Emergence of Lie Groups and Gauge Symmetries from Complex Dynamics

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Abstract

Last decade has seen mounting evidence that *complex dynamics* can shed new light on many open questions of contemporary theoretical physics. Starting from this vantage point, our goal here is to show that Lie groups and the gauge structure of the Standard Model follow from the universal framework of *self-organized criticality* (SOC). In particular, we find that Lie groups and their algebra arise from the flow of spacetime dimensions with the energy scale.

Key words: Lie group, gauge symmetry, Standard Model, Self-Organized Criticality, complex dynamics, Nonequilibrium thermodynamics, minimal fractal manifold.

1. Introduction

When driven far away from thermodynamic equilibrium, complex systems are prone to exhibit *emergent dynamics* stemming from the interplay between nonlinear interaction of components and steady dissipation. As paradigm of this type of emergent behavior, SOC has a vast range of applications extending from astrophysics, natural hazards and magnetospheric physics, to complex networks, internet dynamics, biophysics, and social sciences [1-2].

The relevant observables of the so-called "sandpile" models of SOC relate to the key concept of *avalanche* and include its size s and area a, the avalanche duration t and linear size r. The probability distribution associated with these observables follows the *finite-size scaling* (FSS) ansatz [1].

$$P(\eta, L_\eta) \sim \eta^{-\tau_\eta} \Phi(\eta/\eta_c) \text{ for } \eta >> 1,\ L_\eta >> 1$$

$$\eta_c(L_\eta) \sim L_\eta^{D_\eta} \text{ for } L_\eta >> 1 \tag{1}$$

in which

$$\eta = (s, a, t, r) \tag{2}$$

Here L_η denotes the upper bound of η, whose cutoff value is set by η_c. The parameters τ_η and D_η stand for the avalanche-size exponent and avalanche dimension, respectively, and their specific values determine the *universality* class of the SOC process described by (1). The cutoff function $\Phi(\eta/\eta_c)$ controls the finite-size effects of critical behavior and is defined as

$$\Phi(x) = \begin{cases} \Phi_0 = const;\ |x| >> 1 \\ x^{-\tau_s} \quad ;\ x \to 0 \end{cases} \tag{3}$$

To enable all moments of (1) to exist, the cutoff function must decay sufficiently fast. One obtains the following representation of the cutoff function upon power expanding it around zero,

$$\Phi(x) \sim \begin{cases} \Phi(0) + \Phi'(0)x + \frac{1}{2}\Phi''(0)x^2 + \ldots, & x << 1 \\ \to 0, & x >> 1 \end{cases} \tag{4}$$

With reference to the case $\eta = s$, the avalanche-size probability must be normalized to unity and its average be diverging along with $L \to \infty$, which leads to the following constraints

$$\sum_{s=1}^{\infty} P(s;L) = 1 \qquad \text{for } L < \infty, \tag{5}$$

$$\langle s \rangle = \sum_{s=1}^{\infty} sP(s;L) \to \infty \text{ for } L \to \infty \tag{6}$$

Under the assumption that $\Phi(0) \neq 0$, the behavior of (1) for an infinite system size may be approximated as

$$\lim_{L\to\infty} P(s;L) \sim s^{-\tau_s} \Phi(0) \tag{7}$$

Furthermore, compliance with (5) and (6) requires that the avalanche-size exponent must fall in the range

$$1 < \tau_s \leq 2 \tag{8}$$

The most straightforward analog of $s >> 1$ in the framework of the *minimal fractal manifold* (MFM) is provided by [3-5]

$$s \to \varepsilon^{-1} = (4 - D)^{-1} >> 1 \tag{9}$$

which leads to the following form of the FSS ansatz (1)

$$P(\varepsilon, \varepsilon_c) \sim \varepsilon^{\tau_s} \Phi(\varepsilon_c / \varepsilon) \ , \ \varepsilon << 1 \tag{10a}$$

$$\varepsilon_c(\mu) \sim \mu^{D_S}, \ \mu >> 1 \tag{10b}$$

where μ stands for the running energy scale, expressed in dimensionless form.

