Hidden variables model of the Harmonic Oscillator.

Abstract

A hidden variables matrix mechanics model of the harmonic oscillator is presented as a counter-example in examining fundamental assumptions of quantum mechanics. Solutions are obtained which can be interpreted as describing continuous motion of a particle at all times located at points in space. While this is contrary to the basic postulate of Heisenberg, the experimental results of the standard matrix mechanics treatment are nevertheless reproduced. The proposed model is motivated by the foundational issues raised by Bell. Inequalities violation is however, attributed to the mathematical representation of outcome quantities as metric variables rather than the consensus assumption of local causality. Examining the consequence of this alternative conclusion on an actual quantum system creates an overlapping between Bell inspired foundational issues and the original postulates of Heisenberg and Born. Heisenberg's basic postulates – randomness of transitions and treating the system as an ensemble – are critical. Bohr's assumption that transitions occur instantaneously, together with Heisenberg's non-path postulate where the particle can be measured at spatially separated locations without continuous movement between locations, are discarded. The proposed hidden variables are non-contextual. These results question the commonly accepted inequalities violation conclusions that any local and/or non-contextual hidden variables theory cannot reproduce quantum predictions. That the wave function gives a complete description of the quantum state is likewise not supported by the model. Heisenberg's measurable-only quantities are interpreted as arising from a substructure of periodic endogenous motion of the system.

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1. Introduction

Bell inequalities, together with other reality testing configurations – Hardy non-locality, steering, macroscopic realism – is regarded as the conceptual and empirical framework for exploring quantum foundations [1]. As is well known, Bell was motivated by Einstein-Podosky-Rosen (EPR) questioning the completeness of quantum mechanics. Inequalities violation is now well established experimentally [2-6]. Local causality is generally considered to be the fatal assumption which would then imply that non-locality is a fact of nature [7-8]. However, explaining how non-local influences work physically remains a challenge while conflict with relativity persists. Also, violation only establishes that inequalities are wrong not why they are wrong. Nevertheless, the consensus view is that any local hidden variables theory cannot reproduce the results of quantum mechanics (QM).

There are alternative interpretations, including: a) Bell inequalities lack sufficient generality to be conclusive [9]; b) contextuality is the primitive characteristic [10-16]; c) classicality rather than local causality is the fatal assumption [17]; d) both locality and counterfactual definiteness are assumed [18-19]; and e) Bell's own contrary free particle position-momentum analysis which is in agreement with both QM and experiment [20].

Perhaps a most comprehensive analysis examines Bell, Kochen-Specker and Leggett-Garg configurations simultaneous, which are shown to be different violations of the same underlying mathematical property of a joint probability distribution for all possible measurements [21].

While these alternative explanations have generally not been the focus of intense scrutiny, there is a significant view supporting the contextuality option. Bell-like inequalities based on the assumption of non-contextuality are found to be violated; leading to a second potential conclusion that QM predictions cannot be reproduced by non-contextual hidden variables theories.

As a summary, inequalities violation potentially rules out any of local causality, classicality, non-contextuality and/or counterfactual definiteness. Alternatively, realism, determinism and hidden variables are not excluded. Also, if local causality is not rejected physical theories would then be local, deterministic and counterfactual indefinite [19]. There is some conjecture on the question of realism. Different authors use different definitions, which leads to some confusion. The definition used here is that of Norsen where realism is not an issue [22].

There is a further technical complication that since conclusions are drawn from inequalities being incorrect any inference remains conditional on there been no additional unidentified assumptions. Given this democracy of thought and technical inconclusiveness, Bell and associated inequalities are prudently seen here as invaluable in identifying *possibilities*. Any candidate fatal assumption is then faced with the challenge of explaining its consequences.

This work develops on previous analysis identifying the assumption of representing an outcome quantity by a metric variable [23-24]. At issue is then the mathematical representation of physical quantities. Contrary to the assertion that inequalities are a conjunction of locality and experimental outcomes (in which case locality is the sole assumption) it has been shown that assuming metric variables is an additional assumption [25-27].

Metric variables are the variable type of classical theory. This is due to the metric nature of space and time. Geometrical points in Euclidean geometry obey field algebra with measurability definable within its axiomatic structure [28-29]. Due to the isomorphism with numbers, points in Euclidean geometry can be represented by number-values. To the extent that physical points in space and time can be represented by Euclidean geometrical points, they can likewise be represented by numbers, and can likewise possess a mathematically defined measurability.

This geometrical-mathematical reasoning is more significant than mere abstractionism. It defines a logical procedure for representing a physical quantity by a mathematical entity: in this case location by number-value. For standard QM physical observables are associated with a mathematical representation – operators in Hilbert space - by postulate. To complete the connection between mathematics and empirical reality, experimental outcomes are associated, again by postulate, with eigenvalues-eigenstates of the quantum formalism.

If assuming metric variables is fatal then violation of Bell inequalities can also permit the possibility of a non-metric space with time remaining metric. Such space would be described by Projective geometry whose geometrical points obey division ring algebra, with measurability only definable under special case conditions.

Introducing non-metric space may seem a radical proposal but does have a simple consequence that continuous movement in space and time can also be a feature of QM. The following treatment of the harmonic oscillator using matrix mechanics methods is an example of such an analysis. Examining an actual quantum system has the advantage of a more rigorous testing of fundamental assumptions than reasoning alone of EPR and Bell, while also extending discussion beyond inequalities and their violations.

This analysis is not inconsistent with identifying non-contextuality, counter-factual definiteness or classicality as sources of inequality violation. Nor is there disagreement with Bell’s free particle analysis [24].

While the proposed model does give solutions which can be interpreted to require non-metric space, it is found that the same results can be interpreted more simply without rejecting classical space. Further, the same analysis can proceed directly from a critical examination of Heisenberg's fundamental assumptions without reference to EPR or Bell or the nature of space. In which case, the proposed model is a stand-alone counterexample independent of any opinion on inequalities violation. That notwithstanding, the somewhat contradictory approach chosen here is deliberate. This work is an examination of foundational postulates and is intended to remain within the framework of EPR and questions on the completeness of QM.

2. Harmonic oscillator: physical assumptions

The following discussion is more detailed than would normally be required for such a well-known system. It is not however the mathematical solutions which are the primary focus but rather the underlying foundational assumptions, particularly in relation to Heisenberg and Born [30-31].

Re-visiting matrix mechanics for answers to foundational questions may seem indirect given that the central issue is the meaning of the wave function. However, Heisenberg and Born postulates are as close as possible to classical-mechanical laws, and are thereby more transparent than the Schrodinger-based postulates of standard textbook treatments for exploring possible hidden variables in space and time [30].

