1. Introduction

The sum $\sum_{k=1}^{n} a_{k}b_{n+1-k} = a_{1}b_{n} + a_{2}b_{n-1} + ... + a_{n}b_{1}$, where $n$ are any positive integers, denoted by $R(a_{n}, b_{n})$, are called Reverse Sum of $a_{n}$ and $b_{n}$. Reverse Sum usually appears in Rearrangement Inequality, but not in normal Algebra. Fibonacci Sequence $\{F_{n}\}$ and Lucas Sequence $\{L_{n}\}$ are very similar sequences because they also have recurrence formula, but have $F_{0} = 0$, $F_{1} = 1$ and $L_{0} = 2$, $L_{1} = 0$. Because of that similarity of sequences, we suggest that those sequences can be related as a function of Reverse Sum. In this paper it is shown that $R(F_{n}, L_{n})$ can be written into general form within $\{F_{n}\}$ and some various constants.

2. Preliminaries

These are some important theorems to proof the following main theorem.

**Definition 1:** The sequence $\{F_{n}\}$ is called Fibonacci Sequence if and only if,

$$F_{n} = F_{n-1} + F_{n-2}, F_{0} = 0, F_{1} = 1$$

where $n$ denote integer such that $n \geq 2$.

**Definition 2:** Sequence $\{L_{n}\}$ is called Lucas Sequence if and only if,

$$L_{n} = L_{n-1} + L_{n-2}, L_{0} = 2, L_{1} = 1$$

where $n$ denote integer such that $n \geq 2$.

**Lemma 1:** Consider type IV of recurrence relation $\{a_{n}\}$ like this,

$$a_{n+1} = pa_{n} + qa_{n-1}, n \geq 2, (q \neq 0).$$

Determine $\alpha$ and $\beta$. (3) gives $a_{n+1} = (\alpha + \beta)a_{n} - \alpha\beta a_{n-1}$, so let $\alpha + \beta = p$ and $\alpha\beta = -q$. Thus, $\alpha, \beta$ are the two roots of the quadratic equation $t^{2} - pt - q = 0$, which is called the Characteristic Equation of the given recurrence formula.

(1) $a_{n} = A\alpha^{n} + B\beta^{n}$, if $\alpha \neq \beta$

(2) $a_{n} = (An + B)\alpha^{n}$, if $\alpha = \beta$

where $A, B$ are constants determined by the initial values $a_{1}$ and $a_{2}$.

Now, We start with considering the following Reverse Sum definition

**Definition 3:**
\[ R(F_n, L_n) = F_1L_n + F_2L_{n-1} + \ldots + F_nL_1 \]

We can observe symmetry of the expression such as \( F_1L_n \) and \( F_nL_1 \). Which needed to be established the following lemma;

**Lemma 2:** Suppose Fibonacci Sequence \( \{F_n\} \). Then,

\[
F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]

**Proof:** By Definition 1 gives \( \alpha + \beta = 1 \) and \( \alpha \beta = -1 \) respectively. Considering Characteristic Equation by substituting \( p, q \): \( t^2 - t + 1 = 0 \), gives the solution

\[
\alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}.
\]

That mean \( \alpha \neq \beta \), so

\[
F_n = A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

Because \( F_0 = 0 \) and \( F_1 = 1 \), it is easy to get \( A = \frac{1}{\sqrt{5}} \) and \( B = -\frac{1}{\sqrt{5}} \). Therefore, the lemma have been proved. \( \square \)

**Lemma 3:** Suppose Lucas Sequence \( \{L_n\} \). Then,

\[
L_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]

**Proof:** Claiming Definition 2 gives \( \alpha + \beta = 1 \) and \( \alpha \beta = -1 \) respectively. By similarity of Lemma 2 gives

\[
L_n = A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

From \( L_0 = 2 \) and \( L_1 = 1 \) yield \( A = B = 1 \). Thus, proving have been occured. \( \square \)

**Lemma 4:** Suppose that \( \{F_n\} \) and \( \{L_n\} \) be Fibonacci Sequence and Lucas Sequence. Let \( m \) and \( n \) are positive integer including 0. Then,

\[
F_mL_n + F_nL_m = 2F_{m+n}
\]

