Non-existence of odd harmonic divisor numbers

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Abstract

Let \( n \) be a positive integer and let \( b \) be an odd harmonic divisor number. Let the prime factors of \( b \) which are different from each other be odd primes \( p_1, p_2, \ldots, p_r \) and let the exponent of \( p_k \) be a positive integer \( q_k \). If the product of the series of the prime factors is an integer \( a \),

\[
a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1)
\]

\[
b = \prod_{k=1}^{r} p_k^{q_k}
\]

If \( b \) is a harmonic divisor number, let \( m \) be an integer,

\[
m = \prod_{k=1}^{r} (q_k + 1)
\]

\[an = bm\]

holds. By a research of this paper, let \( a_k \) be an integer and \( b_k \) be an odd integer and if

\[
a_k = a/(p_k^{q_k} + \cdots + 1)
\]

\[
b_k = b/p_k^{q_k}
\]

holds, when \( r \geq 3 \), by a proof which uses the prime factors and the greatest common divisor (GCD) \( C_k \) included in \( b_k \) and \( p_k^{q_k} + \cdots + 1 \), we found that it becomes a contradiction when \( C_k < b_k \) since a least one prime number exists only in the denominator on the left side and it does not in the denominator on the right side. When \( C_k = b_k \), we found that it becomes inconsistent. We have obtained a conclusion that there are no odd harmonic divisor numbers other than 1.

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1. Introduction

In mathematics, a harmonic divisor number, or Ore number (named after Øystein Ore who defined it in 1948), is a positive integer whose divisors have a harmonic mean that is an integer. For example, the harmonic divisor number 6 has the four divisors 1, 2, 3, and 6. Their harmonic mean is an integer:

\[
\frac{4}{(1 + 1/2 + 1/3 + 1/6)} = 2
\]

(Quoted from Wikipedia)

In this paper, we prove that there are no odd harmonic divisor numbers other than 1.

2. Proof

Let \( n \) be a positive integer and let \( b \) be an odd harmonic divisor number. Let the prime factors of \( b \) which are different from each other be odd primes \( p_1, p_2, \ldots, p_r \) and let the exponent of \( p_k \) be a positive integer \( q_k \). If the product of the series of the prime factors is an integer \( a \),

\[
a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1) \quad \text{①}
\]

\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \text{②}
\]

If \( b \) is a harmonic divisor number, let \( m \) be an integer,

\[
m = \prod_{k=1}^{r} (q_k + 1)
\]

\[
an = bm \quad \text{③}
\]

holds. Divide \( m \) and \( n \) by the greatest common divisor and assume that they are relatively prime. Even if this calculation is performed, generality is not lost.

Let \( a_k \) be an integer and \( b_k \) be an odd integer,

\[
a_k = a/(p_k^{q_k} + \cdots + 1)
\]

\[
b_k = b/p_k^{q_k}
\]

From the expression ③,

\[
a_n(p_k^{q_k} + \cdots + 1) = mb_kp_k^{q_k} \quad \text{④}
\]
I. When \( r = 1 \)

\[ n(p_1q_1 + \cdots + 1) = (q_1 + 1)p_1q_1 \]

Let \( n' \) be an integer and if \( n = n'p_1q_1 \) holds,

\[ n'(p_1q_1 + \cdots + 1) = q_1 + 1 \]

Since \( n' \geq 1 \),

\[ (q_1 + 1)/(p_1q_1 + \cdots + 1) \geq 1 \]

\[ q_1 + 1 \geq p_1q_1 + \cdots + 1 \geq p_1q_1 + 1 \]

\[ q_1 \geq p_1q_1 \]

When \( q_1 \geq 1 \) and \( p_1 \geq 3 \), this inequality does not hold obviously. Therefore, odd harmonic divisor numbers do not exist when \( r = 1 \).

