathrm{C}}=\gamma\frac{E^*}{c^2}\boldsymbol{V}_\mathrm{C}$

Some remark:

1) The interest of this proof is to be independent of the closed system condition or other hypothesis: So if we take any system, we can attach a mass to it, even if its internal dynamic don't permit the conservation of its energy. Moreover, it is not necessary for the 4-momentum quantity of the system to be a 4-vector.
2) A simple Lorentz transformation, shows that the 3-momentum is actually the one associated to $P^i(K^*)=\frac{1}{c}\int\iiint_{space-tim}\ T^{ik}\delta\left(n_{lm}x^lx^m\right).d\eta_k(K^*)\,d^4x$, so it is a part of a 4-vector.
3) Since we have defined the mass center in K^*, it allows us to associated to it a true event (the center of the frame) which doesn't change from a frame K to another K', by the relativity of simultaneity. In consequence, we can show that the internal energy, so the mass, is an invariant even in our case.

It will be interesting to derive the energy-momentum tensor with our method, in order to show how the internal degrees of freedom generate the pressure like the formula (35,2) of [Landau-Lifchitz].

References

Landau Lifchitz - The Classical Theory of Field

Janssen, Michel and Mecklenburg, Matthew (2004) *Electromagnetic Models of the Electron and the Transition from Classical to Relativistic Mechanics.*