Spatial-temporal Julia type structures in quantum boundary problems

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Abstract

An initial boundary value problem to a system of linear Schrödinger equations with nonlinear boundary conditions is considered. It is shown that attractor of the problem lies on circles in complex plane. Trajectories tend to fixed points of hyperbolic type with unstable manifold which is formed by saddle points of codimension one. Each element of the attractor are periodic piecewise constant function on pase and amplitude of a wave function in WKB -approximation with finite or infinite points of discontinuities on a period of the Julia type. More exactly, it has been obtained limit solutions of the problem which with accuracy $O(h^2)$ match the exact attractor of the boundary problem, which is independent on $h > 0$ in the zero WKB - approximation. The presented mathematical result are applied to the study of dynamics of two charged particles with opposite impulses, which are confined by two flat walls with surface potentials of double-well type. It is shown that asymptotic behaviour of particles is similar to the behaviour of orbits that arise to well-known logistic map in complex plane. As example, there exist limit periodic nearly piecewise constant distributions of wave functions of Mandelbrot type with Julia type points of 'jumps' for amplitudes and phases of given free charged particles in a confined box with surface nonlinear double-well potential at walls in magnetic field.

Keywords: Shrödinger equations • Gamilton-Jacobi equation • transport equation • initial value boundary problem • asymptotic solutions of relaxation type • periodic piecewise constant distributions • system of difference equations • system of integro-difference equations of the Volterra type • attractor.

1 Introduction

In this paper, asymptotic solutions of a system of linear 'identical' Shrödinger equations with functional nonlinear boundary conditions, and special initial conditions will be considered. Such solutions may be described as $\varphi(x,t)e^{iS(x,t)/h}$, where φ are amplitudes, S are phases,

$h > 0$ is a small parameter. The solutions can be find with accuracy $O(h^2)$, where $O(h^2) \to 0$ as $t \to +\infty$. It is shown that Shrödinger equations can be reduced to a canonical system of Hamilton-Jacobi and transport equations. And the boundary conditions for wave functions can be reduced to the corresponding boundary conditions for the classic canonical system (see, for example, [38, 39]). Initial problem for the canonical system of equations has been considered by Maslov in [38].

It will be proved by method of characteristic that for potential $H(p) = p^2/2$, where p is impulse, solutions have the form

$$\psi(x,t) \to f(t \pm x/p) \quad S(x,t) \to g(t \pm x/p) \tag{1}$$

where $f, g \in R^2$ are vector-functions, ψ is the amplitude and S is the real phase. This observation allows to use the reduction of hyperbolic equation to system of difference equations with continuous time which have piecewise constant periodic solutions if time tends to infinity (see, [11].

To be more concrete, let us consider two 'identical' free particles with Hamiltonian $H_{1,2} = \frac{p_{1,2}^2}{2}$ and impulses $(p_1 = p, p_2 = -p)$. Then the dynamics of these particle satisfies to Hamiltonian equations

$$\dot{q} = H_p, \quad \dot{p} = -H_q \tag{2}$$

where $H = H_1$ or $H = H_2$. If $H_q = 0$ and $H_p = p$ then particles are placed on straight lines $dx/dt = \pm p$ that motivate the form of traveling waves (1). Next, from mechanics it is known that phases of particles can be written as

$$\frac{dS}{dt}[x(t),t)] = \frac{\partial H}{\partial t}[(x(t),t)] + \frac{\partial S}{\partial x}[(x(t),t]\dot{x} \tag{3}$$

along these characteristic $dx(t)/dt = \pm p$. Further, that the phase satisfies to the Hamilton-Jacobi equation

$$\frac{\partial S_{1,2}}{\partial t} + \frac{p_{1,2}^2}{2} = 0 \quad \text{where} \quad \frac{\partial S_{1,2}}{\partial x} = p_{1,2} \tag{4}$$

along trajectories of the Hamiltonian ODE equation, so that

$$dS_k = -\frac{p_k^2}{2}dt \pm pdt, \; k = 1, 2. \tag{5}$$

Formula (5) will be used to prorogate the phase from a boundary into volume of motion of free particle. The similar method will be use to prorogate the amplitude into bulk for the transport equation of the canonical system.