The generic Hamiltonian is:

$$H_o = [p^2 \ + \ x^2]/2 \qquad (1)$$

Where the usual dimensionless quantities have been introduced as:

$$H_o = \frac{H}{\omega},\ p = \frac{P}{\sqrt{\omega m}},\ \ x = X\sqrt{\omega m}\, and\ ħ = 1 \tag{2}$$

At issue is the mathematical representation of position and momentum as non-classical quantities. For the initial assumption of movement in non-metric space, the basic question is how position – still a physical point in space and time – is to be mathematically represented given that points are now subject to division ring rather than field algebra. A possibility is to consider a quaternion representation [23-24]. There are several reasons for this choice: a) quaternions obey division ring algebra; b) quaternions have a matrix representation maintaining consistency with Born-Jordan (BJ) matrix mechanics; and c) quaternions are subject to SU(2) invariance consistent with electron spin.

The analysis can then proceed along similar reasoning to Born-Jordan. A complication arises however with the physical interpretation of variables and solutions. If quaternions represent particle position what is then the meaning of quaternion matrix elements and resulting eigenvalues?

Heisenberg interpreted what Born-Jordan later identified as matrix elements, to be transition amplitudes in what is termed a kinematic re-interpretation [31]. Commencing with the empirically verified Bohr-Einstein frequency relation Heisenberg concluded that transition amplitudes were a function of two indices representing the initial and final states of a transition. He further assumed, following Bohr, that transitions were instantaneous. This assumption has been found to be invalid. Experiments on atomic systems show duration intervals in the order of attosecends [32]. Heisenberg further assumed that classical dynamical equations would still apply to the kinematic re-interpreted quantities. Born-Jordan persisted with this assumption as a core postulate [30]. To be more precise, Born-Jordan showed that assuming a matrix Principle of Least Action classical equations are then applicable. It should nevertheless be clarified that there is ambiguity whether Heisenberg's re-interpreted quantity is position. Mathematically at least the magnitudes of the matrix elements refer to the probabilities of transitions.

For the model being proposed, the previously introduced assumption that the particle is at all times located at a point in physical space and has continuous motion in space and time, will persist but extended to also include transitions. In which case a path, likewise dependent on two indices, can be associated with a transition. Physically then, the matrix elements can be interpreted to refer to classical paths of finite duration. This re-definition simplifies basic physical interpretations: non-metric space is no longer necessary, kinematic re-interpretation is not required, and the position matrix would then refer to an ensemble of transition paths in metric space; *where the ensemble is a feature of the individual particle.*

Associating an ensemble with a single particle does seem perplexing. If however, all ensemble information is not contained by the individual particle it becomes difficult to see how protective measurements, which have been performed experimentally, would be possible [33-34]. Notwithstanding some controversy, the single particle ensemble assumption thereby has empirical justification.

Heisenberg rejected continuous path movement on the instrumentalist argument of non-measurability. However, he also added, rather pragmatically, that a theory based on orbitals did not reproduce experiment. Rejection of orbital-type movement does not necessitate rejection of all other forms of endogenous motion. The important requirement is system stability, which is ensured by Bohr's periodicity condition.

A further possible objection is that re-introducing classical motion contradicts the uncertainty principle. However, that is not the case. The experimentally verified preparation and measurement formulations of the uncertainty relations refer to ensemble distributions and relations between such distributions. The ensemble characteristic of QM is not in question. Ballentine pointed out long ago that these relations do not apply to the individual case, which here would be an individual path [36]. Ballentine's statistical interpretation defines a minimum requirement to meet the benchmark criteria of consistency with the mathematical formulation of QM. The proposed model meets the criteria.

Heisenberg introduced a number of fundamental assumptions which will continue to apply: 1) primacy of the frequency condition, 2) rejection of orbital paths, but still accepting Bohr's periodicity condition, 3) energies, frequencies and transitions amplitudes as measurable quantities, 4) the system is to be treated as an ensemble of transitions for which Born introduced the pivotal matrix representation, and 5) applicability of classical equations of motion to the ensemble position and momentum. To this list is added the experimentally based randomness of transitions. Like Heisenberg and Born a particle picture is assumed.

3. Harmonic oscillator: exploratory model

Quaternion matrices are order 2×2 which differs from the actually infinite matrices of Born-Jordan. However, the frequency condition refers to a unit transition between two arbitrary states which can be represented by 2×2 matrices. Initially, the model is simplified to considering unit transitions, whereas the ensemble postulate requires the collective treatment of matrix mechanics, and use of actually infinite (or at least large but finite) matrices. For the sake of terminology in this section the 2×2 matrices will be referred to as ensemble, although not fully representing Heisenberg's postulate. As will be seen, standard matrix mechanics results are obtained. A more general model incorporating BJ matrices, which correctly represent an ensemble, is also presented. Results are essentially the same.

The Born-Jordan postulate allows classical equations of motion to be applied to the Hamiltonian (1) giving:

$$(\dot{x} + i\,\dot{p}) = -i(x + ip) \qquad (3a)$$

$$(\dot{x} - i\,\dot{p}) = \; i(x - ip) \qquad (3b)$$

Introducing the usual definitions:

$$a = \frac{1}{\sqrt{2}}(x + ip) \qquad (4a)$$

$$a^{*} = \frac{1}{\sqrt{2}}(x - ip) \qquad (4b)$$

Leads to the solutions:

$$a = \frac{1}{\sqrt{2}}(x_o + ip_o)e^{-it} \qquad (5a)$$

$$a^{*} = \frac{1}{\sqrt{2}}(x_o - \; ip_o)e^{it} \qquad (5b)$$

These relations are not obtained from Heisenberg's equation of motion which is not applicable for position 2× 2 matrices. From relations (5) the time dependent expressions are:

$$x(t) = \; x_o\, cost + p_o sint \qquad (6a)$$

$$p(t) = \; p_o\, cost - x_o sint \qquad (6b)$$

These relations have the same form as the standard treatment. Using definitions (5) the Hamiltonian becomes:

$$H_o = \ a^* a + \frac{1}{2}[a, a^*] \qquad (7)$$

Substituting explicitly for position and momentum gives:

$$H_o = \frac{1}{2}(x_o^2 + \ p_o^2) \qquad (8)$$

That the Hamiltonian is time independent is as required by conservation of energy. Born-Jordan established this as a general result for actually infinite matrices, with time-independence further inferring the Hamiltonian energy-matrix is diagonal. For the specific HO Hamiltonian (1) time-independence can be shown by following the BJ relation for the time derivative of a product of two matrices followed by use of classical equations. Following similar reasoning to BJ, it can be further shown that the Hamiltonian is diagonal also for quaternion matrices.