2. Lie groups from dimensional flows

In general, the arbitrarily small deviation from four spacetime dimensions $\varepsilon = 4 - D << 1$ may be configured as a large multivariable set that runs with μ as in

$$\varepsilon = f(\mu) \ , \ \ \varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_N) , \ \ N >> 1 \tag{11}$$

Let q denote the order parameter associated with the multifractal description of (1) [4-5]. A notable property of phase-space trajectories is that they can be represented as groups of continuous transformations [6]. In light of this property, the flow equations

$$\frac{d\varepsilon_i(\mu)}{d\mu} = \beta_i(\varepsilon_1(\mu), \varepsilon_2(\mu), ..., \varepsilon_N(\mu), \mu, q) \tag{12}$$

are equivalent to the group of transformations

$$\varepsilon_i(\mu) = f_i(\varepsilon_1(\mu_0), \varepsilon_2(\mu_0), ..., \varepsilon_N(\mu_0), \mu_0, q) \tag{13}$$

Given (12) and (13), the effect of an infinitesimal shift $\varepsilon + d\varepsilon$ on any scalar phase-space function F amounts to $F + dF$, such that

$$\frac{dF}{d\mu} = XF \tag{14}$$

in which the linear operator X is the *infinitesimal generator of the group* and assumes the form

$$X = \sum_1^N \beta_i(\varepsilon) \frac{\partial}{\partial \varepsilon_i} = \sum_1^N \frac{d\varepsilon_i}{d\mu} \frac{\partial}{\partial \varepsilon_i} \tag{15}$$

Considering that $F(\varepsilon(\mu))$ can be expanded in a converging power series and accounting for the invariance of X under the group it generates, one obtains a generic solution of the flow (12) that can be presented as [6]

$$\varepsilon_i(\mu) = \exp(\mu \sum_1^N \beta_i \, \partial / \partial \varepsilon_i) \, \varepsilon_i \tag{16}$$

The formalism outlined above may be extrapolated to the case of two or more flows of the type (13), namely

$$\beta_k^i(\varepsilon) = \frac{\partial \varepsilon_i}{\partial \mu_k}, \quad k = 1, 2, \ldots, M \tag{17}$$

and

$$X_k = \sum_1^N \beta_k^i(\varepsilon) \frac{\partial}{\partial \varepsilon_i} \tag{18}$$

where the infinitesimal group generators satisfy the commutation rules of the Lie algebra

$$\left[X_i, X_j \right] = \varepsilon_{ijk} X_k \tag{19}$$

In general, an arbitrary multivariable flow is driven by the Lie group operator

$$U(\theta) = \exp(\sum_1^N \theta_i X_i) \tag{20}$$

where θ_i represent a set of continuous parameters. As it is well known, (20) underlines the symmetry attributes of *angular rotation, spin, and the special unitary groups* of the

Standard Model. These findings fall in line with the idea that quantum spin emerges as topological signature of the MFM, a point elaborated upon in [3].

3. Orthogonal groups and the "sum-of-squares" relationship

A particular case of interest derived from (13-14) is provided by the *orthogonal group* $O(N)$. Transformations among the elements of $O(N)$, (y_l), $l = 1, 2, \dots, N$ satisfy the norm-conservation requirement

$$\sum_1^N y_l^2 = \sum_1^N (y')_l^2 \tag{21}$$

With reference to (10), analysis indicates that the most likely value of τ_s for quantum physics and random walk models is [3, 7]

$$\tau_s = 2 \tag{22}$$

Consider now the case where (3) is a valid approximation of the FSS ansatz (1). Upon appropriate normalization $\varepsilon_i \to \varepsilon_{R,i}$, by (3), (5) and (21-22), (10a) leads to a condition that mirrors (21), namely

$$\sum_1^N P(\varepsilon, \varepsilon_c) \Rightarrow \sum_1^N \Phi_0 \varepsilon_i^2 = \sum_1^N \varepsilon_{R,i}^2 = 1 \tag{23}$$

A constraint similar to (23) can be shown to recover the Euclidean formulation of Lorentz symmetry and its properties [8].

