

# A Note on Lattice Theory

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## Abstract

Lattice is a partially ordered set with two operations defined on it that satisfy certain conditions. Lattice theory itself is a branch of abstract algebra. In this paper we present solutions to three classical problems in lattice theory. The solutions are by no means novel.

### Theorem 1

Let  $L$  be a lattice. For  $x, y, z$  in  $L$ ,

$$(1) \quad x \cap (y \cup z) \geq (x \cap y) \cup (x \cap z)$$
$$(2) \quad (x \cup y) \cap (x \cup z) \geq x \cup (y \cap z)$$

### Theorem 2

Let  $L$  be a modular lattice such that  $a \geq b$ ,  $a \cup x = b \cup x$ ,  $a \cap x = b \cap x$ . Then  $a = b$ .

### Theorem 3

Let  $L$  be a modular lattice. Suppose

$$(3) \quad x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$$

for some  $x, y, z$  in  $L$ . Then  $z \cap (x \cup y) = (z \cap x) \cup (z \cap y)$ .

*Proof.*

The outline of this proof can be found in [1]. Since

$$(4) \quad z \cap (x \cup y) \geq (z \cap x) \cup (z \cap y)$$

by (1), it follows that

$$(5) \quad [z \cap (x \cup y)] \cup x \geq [(z \cap x) \cup (z \cap y)] \cup x$$

and

$$(6) \quad [z \cap (x \cup y)] \cap x \geq [(z \cap x) \cup (z \cap y)] \cap x.$$

Moreover,

$$\begin{aligned} [(z \cap x) \cup (z \cap y)] \cup x &\geq (z \cap y) \cup x \\ &= (z \cap y) \cup (x \cup x) \\ &= [(z \cap y) \cup x] \cup x \\ &= [x \cup (z \cap y)] \cup x \\ &= [x \cup (y \cap z)] \cup x \\ &= [(x \cup y) \cap (x \cup z)] \cup x \quad \text{since (3)} \\ &\geq [(x \cup y) \cap z] \cup x \\ &= [z \cap (x \cup y)] \cup x. \end{aligned}$$

Therefore

$$(7) \quad [(z \cap x) \cup (z \cap y)] \cup x \geq [z \cap (x \cup y)] \cup x.$$

On the other hand,

$$\begin{aligned}
[(z \cap x) \cup (z \cap y)] \cap x &\geq (z \cap x) \cap x \\
&= z \cap (x \cap x) \\
&= z \cap x \\
&= z \cap [x \cap (x \cup y)] \\
&= z \cap [(x \cup y) \cap x] \\
&= [z \cap (x \cup y)] \cap x.
\end{aligned}$$

Therefore

$$(8) \quad [(z \cap x) \cup (z \cap y)] \cap x \geq [z \cap (x \cup y)] \cap x.$$

From (5) and (7),

$$(9) \quad [z \cap (x \cup y)] \cup x = [(z \cap x) \cup (z \cup y)] \cup x.$$

Similarly, from (6) and (8),

$$(10) \quad [z \cap (x \cup y)] \cap x = [(z \cap x) \cup (z \cup y)] \cap x.$$

To conclude, from (4), (9), (10)

$$z \cap (x \cup y) = (z \cap x) \cup (z \cup y).$$

by Theorem 2.

**Theorem 4**

For  $x, y, z$  in  $L$ ,  $[(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] = x \cap y$ .

*Proof.*

Since  $(x \cap y) \cup (x \cap z) \geq x \cap y$  and  $(x \cap y) \cup (y \cap z) \geq x \cap y$ ,

$$\begin{aligned}
[(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] &\geq (x \cap y) \cap (x \cap y) \\
&= x \cap y.
\end{aligned}$$

Thus

$$(11) \quad [(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] \geq x \cap y.$$

Moreover,

$$\begin{aligned}
x &\geq x \cap [(x \cap y) \cup z] \\
&= [x \cup (x \cap y)] \cap [(x \cap y) \cup z] \\
&= [(x \cap y) \cup x] \cap [(x \cap y) \cup z] \\
&\geq (x \cap y) \cup (x \cap z) \quad \text{by (2)}
\end{aligned}$$

and hence

$$(12) \quad x \geq (x \cap y) \cup (x \cap z).$$

Furthermore,

$$\begin{aligned}
y &\geq y \cap [(x \cap y) \cup z] \\
&= [y \cup (y \cap x)] \cap [(x \cap y) \cup z] \\
&= [(y \cap x) \cup y] \cap [(x \cap y) \cup z] \\
&= [(x \cap y) \cup y] \cap [(x \cap y) \cup z] \\
&\geq (x \cap y) \cup (y \cap z) \quad \text{by (2)}
\end{aligned}$$

and hence

$$(13) \quad y \geq (x \cap y) \cup (y \cap z).$$

From (12) and (13),

$$(14) \quad x \cap y \geq [(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)].$$

From (11) and (14),

$$[(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] = x \cap y.$$

**Theorem 5**

Let  $L$  be a lattice. If

$$(15) \quad (x \cup y) \cap [z \cup (x \cap y)] = (x \cap y) \cup (y \cap z) \cup (z \cap x)$$

for all  $x, y, z$  in  $L$ , then  $L$  is distributive.

*Proof.*

Suppose  $x \geq y$ . Then

$$\begin{aligned} x \cap (z \cup y) &= (x \cup y) \cap [z \cup (x \cap y)] \\ &= (x \cap y) \cup (y \cap z) \cup (z \cap x) && \text{by (15)} \\ &= y \cup (y \cap z) \cup (z \cap x) \\ &= [y \cup (y \cap z)] \cup (z \cap x) \\ &= y \cup (z \cap x). \end{aligned}$$

Therefore

$$(16) \quad x \cap (z \cup y) = y \cup (z \cap x) \quad \text{whenever } x \geq y.$$

To conclude

$$\begin{aligned} (x \cup y) \cap (x \cup z) &= (x \cup y) \cap (z \cup x) \\ &= (x \cup y) \cap \{z \cup [x \cup (x \cap y)]\} \\ &= (x \cup y) \cap \{z \cup [(x \cap y) \cup x]\} \\ &= (x \cup y) \cap \{[z \cup (x \cap y)] \cup x\} \\ &= x \cup \{[z \cup (x \cap y)] \cap (x \cup y)\} && \text{by (16)} \\ &= x \cup \{(x \cup y) \cap [z \cup (x \cap y)]\} \\ &= x \cup [(x \cap y) \cup (y \cap z) \cup (z \cap x)] && \text{since (15)} \\ &= x \cup \{(x \cap y) \cup [(y \cap z) \cup (z \cap x)]\} \\ &= x \cup \{(x \cap y) \cup [(z \cap x) \cup (y \cap z)]\} \\ &= x \cup \{[(x \cap y) \cup (z \cap x)] \cup (y \cap z)\} \\ &= \{x \cup [(x \cap y) \cup (z \cap x)]\} \cup (y \cap z) \\ &= \{[x \cup (x \cap y)] \cup (z \cap x)\} \cup (y \cap z) \\ &= [x \cup (z \cap x)] \cup (y \cap z) \\ &= [x \cup (x \cap z)] \cup (y \cap z) \\ &= x \cup (y \cap z). \end{aligned}$$

**References**

[1] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer-Verlag, London, 2005.  
 [2] D. E. Rutherford, *Introduction to Lattice Theory*, Oliver and Boyd Ltd., Edinburgh, 1965.