Trigonometry with nested radicals

We will see what we can do with these functions

\[
\begin{align*}
\sqrt{2 - \sqrt{2_1 \pm \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n}}}}} &= 2 \sin \left( \frac{90^{\circ} (2a + 1)}{2^n} \right) \\
\sqrt{2 + \sqrt{2_1 \pm \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n}}}}} &= 2 \cos \left( \frac{90^{\circ} (2a + 1)}{2^n} \right)
\end{align*}
\]

where

\[n = 1,2 \Rightarrow a = 0\]
\[n \geq 2 \Rightarrow 0 \leq a \leq 2^{n-2} - 1\]

If \(n, a\) are known then the signs \(S_k = \pm 1\) are given by relation

\[S_k = (-1)^{\text{round}(a/2^n-k)}, \quad k = 2, 3, \ldots, n - 1\]

If every sign \(S_k\) is replaced with the \(d_k\) digit according to the

\[d_k = (1 - S_k) / 2 \quad (+ = 0, - = 1)\]

relation, then the binary representation of a number

\[b = (d_2 d_3 \ldots d_{n-1})_{(2)}\]

which is closely associated with the \(a\) number will be formed:

- The binary representations of the \(a, b\) numbers always have the same number of digits.
- The numbers \(a, b\) are linked to one another one-by-one, regardless of the value of \(n\).

The following table shows the characteristic matching pattern in the area of the four-digit binary numbers (8-15). There are two ways of transferring groups of numbers, both cross-sectional and parallel. For example, if \(a = 11\) then \(b = 14\) (and vice versa).

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
</tr>
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<tbody>
<tr>
<td>08</td>
<td>12</td>
</tr>
<tr>
<td>09</td>
<td>13</td>
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<tr>
<td>10</td>
<td>14</td>
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<td>11</td>
<td>15</td>
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<td>12</td>
<td>08</td>
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<tr>
<td>13</td>
<td>09</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>11</td>
</tr>
</tbody>
</table>
Example

In practice, calculation of the $S_k$ sign is very easy and can be done without the help of a computer.

For example, if $n = 6$ and $a = 12$ then we will have the following equation of signs:

$$\sqrt{2_1 - \sqrt{2_2 \pm \sqrt{2_3 \pm \sqrt{2_4 \pm \sqrt{2_5 \pm \sqrt{2_6}}}}} = 2 \sin\left(\frac{90^\circ \cdot (2 \cdot 12 + 1)}{2^6}\right)$$

In order to determine the unknown signs, we first divide the $a$ with the numbers $2^{n-2} \cdot 2^{n-3} \ldots 2$ in this order, and we mark under each fraction the quotient rounded to the nearest integer. If this is an even number then you write under the fraction $+$, otherwise you put $-$. Thus, the following table is formed.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>12</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td></td>
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<tr>
<td>−</td>
<td>+</td>
<td>−</td>
<td>+</td>
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</tr>
</tbody>
</table>

So the solution is

$$\sqrt{2_1 - \sqrt{2_2 - \sqrt{2_3 + \sqrt{2_4 - \sqrt{2_5 + \sqrt{2_6}}}}} = 2 \sin\left(\frac{90^\circ \cdot 25}{64}\right)$$

As shown in the previous matching table, it will be $b = 1010_2 = 10_{10}$.

Algorithm for constructing a radical function

With the following algorithm you can construct a radical step-by-step function by inserting the signs $s$ that follow the $2^2$ term. After each insertion you can see how the values of $n, a$ and angle $\omega = 90^\circ \cdot c / 2^n$ are modeled, where

$$c = 2a + 1 = 2^0 \pm 2^1 \pm 2^2 \ldots \pm 2^{n-2}$$
The auxiliary parameter $t$ is dependent on $s$ and takes values 0 and 1.

**Generalization**

We will now extend the radical function so that we can include more general trigonometric terms in it. The following results are not fully proven, so caution is needed!

$$\sqrt{2_1 - \sqrt{2_2 \pm \sqrt{2_3 \pm \cdots \pm \sqrt{2_n \pm 2f(r)}}}} = 2 \sin \omega$$

$$\sqrt{2_1 + \sqrt{2_2 \pm \sqrt{2_3 \pm \cdots \pm \sqrt{2_n \pm 2f(r)}}}} = 2 \cos \omega$$

where

$$f(r) = \sin r, \quad -90^\circ \leq r \leq 90^\circ \text{ or }$$

$$f(r) = \cos r, \quad 0^\circ \leq r \leq 180^\circ$$

$r \in \mathbb{R}$

and

$$\omega = \frac{45^\circ (2a + 1) + (-1)^a (45^\circ - r)}{2^n} \quad \text{if } f(r) = \sin r$$

$$\omega = \frac{45^\circ (2a + 1) - (-1)^a (45^\circ - r)}{2^n} \quad \text{if } f(r) = \cos r$$

where

$$n \geq 2, \quad 0 \leq a \leq 2^{n-1} - 1$$

for which the signs $S_k$ are computed by the relation

$$S_k = (-1)^{\text{round}(a/2^{n-k})}, \quad k = 1, 2, \ldots, n - 1$$
Equations of special form

\[ x = \sqrt{2_1 \pm \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n \pm x}}} \]

In this equation, \( x \) is unknown and all the signs \( S_t \) for \( t = 0, 1, 2, \ldots, n-1 \) are known, with \( n \geq 1 \). We first find the value of an integer \( a \) through the following algorithm

\[ r = \frac{45\degree (2a + 1 - (-1)^a S_0)}{2^n - (-1)^a S_0} \]

Then,

if \( S_0 = +1 \) the solution will be in the form \( x = 2 \cos r \)

if \( S_0 = -1 \) the solution will be in the form \( x = 2 \sin r \)

where

\[ r = \frac{45 \degree (2a + 1 - (-1)^a S_0)}{2^n - (-1)^a S_0} \]

If the equation is

\[ 2 \sin r = \sqrt{2_1 - \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n \pm 2 \sin r}}} \]

where \( r, n \) are known and the signs unknown, then we first determine the integer \( a \) that verifies equality

\[ r = \frac{45 \degree (2a + 1 + (-1)^a)}{2^n + (-1)^a}, \quad 0 \leq a \leq 2^n - 1 \]

so the signs \( S_t \) are taken through the relationship

\[ S_t = (-1)^{\text{round}(a/2^{n-t})}, \quad t = 1, 2, \ldots, n-1 \]

The same procedure is followed to solve the above equation if in it we replace \( \sin r \) with \( \cos r \), except that \( a \) is determined by the relation

\[ r = \frac{45 \degree (2a + 1 - (-1)^a)}{2^n - (-1)^a}, \quad 0 \leq a \leq 2^n - 1 \]

If we have a solution, then we can have infinite of them. For example, let's look at equality
The $101_2 = 5$ motif following $2_2$ corresponds to $a = 6$. If we insert any number of positive signs between the terms $2_2$ and $2_{n-2}$, the value of $b$ will not change (because $00 \ldots 00101_2 = 101_2 = 5 = \text{fixed}$), so the same will apply to the $a$ value. That is, it will be

$$2 \sin \left( \frac{630^\circ}{2^4 + 1} \right) = \sqrt{2_1 - \sqrt{2_2 - \sqrt{2_3 + \sqrt{2_4 - 2 \sin \left( \frac{630^\circ}{2^4 + 1} \right)}}}$$