As example, let us consider 'scattering phase' boundary conditions

$$S_1(0,t) = \Phi_1[S_2(0,t)], \quad S_2(l,t) = \Phi_2[S_1(l,t)], \; t > 0, \tag{6}$$

where $\Phi_1, \Phi_2 \in C^2(I, I)$ for some open bounded interval I. For example, in [42] a device based on a Q-switched oscillator with two nonlinear delayed feedback loops has been propose. It is shown that due to the appropriate phase transformation of the signal, which generate each successive pulse, the phase difference between the two neighboring pulses evolves according to the Bernoulli doubling map. It means that in boundary conditions (6) we can put $\Phi_1 : S \to S$

and $\Phi_2 : S \to 2S$. From [42] it follows that there is a hyperbolic chaotic attractor, which produce a robust, structurally stable chaos. There is possible experimental implementations of the scheme. Boundary conditions (6) permit to find the phase separately from amplitudes, and then to fined amplitudes from the transport equation, where the phase is the known asymptotically periodic piecewise linear function.

Further, using the method of characteristic and relation (5) with help of the boundary conditions we arrive at

$$S_1(0,t) = \Phi_1[S_2(0,t)] = \Phi_1[S_2(0,t)] = \tag{7}$$

$$\Phi_1[S_2(0,t-l/p)] - pl/2 - l] = \Phi_1 \circ [\Phi_2(S_1(0,t) - pl/2 + l)) - pl/2 - l]. \tag{8}$$

Next, we assume for simplicity that $Phi_1 := Id$, where Id is identical map. Then

$$S_2(l,t) = \Phi_2[S_1(l,t)] = \tag{9}$$

$$\Phi_2\big[S_1(0,t-l/p) - \frac{1}{2}p^2\frac{l}{p} + pl\big] = \tag{10}$$

$$\Phi_2\big[\Phi_1(S_2(0,t-l/p))\frac{pl}{2} + pl\big] = \tag{11}$$

$$\Phi_2\big[\Phi_1(S_2(l,t-2l/p) - \frac{1}{2}p^2\frac{l}{p} - pl) + pl\big]. \tag{12}$$

Then from (9) it follows that

$$\Leftrightarrow S_2(l,t) = \Phi_2\big[S_2(l,t-2l/p) + \frac{lp}{2}\big]. \tag{13}$$

Defining $S_2 \to S_2 + \mu$, from (14) we derive at

$$S_2(l,t) = \Phi_2\big[S_2(l,t-2l/p)\big] + \mu, \tag{14}$$

where $\mu = \frac{lp}{2}$. Thus, we obtain the difference equation with continuous time solutions of which can be find by iterations, step by step, if we know an initial function $S_2(l,t)$ for $t \in [-l/p, 0)$. In typical situations, solutions tend to piecewise constant periodic functions with finite, countable or uncountable points of discontinuities on periods [11] at $x = l$. On any points (x,t) solutions can be prolonged along characteristic $dx/dt = -p$. The phase S_2 can be find from the boundary condition $S_2(l,t) = \Phi_2[S_2(l,t)]$, and then $S_2(l,t)$ can be prolonged in the volume along characteristic $dx/dt = p$. Points of discontinuities Γ of S_1 and S_2 match and ones are propagating along characteristic $dx/dt = \pm$, respectively. The set Γ of characteristic is finite or infinite (countable or uncountable, homeomorphic to the Cantor set). We call such limit distributions of phases and amplitudes by asymptotic solutions of relaxation, pre-turbulent or turbulent type, respectively. Thus, $S_2(l,t) \to p_2(t) \in P^+(\Phi_2)$, where $P^+(\Phi_2)$ is a set of attractive circles of fixed points the map $\Phi_2 : I \to I, S_2(l,t) = \Phi_2[S_1(l,t)]$, and hence $S_1(l,t) \to (\Phi_2)^{-1}(p_1)$ for almost all points on I.