Another feature of relation (7) is that non-commutation terms cancel. For this model, the physical origin of non-commutation differs from orthodoxy. Non-commutation is not directly a consequence of the uncertainty principle, nor primitive position and momentum, which remain classical quantities, nor the structure of space as first supposed. Mathematically, it is a consequence of representing the ensemble position and momentum by matrices. Since matrices are introduced to represent the ensemble feature of the system, non-commutation is then a consequence of Heisenberg's ensemble postulate; which is itself sourced in the nature of quantum matter.

Standard forms for quaternions are:

$$q = a + bi\sigma_3 + ci\sigma_2 + di\sigma_1 \tag{9a}$$

$$q = \begin{bmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{bmatrix} \tag{9b}$$

Where a, b, c, d are real or complex coefficients, and σ_i are the usual Pauli spin matrices. Following the form of BJ matrices, in general the ensemble position in 2× 2 matrix representation is:

$$x(t) = \begin{bmatrix} A & b_1\, e^{it} + \; b_2 e^{-it} \\ c_1\, e^{it} + \; c_2 e^{-it} & D \end{bmatrix} \tag{10}$$

At issue are the values of the coefficients. For diagonal elements the transition frequency is zero in which case these elements are constants. Heisenberg further assumed these elements are zero, which would also be the case here – if there is no transition there can be no transition path. However, following Born-Jordan, Newton's 2nd Law of motion gives the most basic equation:

$$\ddot{x} + \; x = 0 \tag{11}$$

Substituting the position matrix (10) leads to zero diagonal elements in agreement with Heisenberg. In which case the initial position and momentum are:

$$x_o = \begin{bmatrix} 0 & c + id \\ -(c - id) & 0 \end{bmatrix} \tag{12a}$$

$$p_o = \begin{bmatrix} 0 & c' + id' \\ -(c' - i\,d') & 0 \end{bmatrix} \tag{12b}$$

Substituting relations (12) into (6) leads to the time dependent matrices.The periodicity condition for the harmonic oscillator is [37]:

$$\oint pdq = 2\pi(n + \tfrac{1}{2}) \tag{13}$$

Substituting position and momentum relations (6a) and (6b) gives:

$$[p_o^2 + x_o^2] = 2(n + \tfrac{1}{2}) \tag{14}$$

The matrix mechanics use of classical equations has been extended to apply also to the periodicity condition. Omitting the identity matrix, which as expected gives a diagonal Hamiltonian, the standard HO energy solutions follow as:

$$H = \hbar\omega(n + \tfrac{1}{2}) \tag{15}$$

Non-commutation of the ensemble quantities follows from relations (6) as:

$$[x(t), p(t)] = [x_o\,, p_o] \tag{16}$$

Clearly, the RHS of this relation is not a function of time. Born-Jordan established this result to be a property of matrices independent of order. Using (12) the non-commutation relation becomes:

$$[x(t), p(t)] = 2(cd' - dc')i\sigma_3 \tag{17}$$

Born-Jordan established that for matrices of any order the RHS matrix should be diagonal, which it is. The presence of the Pauli spin matrix may suggest an incorrect departure from QM. However, non-commutation of 2×2 sub-matrices along the main diagonal of BJ matrices also includes the spin matrix. If as stated non-commutation is a consequence of ensemble-ness, then different mathematical representations will give different non-commutation relations. It is sufficient that there be consistency; which as will be seen is the case. A consequence of (17) is that ensemble position and momentum are not subject to the Heisenberg equation of motion, which again is not a problem provided that standard results are reproduced; which they are.

Using relations (12a) and (12b) together with the periodicity condition (14) gives:

$$c^2 + d^2 + (c')^2 + (d')^2 = -(2n+1) \tag{18}$$

Redefining coefficients gives:

$$c^2 = (\alpha_1 + \beta_1),\ d^2 = (\alpha_2 + \beta_2) \tag{19a}$$

$$(c')^2 = (\alpha_3 + \beta_3),\ (d')^2 = (\alpha_4 + \beta_4) \tag{19b}$$

Where $\alpha_i = \alpha_i\,(n)$ and β_i is a constant, leading to:

$$\sum \alpha_i = -(2n)\ and\ \sum \beta_i = (-1) \tag{20}$$

From (20) both α_i and β_i must be real. According to Heisenberg and Born the RHS of (17) should be a constant. The rationale is that the quantum condition (using Born terminology) refers to a change-of-state, and should therefore be state independent, that is, independent of n. Maintaining consistency with Born-Jordan the constant is set to unity; so that:

$$(cd' - dc') = 1/2 \qquad (21)$$

While this assumption is arbitrary, it is nevertheless appropriate: the purpose of the model is to develop an example of a possible hidden variable substructure reproducing standard QM. This assumption meets the minimalist criteria. As will be found however, the same result can be obtained without direct reference to BJ results by defining an *a priori* condition.

Relation (21) can be re-expressed as:

$$\left[(\alpha_1 + \beta_1)^{1/2}(\alpha_4 + \beta_4)^{\frac{1}{2}} - (\alpha_2 + \beta_2)^{\frac{1}{2}}(\alpha_3 + \beta_3)^{\frac{1}{2}}\right] = 1/2 \qquad (22)$$

At issue are again the values of the coefficients. The LHS of this relation is subject to three constraints: 1) real 2) positive and 3) n independent. Using these conditions it is found that all α_i and β_i must have the same sign, and that all $\alpha_i\ (n)$ are the same function.

The reasoning is as follows. All $(\alpha_i + \beta_i)$ terms having the same sign does not potentially contradict the constraints. Also, the signs of α_i and β_i must be the same within each term. If signs are different then the sign of $(\alpha_i - \beta_i)$ will change depending on $(\alpha_i(n) > \beta_i)$ or $(\alpha_i(n) < \beta_i)$, that is depending on the value of n, which would contravene the third constraint.

Considering the LHS product terms individually, if the two $(\alpha_i + \beta_i)$ terms have different signs their product cannot be real thereby contradicting the first constraint. Alternatively, if the two $(\alpha_i + \beta_i)$ terms have the same sign but are of opposite sign to the two terms of the other product term, the product terms on the LHS will have the same sign, that is either both positive or both negative. In which case, the $\alpha_i$terms, which define the functional n dependence, cannot be eliminated from the LHS expression thereby contradicting the third constraint.

