There are no quasiperfect numbers

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Abstract

Let \( b \) be a quasiperfect number. Let the prime factors of \( b \) which are different from each other be primes \( p_1, p_2, \ldots, p_r \) and let the exponent of \( p_k \) be a positive integer \( q_k \). If the product of the series of the prime factors is an odd integer \( a \),

\[
a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1)
\]

\[
b = \prod_{k=1}^{r} p_k^{q_k}
\]

If \( b \) is a quasiperfect number,

\[
a = 2b + 1
\]

holds. By a research of this paper, let \( a_k \) be an odd integer, \( b_k \) be an integer and \( c_k \) be an even integer and the following equations are assumed to hold.

\[
a_k = a / (p_k^{q_k} + \cdots + 1)
\]

\[
b_k = b / p_k^{q_k}
\]

\[
a_k = c_k p_k + 1
\]

When \( r \geq 2 \), By a proof which uses the product of \( a_k / b_k \), in order for \( b \) to be a quasiperfect number the following expression must be satisfied when \( r \geq 2 \).

\[
b = 1/2
\]

We have obtained a conclusion that there are no quasiperfect numbers since this expression does not hold.

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1. Introduction
In mathematics, a quasiperfect number is a natural number \( n \) for which the sum of all its divisors (the divisor function \( \sigma(n) \)) is equal to \( 2n + 1 \). (Quoted from Wikipedia)

In this paper, we prove that there are no quasiperfect numbers.

2. Proof
Let \( b \) be a quasiperfect number. Let the prime factors of \( b \) which are different from each other be primes \( p_1,p_2,\ldots,p_r \) and let the exponent of \( p_k \) be a positive integer \( q_k \).

If the product of the series of the prime factors is an odd integer \( a \),
\[
a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1) \quad \ldots \quad ①
\]
\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \ldots \quad ②
\]

If \( b \) is a quasiperfect number,
\[
a = 2b + 1 \quad \ldots \quad ③
\]
holds.

Let \( a_k \) be an odd integer and \( b_k \) be an integer,
\[
a_k = \frac{a}{(p_k^{q_k} + \cdots + 1)}
\]
\[
b_k = \frac{b}{p_k^{q_k}}
\]
\( p_k^{q_k} + \cdots + 1 \) is odd since \( a \) and \( a_k \) are odd integers. Thereby, \( q_k \) is an even integer for all \( k \).

From the equation ③,
\[
a_k(p_k^{q_k} + \cdots + 1) = 2b_k p_k^{q_k} + 1 \quad \ldots \quad ④
\]

1. When \( r = 1 \)
\[
p_1^{q_1} + \cdots + 1 = 2p_1^{q_1} + 1
\]
\[
p_1^{q_1-1} + \cdots + 1 = 2p_1^{q_1-1}
\]
\( 1 \equiv 0 \pmod{p_1} \)

It becomes inconsistent. Therefore, quasiperfect numbers do not exist when \( r = 1 \).
II. When \( r \geq 2 \)
\[
p_k q^k + \cdots + 1 = (p_k q^k + 1)/(p_k - 1) < p_k q^k + 1/\left(p_k - 1\right)
\]

When \( p_k = 2 \),
\[
p_k q^k + \cdots + 1 < p_k q^k + 1
\]
\[
a_k(p_k q^k + \cdots + 1) < a_k p_k q^k + 1
\]
From the equation (4),
\[
2b_k p_k q^k + 1 < a_k p_k q^k + 1
\]
\[
a_k/b_k > 2/p_k
\]

When \( p_k \geq 3 \),
\[
p_k q^k + \cdots + 1 < p_k q^k + 1/2
\]
In the same way,
\[
a_k/b_k > 4/p_k
\]

If \( p_1 \geq 2 \),
\[
\prod_{k=1}^{r} \left(a_k/b_k\right) > (2/p_1) \prod_{k=2}^{r} \left(4/p_k\right)
\]
\[
(a/b)^{r-1} > 2^{2r-1} / \prod_{k=1}^{r} p_k
\]
\[
\prod_{k=1}^{r} p_k > 2^{2r-1}/(a/b)^{r-1} \quad \text{... (5)}
\]

Let \( c_k \) be an even integer. From the equation (4),
\[
a_k = c_k p_k + 1
\]

From the inequality (5),
\[
\prod_{k=1}^{r} a_k > \prod_{k=1}^{r} c_k p_k > 2^{2r-1} \prod_{k=1}^{r} c_k / (a/b)^{r-1}
\]
\[
a^{r-1} > 2\prod_{k=1}^{r} c_k \times 4^{r-1}/(a/b)^{r-1} \quad \text{... (6)}
\]
\[
(a^2/(4b))^{r-1} > 2\prod_{k=1}^{r} c_k
\]
\[(2b + 1)^2 / (4b)r^{-1} > 2 \prod_{k=1}^{r} c_k\]

\[(2\sqrt{2} \times b)^2 / (4b) > (2 \prod_{k=1}^{r} c_k)^{1/(r-1)}\]

\[a > 2b > (2 \prod_{k=1}^{r} c_k)^{1/(r-1)}\]

\[a^{r-1} > 2 \prod_{k=1}^{r} c_k\]

From the inequality 6, a set A and a set B each having \(a\) as an element are defined under the following conditions.

\[A: a^{r-1} > 2 \prod_{k=1}^{r} c_k \times 4^{r-1}/(a/b)^{r-1}\]

\[B: a^{r-1} > 2 \prod_{k=1}^{r} c_k\]

Since \(A \Rightarrow B\), \(A \subseteq B\) holds. On the other hand \(A \supseteq B\) holds because \(B \land \neg A = \varnothing\) must be hold. Therefore, \(A = B\) must be satisfied.

\[4^{r-1}/(a/b)^{r-1} = 1\]

\[a/b = 4\]

\[2b + 1 = 4b\]

\[b = 1/2\]

This expression does not hold obviously when \(r \geq 2\). Therefore, quasiperfect numbers do not exist when \(r \geq 2\). From the above I and II, there are no quasiperfect numbers.

3. Acknowledgement

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