The transport equations are

$$\frac{\partial S_k}{\partial x}\frac{\partial \varphi_k}{\partial x}+\frac{\partial \varphi_k}{\partial t}+\frac{1}{2}\varphi\frac{\partial^2 S_k}{\partial x^2}=0, \tag{15}$$

where

$$S(x,t)=S(0,t-x/p)-\frac{p^2}{2}\frac{x}{p}+px. \tag{16}$$

Thus, the phase is linear on characteristics $dx/dt=\pm p$, and equation (15) is

$$p\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_1}{\partial t}=0, \tag{17}$$

$$-p\frac{\partial \varphi_2}{\partial x}+\frac{\partial \varphi_2}{\partial t}=0 \tag{18}$$

with boundary conditions

$$\varphi_1(0,t)=\Psi_1[\varphi_2(0,t),\quad \varphi_2(l,t)=\Psi_2[\varphi_2(l,t),\, t\in R^+. \tag{19}$$

Without loss of generality, we assume that $\Psi_1:=Id$. Then a solution has the form

$$\varphi_1(x,t)=y(t-x/p),\quad \varphi_2(x,t)=y(t+x/p) \tag{20}$$

where $y(t)$ is a solution of the difference equation

$$y(t)=\Psi_2[y(t-x/p)],\, t\in[-x/p,\infty). \tag{21}$$

We suppose that $\Psi_2\in C^2(I,I)$. Then this map is structural stable , $Per(\Psi_2)=P^+\cup P^-$. P^+,P^- are attractive and repelling circles of Ψ_2, where P^+ is finite and P^- is finite or countable on characteristics [11]. Then the problem has asymptotic $2^N l/p$ - periodic piecewise constant solutions of variable $\zeta=t-x/p$, where N ia least common multiple of attractive circles.

As a result, in WKB - approximation with accuracy $O(h^2)$, we have solutions

$$\psi_2(x,t)=\psi_1(t+x/p)e^{iS_2(t+x/p)},\quad \psi_1(x,t)=\Psi_1(t-x/p)e^{iS_1(t-x/p)}. \tag{22}$$

In conclusion, applications to the motions of two free charged particles in box with two flat walls with surface potentials of double-well type will be considered. Notice that if we consider quadratic or logistic maps given on complex plane $\mathcal{C}$ as the boundary conditions, then these conditions motivate the structure of Mandelbrot type on circles S^1 on complex plane for the boundary quantum problem, which are typical for quadratic or logistic maps.

Below it will be shown that the quantum problem can be reduced to the study of iterations of the complex map: $z\to\Phi(z)$, where $\Phi:=\Phi_1\circ\Phi_2$ is a superposition of maps. We choose $\Phi[z(\varphi,S)]\to z[\Phi_1(\varphi),\Phi_2(S)]$ in polar coordinates. It means that $\Phi: S^{(1)}\to S^{(2)}$, where a circle $S^{(\cdot)}\in\mathcal{C}$ in the behavior of orbits for corresponding dynamical systems has been studied in ([27], Eq. (17)).

2 Problem statement

We consider the two free particles with opposite impulses p and $-p$ which move on interval $0 < x < l$, so that at ends on the interval there is action of surface nonlinear potentials. Then in the bulk of a pattern an evolution of particles satisfies to the two linear Schrodinger equations

$$-ih\frac{\partial\psi_1}{\partial t} - \frac{h^2}{2}\frac{\partial^2\psi_1}{\partial x^2} = 0, \tag{23}$$

$$-ih\frac{\partial\psi_2}{\partial t} - \frac{h^2}{2}\frac{\partial^2\psi_2}{\partial x^2} = 0. \tag{24}$$

Interaction of free particles with 'surface' potentials is described by the differential boundary conditions

$$\frac{\partial\psi_1}{\partial t} = F_1[\psi_1,\psi_2], \quad \frac{\partial\psi_2}{\partial t} = F_2[\psi_1,\psi_2] \quad \text{as} \quad x = 0,\, t > 0, \tag{25}$$

$$\frac{\partial\psi_1}{\partial t} = G_1[\psi_1,\psi_2], \quad \frac{\partial\psi_2}{\partial t} = G_2[\psi_1,\psi_2] \quad \text{as} \quad x = l,\, t > 0. \tag{26}$$

We use also the initial conditions and with the initial conditions

$$\psi_k(x,0) = \varphi_k(x,0)e^{iS_k(x,0)}, \tag{27}$$

where $S \to S/h$.