For $n = 0$ it follows from (20) that $\sum \alpha_i(0) = 0$, since all α_i terms have the same sign, the $\alpha_i\,(0)$ terms are all zero. The two conditions become:

$$[(\beta_1)^{1/2}(\beta_4)^{1/2} - (\beta_2)^{1/2}(\beta_3)^{1/2}] = 1/2 \qquad (23a)$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = (-1) \qquad (23b)$$

All β_i must then be negative. By inspection a solution is:

$$\beta_1 = \beta_4 = 0 \; and \; \beta_2 = \beta_3 = \beta = (-\tfrac{1}{2}) \qquad (24)$$

A numerical analysis of (23a) and (23b) gives the same result. Solution (24) for β applies for all n values, in which case relation (22) then becomes:

$$\left[(\alpha_1)^{1/2}(\alpha_4)^{1/2} - (\alpha_2 + \beta)^{1/2}\,(\alpha_3 + \beta)^{1/2}\right] = 1/2 \qquad (25)$$

The LHS must be independent of n, to illuminate this functional dependence requires that all α_i are the same function. From relation (20) and (24) the solutions become:

$$\alpha_i = \alpha(n) = -\left(\frac{n}{2}\right) \; and \; \beta = (-\tfrac{1}{2}) \tag{26}$$

The quaternion coefficients are then:

$$c = d' = i(\tfrac{n}{2})^{1/2} \; and \; c' = d = i((n+1)/2)^{1/2} \tag{27}$$

Using the basic relation $p(t) = \dot{x}\,(t)$ as an *a priori* condition leads to the same results. The initial position and momentum become:

$$x_o = (\tfrac{i}{\sqrt{2}})\begin{bmatrix} 0 & \sqrt{n} + i\sqrt{n+1} \\ -\sqrt{n} + i\sqrt{n+1} & 0 \end{bmatrix} \tag{28a}$$

$$p_o = (\tfrac{i}{\sqrt{2}})\begin{bmatrix} 0 & \sqrt{n+1} + i\sqrt{n} \\ -\sqrt{n+1} + i\sqrt{n} & 0 \end{bmatrix} \tag{28b}$$

Both matrices are hermitian, in which case their respective eigenvalues are real. Substituting relations (28) into relations (6) gives the time dependent ensemble position and momentum as:

$$x(t) = \left(-\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 0 & -i\sqrt{n}\, e^{it} + \sqrt{n+1}e^{-it} \\ i\sqrt{n}\, e^{-it} + \sqrt{n+1}e^{it} & 0 \end{bmatrix} \tag{29a}$$

$$= \left(-\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 0 & \sqrt{n+1}\, e^{-it} \\ \sqrt{n+1}e^{it} & 0 \end{bmatrix} + \left(-\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 0 & -i\sqrt{n}e^{it} \\ i\sqrt{n}\, e^{-it} & 0 \end{bmatrix}$$

$$p(t) = \left(-\frac{i}{\sqrt{2}}\right)\begin{bmatrix} 0 & -\sqrt{n+1}\,e^{-it} - i\sqrt{n}e^{it} \\ \sqrt{n+1}e^{it} - i\sqrt{n}\,e^{-it} & 0 \end{bmatrix} \quad (29b)$$

$$= \left(-\frac{i}{\sqrt{2}}\right)\begin{bmatrix} 0 & -\sqrt{n+1}\,e^{-it} \\ \sqrt{n+1}e^{it} & 0 \end{bmatrix} + \left(-\frac{i}{\sqrt{2}}\right)\begin{bmatrix} 0 & -i\sqrt{n}e^{it} \\ -i\sqrt{n}\,e^{-it} & 0 \end{bmatrix}$$

Again, both quantities are hermitian. Off-diagonal elements describe periodic motion with frequency that of the propagating virtual photon, which, together with their total energy are:

$$x_{HV}(t) = i\sqrt{n/2}\,e^{it} - \sqrt{(n+1)/2}e^{-it} \quad (30a)$$

$$p_{HV}(t) = i\sqrt{(n+1)/2}\,e^{-it} - \sqrt{n/2}e^{it} \quad (30b)$$

$$E_{HV} = -i\hbar\omega\sqrt{n(n+1)} \quad (30c)$$

$$where\ p_{HV}(t) = \dot{x}_{HV}(t) \quad (30d)$$

Similar relations hold for the conjugate matrix element with corresponding energy: $+i\hbar\omega\sqrt{n(n+1)}$. Obviously all quantities are complex, which suggests non-physicality. However, that is not the case. It is not a classical point-particle whose motion is being described but rather a transitioning quantum entity propagated by absorption and emission of virtual quanta.

Complex classical trajectories are the subject of current research, showing a connection with QM by reproducing quantum effects, notably tunnelling [38-39]. The possibility that complex classical systems can be viewed as possible hidden variables formulations has been suggested [39]. Interestingly, it can be inferred from protective measurements that the one-particle wave function is formed by an ergodic particle motion [34].

Mathematically, these relations are a consequence of the periodicity condition, together with SU(2) invariance. That matrices are hermitian is likewise a consequence of the same two basic conditions.

The position matrix describes both stationary state behaviour, by real eigenvalues, as well as non-stationary state behaviour by complex transition paths. A stationary state must be balanced by emission and absorption of virtual quanta; in which case there is no net complex energy contribution, since the complex conjugate energy terms cancel.

The presence of non-stationary state behaviour concurs with Heisenberg's rejection of an orbital-type system.

The first term of the position and momentum matrices, relation (29a) and (29b) reproduce the matrix elements along the main diagonal of Born-Jordan to a phase factor. This result, together with the energy relation (15), suggests the measurable-only quantities of QM can be reproduced by the model. With the extended model using actually infinite matrices to fully incorporate the ensemble postulate, matrix elements are found to be equivalent with Born-Jordan.

