On the Riemann hypothesis

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A proposed proof of the Riemann hypothesis.

1. Introduction

The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(1)

for $\sigma = \text{Re}(s) > 1$. For other values of $s$ it is defined uniquely by analytic continuation, see [1]. The function $\zeta(s)$ has trivial zeros at $s = -2l$ for $l \in \mathbb{N} = \{1, 2, 3, \ldots\}$. It is known that the nontrivial zeros $s = \sigma + it$ of $\zeta(s)$ satisfy the following properties.

I: If $s = \sigma + it$ is a nontrivial zero of $\zeta(s)$ then $s = \sigma - it$ is a nontrivial zero of $\zeta(s)$.

II: If $s = \sigma + it$ is a nontrivial zero of $\zeta(s)$ then $\sigma \in (0, 1)$.

III: If $s = \sigma + it$ is a nontrivial zero of $\zeta(s)$ then $s = 1 - \sigma + it$ is a nontrivial zero of $\zeta(s)$.

2. Proof of the Riemann hypothesis

Theorem

All nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Proof

In light of [2] consider

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log_e(2\pi) - \frac{1}{2} \log_e(1 - x^{-2})$$

(2)

for $x \in (n + 1, n + 2)$ and $n \in \mathbb{N}$. Here $\psi(x)$ is a weighted prime counting function

$$\psi(x) = \sum_{p \leq x} \log_e p$$

(3)

where $p$ is prime and the sum is over all prime powers. The sum in the second term on the right of (2) is over all $\rho$ such that $s = \rho$ is a nontrivial zero of $\zeta(s)$. The exact function $\psi(x)$ is constant on the domain between any two consecutive integers. The approximation of $\psi(x)$ with finitely many $\rho$ values displays a Gibbs phenomenon. Differentiating (2) with respect to $x$ yields

$$\psi'(x) = 1 - \sum_{\rho} x^{\rho - 1} - \frac{1}{x^3 - x}.$$  (4)

Differentiating (2) twice with respect to $x$ yields

$$\psi''(x) = - \sum_{\rho} (\rho - 1)x^{\rho - 2} + \frac{3x^2 - 1}{(x^3 - x)^2}.$$  (5)

Rearranging (5) yields

$$\psi''(x)(x^3 - x)^2 = - \sum_{\rho} (\rho - 1)x^\rho(x^2 - 1)^2 + 3x^2 - 1.$$  (6)
Now
\[ \sum_{\rho} (\rho - 1)x^\rho = \sum_\beta \sum_\gamma (\beta + iy - 1)x^{\beta + iy}. \] (7)

On using Euler’s identity
\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) \] (8)
equation (7) becomes
\[ \sum_{\rho} (\rho - 1)x^\rho = \sum_\beta \sum_\gamma (\beta + i\gamma - 1)x^{\beta + i\gamma}[\cos(\gamma \log_e x) + i \sin(\gamma \log_e x)] \] (9)

which expands to
\[ \sum_{\rho} (\rho - 1)x^\rho = \sum_\beta \sum_\gamma x^\beta[\cos(\gamma \log_e x)(\beta - 1) - \sin(\gamma \log_e x)\gamma] \]
\[ + i \sum_\beta \sum_\gamma x^\beta[\sin(\gamma \log_e x)(\beta - 1) + \cos(\gamma \log_e x)\gamma]. \] (10)
The second term on the right of (10) disappears due to I. Then (10) becomes
\[ \sum_{\rho} (\rho - 1)x^\rho = \sum_\beta \sum_\gamma x^\beta[\cos(\gamma \log_e x)(\beta - 1) - \sin(\gamma \log_e x)\gamma]. \] (11)

Equation (6) is then
\[ \psi''(x)(x^3 - x)^2 = - \sum_\beta \sum_\gamma x^\beta[\cos(\gamma \log_e x)(\beta - 1) - \sin(\gamma \log_e x)\gamma](x^2 - 1)^2 + 3x^2 - 1. \] (12)

Let \( x = y + c \) where \( 0 \leq y \ll 1 \) and \( c \) is a constant such that \( x \in (n + 1, n + 2) \). Then (12) implies
\[ \psi''(y + c)[(y + c)^3 - (y + c)]^2 = - \sum_\beta \sum_\gamma (y + c)^\beta[\cos(\gamma \log_e(y + c))(\beta - 1) \]
\[ - \sin(\gamma \log_e(y + c))\gamma][(y + c)^2 - 1]^2 + 3(y + c)^2 - 1. \] (13)

On using a Taylor expansion (13) becomes
\[ \psi''(y + c)[(y + c)^3 - (y + c)]^2 = - \sum_\beta \sum_\gamma (y + c)^\beta[\cos(\gamma \log_e c)(\beta - 1) - \sin(\gamma \log_e c)\gamma \]
\[ + [- \sin(\gamma \log_e c)\frac{\gamma}{c}(\beta - 1) - \cos(\gamma \log_e c)\frac{\gamma^2}{c}](y + O(y^2))][(y + c)^2 - 1]^2 + 3(y + c)^2 - 1. \] (14)

Equating like coefficients of \((y + c)\) in (14), for \( \beta \in (0, 1) \) in accordance with II, yields a linear polynomial equation for \( \beta \). Therefore only one \( \beta \) value is possible and it is \( \beta = \frac{1}{2} \) by III. Therefore the Riemann hypothesis is true. □

References