Let us assume that the differential form, which correspond to ODE (25), (26, is exact. It means that

$$F_2[\psi_1,\psi_2]d\psi_1 - F_1[\psi_1,\psi_2]d\psi_2 = dW_1[\psi_1,\psi_2] = 0, \tag{28}$$

$$G_2[\psi_1,\psi_2]d\psi_1 - G_1[\psi_1,\psi_2]d\psi_2 = dW_2[\psi_1,\psi_2] = 0. \tag{29}$$

Further, we suppose that the functional relations

$$dW_1[\psi_1,\psi_2] = 0, \quad dW_2[\psi_1,\psi_2] = 0 \tag{30}$$

are globally solvable on some interval I, so that

$$\psi_1 = \Phi_1[\psi_2], \quad \psi_2 = \Phi_2[\psi_1]. \tag{31}$$

Then differential boundary conditions (25),(26) can be reduced to the functional boundary conditions

$$\psi_1 = \Phi_1[\psi_2] \quad \text{as} \quad x = 0,\, t > 0, \tag{32}$$

$$\psi_2 = \Phi_2[\psi_1] \quad \text{as} \quad x = l,\, t > 0, \tag{33}$$

where $\psi_1(0,0) = \Phi_1[\psi_2(0,0)]$, $\psi_2(0,l) = \Phi_1[\psi_2(0,l)]$, and similar conditions may be written on first and second derivatives of functions ψ_1, ψ_1.

It will be proved that with accuracy $O(h^2)$ solutions can be find as

$$\psi_k := e^{iS_k(x,t)/h}\varphi_k(x,t),\ k=1,2, \tag{34}$$

where $S_k(x,t)$ and $\varphi(x,t)$ are smooth and real functions. Substituting (34) into (23),(24), we obtain that

$$\left(\frac{\partial S_k}{\partial t}+V(x)+\frac{1}{2}(\nabla S)^2\right)\varphi+(-ih)\left(\frac{\partial S_k}{\partial x}\frac{\partial \varphi_k}{\partial x}+\frac{\partial \varphi_k}{\partial t}+\frac{1}{2}\varphi_k\frac{\partial \varphi_k^2}{\partial x^2}\right)+\frac{(-ih)^2}{2}\frac{\partial \varphi_k^2}{\partial x^2}=0. \tag{35}$$

From (35) it follows that $S(x,t)$ is a solution of the Hamilton-Jacobi equation

$$\frac{\partial S_k}{\partial t}+V(x)+\frac{1}{2}\left(\frac{\partial S_k}{\partial x}\right)^2=0 \tag{36}$$

and $\varphi(x,t)$ is a solution of the transport equation

$$\frac{\partial S_k}{\partial x}\frac{\partial \varphi_k}{\partial x}+\frac{\partial \varphi_k}{\partial t}+\frac{1}{2}\varphi\frac{\partial^2 S_k}{\partial x^2}=0. \tag{37}$$

We assume that maps Φ_1,Φ_2 have the form

$$\Psi_1[e^{iS}\varphi]:=(S,\varphi)\to(e^{if_1(S)}),g_1(\varphi),\quad \Psi_2[e^{iS}\varphi]:=(S,\varphi)\to(e^{if_1(S)},g_2(\varphi)), \tag{38}$$

where $S\to S/h$. It means that Φ_1,Φ_2 transform circle into another circle, so that there is decomposition of the transformation in (S,φ) plane of $O\varphi$ and OS - directions. Then boundary conditions (32),(33) can be rewritten as

$$S_1(0,t)=f_1[S_2(0,t)]\quad S_2(l,t)=f_2[S_1(l,t)], \tag{39}$$

$$\varphi_1(0,t)=g_1[\varphi_2(0,t)]\quad \varphi_2(l,t)=g_2[\varphi_1(l,t)]. \tag{40}$$

Asymptotic behavior of phases

Note that from mechanics it is known the formula

$$\frac{dS[x(t),t]}{dt}=\frac{\partial S}{\partial t}[x(t),t]+\frac{\partial S}{\partial x}[x(t),t]\frac{dx(t)}{dt} \tag{41}$$

which is true along trajectories $dx(t)dt=\pm p$. Next, the main role plays the main relation of anaclitic mechanics $p:=\partial S/\partial x$, which allows, using additionally the Hamilton-Jacobi equation with Hamiltonian $H(p):=p^2/2$ rewrite (72) as

$$S[x(t),t]=S[x(t_0),t_0]-\int_{t_0}^{t}V[p(s-t_0)+x(t_0)]ds-\frac{1}{2}p^2(t-t_0)+p[x(t)-x(t_0)]. \tag{42}$$