II. When \( r \geq 2 \)

From the equation \( ④ \),

\[ na_k(p_k^{q_k+1} - 1) = mb_kp_k^{q_k}(p_k - 1) \]

\[ ((na_k - mb_k)p_k + mb_k)p_k^{q_k} = na_k \]

Since \( na_k/p_k^{q_k} \) is an integer, let \( c_k \) be an integer.

\[ ((na_k - mb_k)p_k + mb_k) = na_k/p_k^{q_k} = c_k \]

When \( p_k \geq 3 \),

\[ p_k^{q_k-1} + \cdots + 1 = (p_k^{q_k} - 1)/(p_k - 1) < p_k^{q_k}/2 \]

From the equation \( ④ \),

\[ mb_k - na_k = c_k(p_k^{q_k} + \cdots + 1) - c_kp_k^{q_k} = c_k(p_k^{q_k-1} + \cdots + 1) \]

\[ mb_k - na_k < c_kp_k^{q_k}/2 = na_k/2 \]

\[ mb_k < 3na_k/2 \]

\[ a_k/b_k > 2m/(3n) > 2/3 \quad ⑤ \]

When \( r = 2 \),

\[ a_1 = p_2^{q_2} + \cdots + 1 \]

\[ b_1 = p_2^{q_2} \]

\[ a_1/b_1 = (p_2^{q_2} + \cdots + 1)/p_2^{q_2} = (p_2^{q_2+1} - 1)/(p_2^{q_2}(p_2 - 1)) < p_2/(p_2 - 1) \]

If \( p_1 < p_2 \), since \( p_2 \geq 5 \) holds,

\[ a_1/b_1 < 5/4 \]

This inequality contradicts the inequality \( ⑤ \). Therefore, there are no odd harmonic divisor numbers when \( r = 2 \).
III. When \( r \geq 3 \)

From the equation (6),
\[
\frac{n a_k}{b_k} = m p_k q_k / (p_k q_k + \cdots + 1) \quad \cdots (6)
\]

When \( m \) is divided by \( p_k q_k + \cdots + 1 \), let \( m' \) be an integer,
\[
m' = m p_k q_k / (p_k q_k + \cdots + 1)
\]
\[
a_k = m' / n \times b_k
\]
hold.

The equation (6) is an equation for obtaining \( m'/n \)-multiperfect numbers. When \( m \) is divisible by \( p_k q_k + \cdots + 1 \) for plural \( q_k \) with the same \( p_k \), \( m \) must be divided by the number of one of \( q_k \). By repeating this operation, it is possible to prevent \( m \) from being divisible by \( p_k q_k + \cdots + 1 \) for all \( k \).

A case where \( m \) cannot be divided by \( p_k q_k + \cdots + 1 \) for all \( k \) is considered. At this time, the right side is not an integer. \( p_k q_k + \cdots + 1 \) is the product of the prime factors \( p_1 \) to \( p_r \) excluding \( p_k \) and the prime factors of \( m \). Let \( C_k \) be the greatest common divisor (GCD) of the denominators on both sides. When the denominator on both sides are divided by \( C_k \), if \( m C_k \) becomes a multiple of the denominator on the right side, let \( s_k \) be an integer,
\[
m C_k = s_k (p_k q_k + \cdots + 1)
\]
this equation is assumed to be hold, the value of the left side of the equation (6) is \( s_k p_k q_k / C_k \). If this value is assumed to be an integer, \( s_k \) is a multiple of \( C_k \) since \( p_k \) does not exist as the prime factor of \( C_k \). However, this contradicts the condition that \( m \) is not divided by \( p_k q_k + \cdots + 1 \). Therefore, when \( C_k \) is transposed from the denominator on the left side to the right side, the right side does not become an integer.

Let \( P_k \) be an odd integer and \( P_k = b_k / C_k \) holds. When \( b_k > C_k \), if the numerator on the left side is a multiple of \( P_k \), it becomes contradiction since the left side is an integer and the right side is not. Therefore, when the left side is reduced, at least one of the prime factors \( p_{sk} \) of \( P_k \) remains in the denominator. At this time, it becomes inconsistent since the prime number \( p_{sk} \) does not exist in the denominator on the right side.
When \( b_k = c_k \) and the denominators on both sides are divided by \( b_k \), a contradiction arises from the above proof since the left side is an integer and the right side is not. Therefore, there are no odd harmonic divisor numbers when \( r \geq 3 \).

From the above I, II, and III, there are no odd harmonic divisor numbers other than 1.

3. Acknowledgement
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