Dimensional regularization arguments imply that, to a leading-order approximation, the deviation from four spacetime dimensions amounts to $\varepsilon = O\,(m/\Lambda_{UV})^2$, where m and Λ_{UV}

are the particle mass measured near the Fermi scale and the ultraviolet cutoff, respectively. The same line of arguments suggests that the parameter Φ_0 in (23) matches the ratio [3]

$$\Phi_0 = (\frac{\Lambda_{cc}^{1/4}}{M_{EW}})^2 = (\frac{M_{EW}}{M_{Pl}})^2 \tag{24}$$

in which $M_{EW}, \Lambda_{cc}^{1/4}$ and M_{Pl} stand for the Fermi scale, the cosmological constant scale and the Planck scale, respectively. Under these conditions, (23) recovers the so-called *sum-of-squares* relationship linking the square of elementary particle masses to the square of the Fermi scale [3, 7]. The sum-of-squares relationship, along with the requirement of marginal stability of Renormalization Group trajectories, lock in the *flavor structure* of the Standard Model and substantially reduce its number of free parameters [3, 7, 11]. It is also worth noting that the symmetry described by (15) and (18) is naturally associated with *local scale invariance,* which typically underlies the geometry of fractal and multifractal objects [12]

4. Bifurcations and the U(1) x SU(2) => SU(3) transition

As pointed out in [9-10], the so-called Feigenbaum-Sharkovskii-Magnitskii (FSM) paradigm of universal transition to chaos in nonlinear dissipative systems points to the sequential generation of the Standard Model families. According to this scenario, the quartet of electroweak bosons turns into the gluon octet and lepton multiplet into the quark multiplet according to

$$\begin{pmatrix} \gamma & W^- & W^+ & Z^0 \end{pmatrix} \Rightarrow (gluons_{1-8}) \tag{25}$$

$$\left(\nu_e \quad \nu_\mu \quad \nu_\tau \quad e \quad \mu \quad \tau\right) + antiparticles \Rightarrow \left(u \quad d \quad c \quad s \quad b \quad t\right)_{r,g} + antiparticles \qquad (26)$$

Relationships (25-26) show that the dynamical transition $U(1)\times SU(2) \Rightarrow SU(3)$ is a transformation of a *stable cycle of period 4* in the electroweak sector to a *stable cycle of period 8* in the strong sector. Note that there are 12 distinct leptons and 24 distinct colored quarks in (25-26). It is thus reasonable to conjecture that transition of leptons to quarks occurs through a bifurcation leading to a stable cycle of period 24 from a stable cycle of period 12.

Finally, two important remarks are in order:

a) color and electrical charge conservation constrain the number of independent flavors generated through bifurcations. For example, taking R and G to represent independent color states, color conservation prohibits formation of distinct flavors of type B since $R+G+B=1$, by definition.

b) there is a natural mixing of cycles prior to their complete separation through bifurcation. As a result of this mixing, it is conceivable that the transition (25-26) allows leptons and quarks to couple through electroweak gauge bosons while forbiding leptons to couple to gluons.

References

1. Christensen, K. and Moloney, N. R., "Complexity and Criticality", Imperial College Press, 2005.

2. https://arxiv.org/pdf/1310.5527.pdf

3. Available at the following sites:

http://www.aracneeditrice.it/aracneweb/index.php/pubblicazione.html?item=9788854889972

https://www.researchgate.net/publication/278849474_Introduction_to_Fractional_Field_Theory_consolidated_version

4. Available at the following site:

https://www.researchgate.net/publication/340819356_Emergence_of_Lagrangian_Field_Theory_from_Self-Organized_Criticality#fullTextFileContent

5. Available at the following site:

https://www.researchgate.net/publication/341110733_Complex_Dynamics_as_Foundation_of_Relativistic_Spacetime

6. McCauley, J. L., "Chaos, Dynamics and Fractals", Cambridge Univ. Press, 1993.

7. https://www.researchgate.net/publication/333677753_Derivation_of_the_Sum-of-Squares_Relationship

8. https://www.prespacetime.com/index.php/pst/article/view/1447/1381

9. Available at the following site:

https://www.researchgate.net/profile/Ervin_Goldfain/publication/228625050_Chaotic_Dynamics_of_the_Renormalization_Group_Flow_and_Standard_Model_Parameters_Intl/links/54157cf50cf2bb7347db3d38/Chaotic-Dynamics-of-the-Renormalization-Group-Flow-and-Standard-Model-Parameters-Intl.pdf

10. Available at the following site:

https://www.researchgate.net/publication/228354507_Bifurcations_and_pattern_formation_in_particle_physics_a_model_study

11. https://www.sciencedirect.com/science/article/pii/S1007570407000950

12. https://www.sciencedirect.com/science/article/abs/pii/S0960077904002942