Position and momentum eigenvalues and respective eigenvectors are:

$$x_{evalue}(t) = (\pm)[\left(n + \tfrac{1}{2}\right) + \sqrt{n(n+1)}\,\sin(2t)]^{½} \quad (31a)$$

$$where\ x_{evalue}(t) = f\left(x_{HV}(t), \overline{x_{HV}(t)}\right) \quad (31b)$$

$$x_{evector}(t) = \begin{bmatrix} (\pm)[\left(n + \tfrac{1}{2}\right) + \sqrt{n(n+1)}\,\sin(2t)]^{½} \\ -i\sqrt{\tfrac{n}{2}}\; e^{-it} - \sqrt{(n+1)/2}e^{it} \end{bmatrix} \quad (31c)$$

$$p_{evalue}(t) = (\pm)[\left(n + \tfrac{1}{2}\right) - \sqrt{n(n+1)}\,\sin(2t)]^{½} \quad (31d)$$

$$where\ p_{evalue}(t) = g(\dot{x}_{HV}(t), \overline{\dot{x}_{HV}(t)}) \quad (31e)$$

$$p_{evector}(t) = \begin{bmatrix} (\pm)[\left(n + \tfrac{1}{2}\right) - \sqrt{n(n+1)}\,\sin(2t)]^{½} \\ \left(-\tfrac{i}{\sqrt{2}}\right)(\sqrt{n+1}e^{it} - i\sqrt{n}\; e^{-it}) \end{bmatrix} \quad (31f)$$

Eigenvectors are likewise functions of the corresponding hidden variables. As expected both eigenvalues are real since their respective matrices are hermitian.

These solutions are not obtained by Born-Jordan in which case they are not described by standard matrix mechanics, and so, would also not be described by the wave function.

Other properties are: 1) position and momentum eigenvalues are not related by classical definitions; 2) since the respective matrices do not commute their eigenvalues are not simultaneous, meaning they are not simultaneously measurable; and 3) the position eigenvalue describes position as a continues function of time, that is, *describing continues movement in space and time*.

The ground state is of special interest. The position eigenvalue (31a) is not a function of time thereby reproducing zero point oscillations about the equilibrium. Alternatively, if this result is assumed as an *a priori* condition the non-commutation constant of relation (17) is found to be ± 1. Both plus or minus values reproduce the same eigen-path solutions (31) as well as all measurable-only values of Born-Jordan. A further feature of the ground state is that the energy of the hidden variables classical transition path is zero – consistent with the classical H.O. There may be the suggestion of a violation of conservation of energy. However, individual kinetic and potential energy components are non-zero resulting in the well-known ½ħω quantum ground state energy eigenvalue.

Ensemble position and momentum (29) can also be expressed in Pauli spin-matrix representation:

$$x(t) = \left(-\frac{1}{\sqrt{2}}\right)\{\left[\sqrt{n}cost + \sqrt{n+1}\,sint\right]\sigma_2$$

$$+ [\sqrt{n+1}\,cost + \sqrt{n}sint]\sigma_1\} \tag{32a}$$

$$p(t) = \left(-\frac{1}{\sqrt{2}}\right)\{\left[\sqrt{n+1}cost - \sqrt{n}\sqrt{sint}\right]\sigma_2$$

$$+ [\sqrt{n}cost - \sqrt{n+1}sint]\sigma_1\} \qquad (32b)$$

It should be emphasized that relations (31) and (32) refer to different physical quantities – the first refers to an *individual* case while the second refers to an *aggregate* property. Further, relations (32) refer to points on a non-Euclidean mathematical structure subject to division ring algebra – Dirac q-numbers. Geometrically, the points are subject to the axioms of projective geometry, excluding that of order. Their structure is also SU(2) invariant. A projective mathematical structure circumvents the need for physical space to be projective as first supposed.

Classical theory is defined on a Euclidean mathematical structure, geometrically the same as the physical space of the phenomena which the theory describes. Consequently, both observable variables and predicted measurable outcomes of classical theory are the same: both metric and both subject to the equations of motion. Relation (32) suggests the mathematical structure of quantum theory is more complicated. Only the ensemble quantities are subject to classical equations whereas the eigenvalues are not. That observable and measurement outcome variables are subject to different mathematical structures, and so different algebra, has consequences for Bell inequalities.

Three definitions of position and momentum are introduced: the unit ensemble matrix, the matrix elements and the eigenvalues. As discussed, the ensemble quantities (or rather unit ensemble) are a consequence of the postulates of

Heisenberg and Born. These are the quantum observables position and momentum, which are non-contextual and non-metric, that is, do not take number-values and are do not possess measurability as a geometrical-mathematical property. Assuming mathematical properties have correspondence with reality the ensemble quantities are then physically non-measurable. The matrix element position and momentum, which are termed hidden variables, are the usual classical variables: metric, non-contextual and counterfactual definite, including the geometrical-mathematical property of measurability.

The eigenvalue quantities are the measurement outcome variables of QM; in which case are directly relevant to Bell inequalities. These quantities are: non-simultaneously metric, contextual and can be counterfactual indefinite, as well as not being subject to classical equations of motion. They do not have simultaneously defined measurability. Consequently, while the measurement outcome of either variable will give a number-value, the inferred value of the non-measured other quantity cannot be reasoned counterfactually.

Bell inequalities involve an actual measurement outcome and the inferred outcome of the non-measured quantity as if it were measured, for two incompatible observables. Values are then substituted into a mathematical modelling of an experimental configuration. The modelling is wrong. For the variables just described, Bell's inequalities are incorrect irrespective of other assumptions.

The preferred inequalities violation conclusion - that no local and/or non-contextual hidden variables theory can reproduce quantum predictions - is not supported by this model. That a local, deterministic hidden variables physical theory must be counterfactual indefinite is likewise not supported. Counterfactual definite hidden variables are possible where the variables reproduce quantum outcome variables rather than directly the experimental outcomes.

Non-measurability of observables may suggest a contradiction with quantum orthodoxy, at least from an instrumentalist perspective. However, the question is more complex. Observables refer to aggregate properties of an ensemble of individual transition paths, which are measurable. In which case, the source of non-measurability is not generic but rather that an observable refers to a collective property. It is meaningful, for example, to consider the age of a population as a tangible property yet only the ages of individual members are measurable.

Moreover, any instrumentalist concerns would also impact on Heisenberg's definitions. It is the aggregate characteristic of these observables, not whether the aggregates are trans-amplitudes or trans-paths, which gives rises to non-numeracy and non-measurability. Violation of instrumentalism in this model is then no different to that already present in standard matrix mechanics.

The relation between observable and corresponding eigenvalue outcome raises issues in interpreting the eigenvalue-eigenstate half link [34-35]. A fundamental question is whether the half link implies that a QM observable is a property of the system. Gao states that *"the link only says that an observable is a property of a physical system when the system is in an eigenstate of the observable, and it does not say that the observable is a property of the system for all cases, for example, when the system is not in an eigenstate of the observable"*. The relation between observables and outcome is summarised as: *"when a physical system is in an eigenstate of an observable, the system has a property represented by the eigenvalue associated with the eigenstate. This property is certainly not the observable itself, although we may say that it is the observable possessing the corresponding eigenvalue"*.

