Non-existence of odd almost perfect numbers

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Abstract

Let $b$ be an odd almost perfect number. Let the prime factors of $b$ which are different from each other be odd primes $p_1, p_2, \ldots, p_r$ and let the exponent of $p_k$ be a positive integer $q_k$. If the product of the series of the prime factors is an odd integer $a$, 

$$a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1)$$

$$b = \prod_{k=1}^{r} p_k^{q_k}$$

If $b$ is an almost perfect number, 

$$a = 2b - 1$$

holds. By a research of this paper, let $a_k$ and $b_k$ be odd integers and $c_k$ be an even integer and the following equations are assumed to hold.

$$a_k = a/(p_k^{q_k} + \cdots + 1)$$

$$b_k = b/p_k^{q_k}$$

$$a_k = c_k p_k - 1$$

When $r \geq 2$, By a proof which uses the product of $a_k/b_k$, in order for $b$ to be an odd almost perfect number the following inequality must be satisfied when $r \geq 2$.

$$b \leq 16/17$$

We have obtained a conclusion that there are no odd almost perfect numbers other than 1 since this inequality does not hold.

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1. Introduction

In mathematics, an almost perfect number (sometimes also called slightly defective or least deficient number) is a natural number $n$ such that the sum of all divisors of $n$ (the sum-of-divisors function $\sigma(n)$) is equal to $2n - 1$. The only known almost perfect numbers are powers of 2 with non-negative exponents (sequence A000079 in the OEIS). Therefore the only known odd almost perfect number is $2^0 = 1$.

(Quoted from Wikipedia)

In this paper, we prove that there are no odd almost perfect numbers other than 1.

2. Proof

Let $b$ be an odd almost perfect number. Let the prime factors of $b$ which are different from each other be odd primes $p_1, p_2, \ldots, p_r$, and let the exponent of $p_k$ be a positive integer $q_k$. If the product of the series of the prime factors is an odd integer $a$,

$$a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1) \quad ①$$

$$b = \prod_{k=1}^{r} p_k^{q_k} \quad ②$$

If $b$ is an almost perfect number,

$$a = 2b - 1 \quad ③$$

holds.

Let $a_k$ and $b_k$ be odd integers,

$a_k = a/(p_k^{q_k} + \cdots + 1)$

$b_k = b/p_k^{q_k}$

$p_k^{q_k} + \cdots + 1$ is odd since $a$ and $a_k$ are odd integers. Thereby, $q_k$ is an even integer for all $k$.

From the equation $③$,

$$a_k(p_k^{q_k} + \cdots + 1) = 2b_k p_k^{q_k} - 1 \quad ④$$

1. When $r = 1$

$p_1^{q_1} + \cdots + 1 = 2p_1^{q_1} - 1$

$1 \equiv -1 \quad (\text{mod } p_1)$

It becomes inconsistent since $p_1 \geq 3$. Therefore, odd almost perfect numbers do not exist when $r = 1$. 

2
When $r \geq 2$

$$p_k^{q_k + \cdots + 1} = (p_k^{q_k+1} - 1)/(p_k - 1) < p_k^{q_k+1}/(p_k - 1)$$

When $p_k \geq 3$,

$$p_k^{q_k + \cdots + 1} < p_k^{q_k+1}/2$$

$$a_k(p_k^{q_k + \cdots + 1}) < a_k p_k^{q_k+1}/2$$

From the equation ④,

$$2b_k p_k^{q_k} - 1 < a_k p_k^{q_k+1}/2$$

Since $p_k \geq 3$ and $b_k p_k^{q_k} \geq 9$,

$$15b_k p_k^{q_k}/8 < a_k p_k^{q_k+1}/2$$

$$a_k/b_k > 15/(4p_k)$$

$$\prod_{k=1}^{r} a_k/b_k < \prod_{k=1}^{r} (15/(4p_k))$$

$$\prod_{k=1}^{r} p_k < (15/4)^{r}/(a/b)^{r-1} \ldots (5)$$

Let $c_k$ be an even integer. From the equation ④,

$$a_k = c_k p_k - 1$$

From the inequality ⑤,

$$\prod_{k=1}^{r} a_k < \prod_{k=1}^{r} c_k p_k < (15/4)^{r} \prod_{k=1}^{r} c_k / (a/b)^{r-1}$$

$$a^{r-1} < (15/4)^{r} \prod_{k=1}^{r} c_k / (a/b)^{r-1} \ldots (6)$$

$$(4a^2/15b)^{r-1} < 15/4 \times \prod_{k=1}^{r} c_k$$

$$(a/4)^{r-1} < (8a^2/15(a + 1))^{r-1} < 15/4 \times \prod_{k=1}^{r} c_k$$

$$a^{r-1} < 15/4 \times \prod_{k=1}^{r} c_k \times 4^{r-1}$$

When the inequality ⑥ holds, this inequality must be satisfied.

$$(15/4)^{r}/(a/b)^{r-1} \geq 15/4 \times 4^{r-1}$$

$$(15/4)/(a/b) \geq 4$$

$$a/b \leq 15/16$$
\[16(2b - 1) \leq 15b\]
\[b \leq 16/17\]
This inequality does not hold obviously when \(r \geq 2\). Therefore, odd almost perfect numbers do not exist when \(r \geq 2\). From the above I and II, there are no odd almost perfect numbers other than 1.

3. Acknowledgement
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4. References
Hiroyuki Kojima "The world is made of prime numbers" Kadokawa Shoten, 2017
Fumio Sairaiji·Kenichi Shimizu "A story that prime is playing" Kodansha, 2015
The Free Encyclopedia Wikipedia
Kouji Takaki "Proof that there are no odd perfect numbers". 2020
Kouji Takaki "Non-existence of odd n-multiperfect numbers". 2020
Kouji Takaki "There are no quasiperfect numbers". 2020