**Proof:** Make it easy by assuming \( \Psi_1 = \frac{1 + \sqrt{5}}{2}, \Psi_2 = \frac{1 - \sqrt{5}}{2} \) and \( \eta = \frac{1}{\sqrt{5}} \). Then:

\[
F_mL_n + F_nL_m = (\eta \Psi_1^m - \eta \Psi_2^m)(\Psi_1^n + \Psi_2^n) + (\eta \Psi_1^n - \eta \Psi_2^n)(\Psi_1^m + \Psi_2^m)
= 2\eta \Psi_1^m \Psi_2^n - 2\eta \Psi_1^n \Psi_2^m
= 2(\eta \Psi_1^{m+n} - \eta \Psi_2^{m+n}) = 2F_{m+n} \quad \square
\]

**Definition 4:** Let \( x \) be a real number. \( \lfloor x \rfloor \) denote *Floor Function* or integer part of \( x \). For example, \( \lfloor 5.08 \rfloor = 5 \) and \( \lfloor 7 \rfloor = 7 \).

**Definition 5:** Let \( x \) be a real number. Then

\[
\{x\} = x - \lfloor x \rfloor
\]

where \( \{x\} \) denote *Decimal Part* of \( x \).

**Definition 6:** Let \( a, b \) be integers. \( a \equiv_m b \) meaning \( a - b \) is divisible by \( m \).
3. Main Theorem

By applying all Definitions and Lemma,

**Theorem 1:** Let \( n \) be positive integer including 0. Then:

\[
R(F_n, L_n) = (n + 2\{\frac{n}{2}\})F_{n+1}
\]

**Proof:** We separate the value of \( n \) into 2 classes below,

(1) \( n \equiv_2 0 \) implies

\[
R(F_n, L_n) = F_1L_n + \ldots + F_{\frac{n}{2} - 1}L_{\frac{n}{2} + 1} + F_{\frac{n}{2} + 1}L_{\frac{n}{2} - 1} + \ldots + F_nL_1
\]

from there, using Lemma 4 yields

\[
R(F_n, L_n) = (F_1L_n + F_nL_1) + (F_2L_{n-1} + F_{n-1}L_2) + \ldots + (F_{\frac{n}{2} - 1}L_{\frac{n}{2} + 1} + F_{\frac{n}{2} + 1}L_{\frac{n}{2} - 1}) = \frac{n}{2}(2F_{n+1}) = nF_{n+1}.
\]

(2) \( n \equiv_2 1 \)

so \( n + 1 \equiv_2 1 + 1 \equiv_2 2 \equiv_2 0 \) gives

\[
R(F_n, L_n) = F_1L_n + \ldots + F_{\frac{n+1}{2} - 1}L_{\frac{n+1}{2} + 1} + \ldots + F_nL_1
\]

likewise (1), \( R(F_n, L_n) = (F_1L_n + F_nL_1) + (F_2L_{n-1} + F_{n-1}L_2) + \ldots + F_{\frac{n+1}{2} - 1}L_{\frac{n+1}{2} + 1} \).

Because \( F_{\frac{n+1}{2} - 1}L_{\frac{n+1}{2} + 1} = \frac{1}{2}F_{\frac{n+1}{2}}L_{\frac{n+1}{2} + 1} + \frac{1}{2}F_{\frac{n+1}{2}}L_{\frac{n+1}{2} - 1} = \frac{1}{2}(2F_{n+1}) = F_{n+1} \), so

\[
R(F_n, L_n) = \frac{n}{2}(2F_{n+1}) + F_{n+1} = (n + 1)F_{n+1}.
\]

But you may wonder that where Decimal Part came from.

Because (1) we got \( R(F_n, L_n) = nF_{n+1} = (n + 0)F_{n+1} = (n + 2\{\frac{n}{2}\})F_{n+1} \)

and (2) makes \( R(F_n, L_n) = (n + 1)F_{n+1} = (n + 2\{\frac{n}{2}\})F_{n+1} \)

Then the theorem have been proven. \( \square \)