The second conclusion, observable and eigenvalue referring to different properties, concurs with the definitions being presented. On the original question however, the observable represents a property of the system irrespective of whatever eigenstate the system is in. The eigenvalue and corresponding eigenstate are individual characteristics which alter: if the system is in a position eigenstate it cannot be in a momentum eigenstate. However, the momentum observable will still represent the ensemble momentum property irrespective of the system being in a position eigenstate. The observable will only cease to represent the ensemble property of the system if the system ceases to be an ensemble.

Measurements of eigen-paths are subject to Bohr-type quantum observations to the extent that outcomes are apparatus-interactive [40]. This does not however re-admit neo-Copenhagen-ism. An eigen-path measurement involves observing the physical quantities mathematically represented by the eigenvalue-eigenvector relations (31) which characterise *aggregates.* The critical issue is not whether classical and quantum operational measurements differ - they would be expected to be different since quantum observations decode more information - but rather whether quantum measurements necessitate a different reality. The answer to this foundational question is “no”. Eigenvalues-eigenvectors of (31) are sourced on classical paths. Their measurement, while operationally different, would still be founded on the same reality.

Recently, it has been shown in wave-picture configuration that a quantum measurement can be understood as still be interactive with the measuring apparatus but without introducing *alternative realities* [41].

4. Harmonic oscillator: actually infinite order matrices

It is possible to extend the exploratory model to satisfy Heisenberg’s ensemble postulate requiring that transitions be treated collectively. To do so involves extending the BJ matrices to include the extra terms suggested by the model.

Since the following results are obtained by mainly replicating the reasoning of Born-Jordan only summary results are given.

Accordingly, the ensemble quantities (in BJ matrix notation) are:

$$\boldsymbol{q} = \boldsymbol{q_1} + \boldsymbol{q_2} = (q_1(n,m)e^{2\pi i v(n,m)t} + iq_2(n,m)e^{-2\pi i v(n,m)t}) \qquad (33a)$$

$$\boldsymbol{p} = \boldsymbol{p_1} + \boldsymbol{p_2} = (p_1(n,m)e^{2\pi i v(n,m)t} + ip_2(n,m)e^{-2\pi i v(n,m)t}) \qquad (33b)$$

The first terms have the same form as standard matrices and the second terms are those suggested by the unit model. These relations are only applicable in general if it is assumed that the endogenous motion is the same as that of the HO.

Born-Jordan introduced a diagonal matrix **W,** whose elements are the energies defined by the frequency condition, providing connection with experiment. Also introduced is a general matrix function **g** = **g(p, q**) which in the modified form is:

$$\boldsymbol{g} = \boldsymbol{g_1} + \boldsymbol{g_2} = (g_1(n,m)e^{2\pi i v(n,m)t} + ig_2(n,m)e^{-2\pi i v(n,m)t}) \qquad (34)$$

Differentiating w.r.t. time gives:

$$\dot{\boldsymbol{g}} = 2\pi i(v(n,m)\big(g_1(n,m)e^{2\pi i v(n,m)t} - ig_2(n,m)e^{-2\pi i v(n,m)t}\big)) \qquad (35)$$

Following BJ gives:

$$\dot{\boldsymbol{g}} = i/\hbar[\boldsymbol{W}, (\boldsymbol{g_1} - \boldsymbol{g_2})] \qquad (36)$$

For the Hamiltonian energy-matrix the relation is then:

$$\dot{\boldsymbol{H}} = i/\hbar[\boldsymbol{W}, (\boldsymbol{H_1} - \boldsymbol{H_2})] \tag{37}$$

If $\dot{\boldsymbol{g}} = 0$ BJ showed that the standard general matrix must be diagonal. This result does not apply to the modified matrix. From relation (35) the conditions for non-zero, non-diagonal elements are:

$$for\ n \neq m, v(n,m) \neq 0:$$

$$g_1(n,m)e^{2\pi i v(n,m)t} - i g_2(n,m)e^{-2\pi i v(n,m)t} = 0 \tag{38}$$

From relation (38) the non-diagonal elements of (34) would then be the sum of two equal terms. In which case $\boldsymbol{g_1} - \boldsymbol{g_2}$ of relation (36) is a diagonal matrix, so that the RHS equals zero consistent with the LHS (if $\dot{\boldsymbol{g}} = 0$). For the energy-matrix (37) conservation of energy can be preserved while the matrix is not diagonal. This result differs from Born-Jordan.

From the periodicity condition Born-Jordan derived Heisenberg's quantum condition, which applies only for diagonal elements. Following the same reasoning gives the modified relation:

$$\sum_k \{[p_1\,(n,k)q_1(k,n) - q_1\,(n,k)p_1(k,n)] - [p_2\,(n,k)q_2(k,n) - q_2(n,k)p_2(k,n)]\} = (-i\hbar) \tag{39}$$

Born-Jordan then showed that all off-diagonal elements for the standard matrices are zero. Following the same steps gives the same results for the modified matrices. Accordingly, the quantum condition (in BJ terminology) becomes:

$$[\boldsymbol{q_1}, \boldsymbol{p_1}] - [\boldsymbol{q}_2, \boldsymbol{p}_2] = i\hbar \tag{40}$$

That this relation differs from the iconic quantum condition may seem incompatible. At least for the HO however, equivalence with standard matrix mechanics will be established.

The following treatment initially follows a standard textbook approach which uses the special-case conditions of the HO [42]. Specific features are: non-zero matrix elements are pair-wise on either side of the main diagonal; a single non-zero element in the lowest row; complex conjugate pair off-diagonal elements; and the applicability of the basic relation where momentum is the time derivative of position. Using these characteristics the nth diagonal element of the Hamiltonian is:

$$E_n = \omega^2[|q_1(n, n')|^2 + |q_1(n, n'')|^2 + |q_2(n, n')|^2 + |q_2(n, n'')|^2] \tag{41}$$

Primes refer to pair-wise elements. A second relation follows from the nth diagonal of condition (39) as:

$$\hbar/2 = \omega[|q_1(n, n')|^2 - |q_1(n, n'')|^2 + |q_2(n, n')|^2 - |q_2(n, n'')|^2] \tag{42}$$

