Abstract. This note is a proof of a strengthened form of the strong Goldbach conjecture.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1 and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Theorem (Strengthened strong Goldbach conjecture (SSGB)). Every even integer greater than 6 can be expressed as the sum of two different primes.

Proof. We define the set \( S_3 := \{ (p_k, m_k, q_k) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p+q) / 2 \} \).

The whole range of natural numbers starting from 3 can be expressed by the triple components of \( S_3 \), since every integer \( x \geq 3 \) can be written as some \( p_k \) with \( k = 1 \) when \( x \) is prime, as some \( p_k \) with \( k \neq 1 \) when \( x \) is composite and not a power of 2, or as \( (3+5)k / 2 \) when \( x \) is a power of 2, where \( p \in \mathbb{P}_3, k \in \mathbb{N} \).

SSGB is equivalent to saying that every integer \( x \geq 4 \) is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a middle component \( m_k \) of \( S_3 \).

Let us assume \( \neg \)SSGB now. This means that there is at least one \( n \geq 4 \) such that for each fixed \( k \geq 1 \) \( nk \) is different from all the \( m_k \) generated in \( S_3 \). According to the above three types of expression by \( S_3 \) triple components, for any \( n \geq 4 \) given by \( \neg \)SSGB we have

\[
\forall k \in \mathbb{N} \quad \exists (p'_k, m'_k, q'_k) \in S_3 \quad nk = p'_k \lor nk = m'_k = 4k'.
\]

So, every \( nk \) given by \( \neg \)SSGB equals a component of some \( S_3 \) triple that exists by definition. Then, under the assumption \( \neg \)SSGB, the set \( S_3 \) consists of the triples where one of the \( nk \)'s equals one of the three components and of those triples where none of the \( nk \)'s equals a component. This implies that under \( \neg \)SSGB the set \( S_3 \) has the same content as if we do not assume the existence of \( n \). So, in fact there is no such \( n \) and we obtain

\( \neg \)SSGB \( \Rightarrow \) SSGB. This proves the theorem. \( \square \)

Remark. The above shows that the set \( S_3 \) remains the same in the case \( n \) exists and in the case \( n \) does not exist. On the other hand, the numbers \( m \) in the components \( m_k \) take all integer values \( x \geq 4 \) when \( n \) does not exist, whereas at least one value is missing when \( n \) exists. This causes an arithmetic antinomy ².

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2 https://www.academia.edu/38898570/The_Inconsistency_of_Arithmetic