The first two terms of both conditions replicate Born-Jordan. For an index n_0 there is only one non-zero element so that $q(n_0, n_0'') = 0$ in which case:

$$E_n = \omega^2[|q_1(n_0, n_0')|^2 + |q_2(n_0, n_0')|^2] \tag{43a}$$

$$ħ/2 = \omega[|q_1(n_0, n_0')|^2 + |q_2(n_0, n_0')|^2] \tag{43b}$$

This leads to the standard ground state energy:

$$E_{n_0} = ħ\omega/2 \tag{44}$$

In which case the energy levels for stationary states are the usual:

$$E_n = (n + ½)ħ\omega \tag{45}$$

Substituting into (41) and (42) gives:

$$[|q_1(n, n+1)|^2 + |q_2(n, n+1)|^2] = (\frac{ħ}{2\omega})(n+1) \tag{46}$$

This leads to:

$$q_1(n, n+1) = \sqrt{\frac{ħ}{2\omega}}a(n)e^{-i\omega t} \tag{47a}$$

$$q_2(n, n+1) = \sqrt{\frac{ħ}{2\omega}}b(n)e^{i\omega t} \tag{47b}$$

$$where\ a(n)_{BJ}^2 \equiv a(n)^2 + b(n)^2 = n + 1 \tag{47c}$$

Corresponding momentum relations can be obtained by taking the time derivative of position. Taking complex conjugates give the corresponding complex conjugate elements. In summary the matrix elements are:

$$q(n, n+1) = (\frac{\hbar}{2\omega})^{½}[a(n)e^{-i\omega t} + ib(n)e^{i\omega t}] \quad (48a)$$

$$p(n, n+1) = (\frac{\hbar\omega}{2})^{½}[-ia(n)e^{-i\omega t} - b(n)e^{i\omega t}] \quad (48b)$$

Embedded in the extended BJ matrices are the quaternion unit transition matrices arising from complex conjugate pairs representing the n ⇄ m transition such that:

$$\boldsymbol{Q} = (\frac{\hbar}{2\omega})^{½}\begin{bmatrix} 0 & a(n)e^{-i\omega t} + ib(n)e^{i\omega t} \\ a(n)e^{i\omega t} - ib(n)e^{-i\omega t} & 0 \end{bmatrix} \quad (49a)$$

$$\boldsymbol{P} = (\frac{\hbar\omega}{2})^{½}\begin{bmatrix} 0 & -ia(n)e^{-i\omega t} - b(n)e^{i\omega t} \\ ia(n)e^{i\omega t} - b(n)e^{-i\omega t} & 0 \end{bmatrix} \quad (49b)$$

Their commutation relation is:

$$[\boldsymbol{Q}, \boldsymbol{P}] = i\hbar(a(n)^2 - b(n)^2)\sigma_3 \quad (50)$$

As previously discussed the RHS constant is set to unity to maintain consistency with the ground state order 2 sub-matrix of standard Born-Jordan. In which case a second relation follows as:

$$a(n)^2 - b(n)^2 = 1 \quad (51)$$

Relations (47c) and (51) give the coefficients as:

$$a(n)^2 = \left(\frac{n}{2} + 1\right) and\ b(n)^2 = n/2 \qquad (52)$$

Alternatively, the same values can be obtained directly from the two commutation relations. Again, using the special case conditions of the HO to define the general structure of the position and momentum matrices, relation (40) generates recursion relations for the coefficients $a(n)^2, b(n)^2, a(n+1)^2, b(n+1)^2$ which together with relation (51) lead to relations (52). All matrices are then fully determined.

The position eigenvalue for (49a) is then:

$$Q_{evalue}(n \rightleftarrows m, t) = (\pm)(\frac{\hbar}{\omega})^{½}[\frac{(n+1)}{2} - ½\sqrt{n(n+2)}\sin(2t)]^{½} \qquad (53)$$

Although there is a slight mathematical difference to the corresponding eigenvalue of the exploratory model (31a), the physical explanation is the same. The above discussion in relation to Bell inequalities remains unchanged. Whether this mathematical solution has physical reality is the obvious question. Heisenberg's measurable-only quantities - which do describe *reality* - are generated from the same coefficients.

The position and momentum matrices are:

$$\boldsymbol{q} = (\frac{\hbar}{2\omega})^{½} \begin{bmatrix} 0 & e^{-i\omega t} & 0 & 0 & \cdots \\ e^{i\omega t} & 0 & \sqrt{\frac{3}{2}}e^{-i\omega t} + i\sqrt{\frac{1}{2}}\, e^{i\omega t} & 0 & \ldots \\ 0 & \sqrt{\frac{3}{2}}e^{i\omega t} - i\sqrt{\frac{1}{2}}\, e^{-i\omega t} & 0 & \sqrt{2}\, e^{-i\omega t} + ie^{i\omega t} & .. \\ 0 & 0 & \sqrt{2}\, e^{i\omega t} - ie^{-i\omega t} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (54a)$$

$$\boldsymbol{p} = i(\frac{\hbar\omega}{2})^{½} \begin{bmatrix} 0 & -e^{-i\omega t} & 0 & 0 & \cdots \\ e^{i\omega t} & 0 & -\sqrt{\frac{3}{2}}e^{-i\omega t} + i\sqrt{\frac{1}{2}}\, e^{i\omega t} & 0 & \ldots \\ 0 & \sqrt{\frac{3}{2}}e^{i\omega t} + i\sqrt{\frac{1}{2}}\, e^{-i\omega t} & 0 & -\sqrt{2}\, e^{-i\omega t} + ie^{i\omega t} & \ldots \\ 0 & 0 & \sqrt{2}\, e^{i\omega t} + ie^{-i\omega t} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (54b)$$

As would be expected, these matrices are consistent with commutation relation (40). The Hamiltonian energy-matrix is:

$$\boldsymbol{H} = (\frac{\hbar\omega}{2}) \begin{bmatrix} 1 & 0 & \frac{i\sqrt{2}}{2} & 0 & \cdots \\ 0 & 3 & 0 & i\left(1 + \sqrt{\frac{3}{2}}\right) & \cdots \\ -\frac{i\sqrt{2}}{2} & 0 & 5 & 0 & \cdots \\ 0 & -i\left(1 + \sqrt{\frac{3}{2}}\right) & 0 & 7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (55)$$

Clearly, this matrix is time independent as required by conservation of energy but it is not diagonal. As discussed, these are not contradictory properties. Splitting the matrix into the sum of a diagonal and a non-diagonal matrix gives:

$$\boldsymbol{H} = \boldsymbol{H_{BJ}} + \boldsymbol{H_{non-diagonal}} \quad (56)$$

The diagonal matrix, as will be discussed, is equivalent to the standard matrix where the elements (following BJ) can be identified with the experimental stationary state energies of the frequency condition. That non-diagonal elements are complex may suggest non-physicality. However, that is not the case. As discussed with the unit model, the endogenous motion generates both stationary and non-stationary behaviour. It is to be expected that the energy-matrix would have elements corresponding to both forms.

Equivalence with Born-Jordan can be established by considering the magnitude of complex conjugate pair elements. Using standard definitions:

$$|q(n,m,t)|^2 = \frac{\hbar}{2\omega}[a(n)^2 + b(n)^2 + a(n)b(n)sin2\omega t] \qquad (56)$$

Taking the mean value over the period of the endogenous motion gives the equivalence:

$$\langle|q(n,m,t)|^2\rangle = \frac{\hbar}{2\omega}[a(n)^2 + b(n)^2] \equiv \frac{\hbar}{2\omega}a(n)^2_{BJ} \equiv |q(n,m)|^2_{BJ} \qquad (57a)$$

$$which\ means\ |q_1(n,m)|^2 + |q_2(n,m)|^2 \equiv |q(n,m)|^2_{BJ} \qquad (57b)$$

Alternatively, relation (57b) can be obtained directly from relation (39), with momentum again defined as the time derivative of position. Combing with condition (51), the position and momentum matrix expressions (54) can then be obtained *by analogy* from the standard BJ matrices.

Born-Jordan proved that energy-conservation and the frequency condition can be established theoretically from the canonical equations of matrix mechanics. This is necessary to connect mathematical solutions with experimental energies. Combining the diagonal condition (41) with the equivalence condition (57a) leads to the equivalence $H_{n,n} \equiv H_{n,n}^{BJ}$ for the specific case of the HO. The diagonal of the energy-matrix (55) can then likewise be identified with experimental energies.

Since the RHS of (57) is also the mean square of the position eigenvalue (53), equivalence can be expressed as:

$$\langle [Q_{evalue}(n \rightleftarrows m,t)]^2 \rangle \ \equiv |q(n,m)|_{BJ}^2 \tag{58a}$$

$$where\ q(n,m)_{BJ} \ \equiv \int_{-\infty}^{+\infty} \emptyset_n^* \, q \emptyset_m dq \tag{58b}$$

Relation (58a) makes more explicit the connection between Heisenberg's measureable-only transition amplitudes and the proposed motion. Schrodinger's fundamental equivalence between matrix and wave mechanics (58b) is also mediated via transition amplitudes, thereby suggesting a possible equivalence between the proposed hidden variables and the wave function as:

$$\oint [Q_{evalue}(n \rightleftarrows m,t)]^2 \, dt \equiv \left| \int_{-\infty}^{+\infty} \emptyset_n^* \, q \emptyset_m dq \right|^2 \tag{59}$$

This relation infers the hypothetical relation:

$$Q_{evalue}(n \rightleftarrows m, t) \equiv \pm[\left|\int_{-\infty}^{+\infty} \emptyset_n^* \, q \emptyset_m dq\right|^2 - q_1(n,m) q_2(n,m) \sin 2\omega t]^{1/2}$$

(60)

If the wave function in (59) is interpreted to express an irreducible randomness, as stated by orthodoxy, there is a mutually-exclusive contradiction with the LHS which is essentially an averaging over neo-classical endogenous paths. If however, the wave function is interpreted statistically, that is, describing properties of distributions, contradiction is averted since the underlying reality can be the same. Further, a statistical interpretation is compatible with both Heisenberg's ensemble postulate and identifying ensemble position and momentum as aggregate quantities. Interestingly, the RHS (the wave mechanics definition) of (59) also refers to a unit transition.

Relation (59) defines a nexus between the wave function and the centre-point of the periodic endogenous motion. While the eigen-path (being classical) is deterministic its centre-point is statistical. This means that randomness and continuity are co-existent yet distinct features. Randomness is attributed to the internal workings of the quantum particle not its space and time movement.

On first impression it may be concluded that the model is diametrically conflicted with the orthodox interpretation of QM. There is however a more nuanced view. Minimalist orthodoxy defines a physical theory to be a self-consistent mathematical logic reproducing experimental outcomes of defined observables. The model meets this minimalism. Departure from orthodoxy is in the realm of foundations which is often seen as *philosophy*. Relation (60) implies an empirical prediction: a mathematically defined path possessing measurability as a mathematical property. Orthodoxy negates the physical reality of a path. There is then at least an in-principle basis for an *empirical* test.

Recent experiments provide qualitative comparison with empirical data. Transitions have been measured in the scale of attoseconds duration [32]. For atomic systems (assuming an average particle speed that of that of light weighted by the fine structure constant) the endogenous motion will be within the size of the atom. A more recent experiment on state-to-state transitions has reported that a completed evolution from ground to excited state is continuous and deterministic while its occurrence is random and discrete [43].

5. Conclusion

This investigation began with identifying the mathematical representation of outcome variables as a possible fatal assumption in Bell inequalities violation alternative to local causality. In this work the consequences of this alternative conclusion are tested in a hidden variables matrix mechanics model of the harmonic oscillator.

Examining the foundational questions of EPR and Bell has (at least in this investigation) led to re-visiting the foundational postulates of Heisenberg and

Born. A number of changes to basic matrix mechanics assumptions are introduced. Heisenberg’s transition amplitudes are re-interpreted as transition paths, thereby negating the non-path postulate. A second initial modification is to replace actually infinite order matrices by quaternions in matrix representation to describe a unit transition between two arbitrary states. Bohr’s assumption that transitions occur instantaneously has been invalidated by experiment, and is rejected.

Apart from these changes, all other physical postulates of Heisenberg and Born are used in an exploratory model. Results of the standard treatment are reproduced.

The model gives an extended mathematical description of continuous endogenous motion in space and time.

An additional model using actually infinite matrices, which then meets Heisenberg’s requirement that the system is to be treated collectively gives the same results. The quaternion unit transition matrices remain embedded in the extended Born-Jordan matrices.

Recent experiments give at least qualitative support for features of the model.

Historically, the issue of quantum foundations was crystallised by EPR and questions on the completeness of the wave function. While the statistical nature of QM is not questioned, the reported eigen-path is not described by the wave function.

For the harmonic oscillator, this analysis shows that a local, non-contextual hidden variables model is capable of reproducing the results of standard QM, is consistent with violation of Bell inequalities, and maintains Bohr-type apparatus-interactive measurement of experimental outcomes without introducing a complex metaphysics. Equivalence with both standard matrix and wave mechanics is identified. Since local causality need not be rejected consistency with relativity is preserved.

Despite the consistency with matrix mechanics, the model nevertheless remains suggestive; conclusions are conditional on a more general treatment including analysis of other quantum systems.

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