Next, from (42), using additionally boundary conditions, we arrive at

$$S_1(l,t)=f_1\circ f_2[S_1(l,t)]+\mu \tag{43}$$

where

$$\mu = -\left(V_1 - \frac{p^2}{2}\right)\frac{l}{p} - \left(V_2 - \frac{p^2}{2}\right)\frac{l}{p}. \tag{44}$$

Indeed, integrating the differential form from a point (l,t) to a point $(0, t-l/p)$ along $dx(t)/dt = p$, we obtain that

$$S_1(l,t) = S_1(0, t-l/p) - \left(V_1 - \frac{p^2}{2}\right)\frac{l}{p}. \tag{45}$$

Indeed, using the boundary conditions at $x = 0$ and $x = l$, we get that

$$S_1(l,t) = f_1[S_1(0, t-l/p)] - \left(V_1 - \frac{p^2}{2}\right)\frac{l}{p} + pl = \tag{46}$$

$$f_1\Big[S_2(0, t-2l/p) - \left(V_2 - \frac{p^2}{2}\right)\frac{l}{p} - pl\Big] - \left(V_1 - \frac{p^2}{2}\right)\frac{l}{p} + pl =$$

$$f_1 \circ \Big[f_2(S_1(l, t-2l/p)) - \left(V_2 - \frac{p^2}{2}\right)\frac{l}{p} - pl\Big] - \left(V_1 - \frac{p^2}{2}\right)\frac{l}{p} + pl.$$

Denote

$$\mu_2 = \left(V_2 - \frac{p^2}{2}\right)\frac{l}{p} - pl, \quad \mu_1 = \left(V_1 - \frac{p^2}{2}\right)\frac{l}{p} + pl, \tag{47}$$

$$f_{1,\mu_1} = \Big[f_2(S_1(l, t-2l/p)) - \mu_1\Big], \quad f_{2,\mu_2} = f_{1,\mu_1} - \mu_2. \tag{48}$$

Let $G := f_{1,\mu_1} \circ f_{2,\mu_2} : I \to I$. Assume that $G \in C^2(I, I$ is structural stable. Such maps form open dense subset. Then a set $Per\, G$ of periodic points of G is decomposed in the union P^+ - points of attractive circles and P^- - points of repelling circles, where the set P^- is finite or countable ([11],p.234)). In typical cases, from these assumptions it follows that the difference equation

$$S_1(t) = G[S_1(t - 2l/p)] \tag{49}$$

has 2^{4N} - asymptotic periodic solutions with finite, countable or uncountable points of discontinuities Γ on a period. The structure of set Γ depends on the structure of the delimiter $D := \bigcup\bigcup_{n/ge0} G^{-n}\bar{P}^-$, where $\bar{P}^-$ is the closure of P^-. Here, D is nowhere dense closed set of measure zero, which is finite (particularly, empty), countable or countable.

We call such solutions by solutions of relaxation, pre-turbulent and turbulent type (see, Fig.1). Thus a phase of the initial boundary value problem can be determined independently on amplitudes.

Assume that $\Psi_\mu : I \to I$, where I is bounded open interval, the map is structural stable and $\Psi_\mu \in C^2(I, I)$. Then a set $Per\, \Psi_\mu$ of periodic points is decomposed on a set of attractive points P^+ and a set P^- of repelling circles, so that $Per\, \Psi_\mu = P^+ \bigcup P^-$, where

P^- is finite or countable, and a set P^+ is finite. Below will be considered a case when P^- is finite. Let $D = \bigcup_{n\geq 0} \Psi_\mu^{-n} \bar{P^-}$ is the separator of the map Ψ_μ. The separator is countable, or uncountable nowhere dense closed set of measure zero [11]. The separator D is uncountable if and only if the map Ψ_μ has circles with periods $\neq 2^i, i = 0, 1, 2, ...$ The separator D is uncountable if and only if the map Ψ_μ has circles with periods $\neq 2^i, i = 0, 1, 2, ...$

Let $h_1(t)$ is an initial function given on $[-l/p, 0)$. Let us define $\Gamma = \Psi_\mu^{-1}$ and assume that $\dot{h_1}(t) \neq 0,\ t \in \Gamma$. Then Γ is closed nowhere dense set. Solutions can be found by iterations of the initial function $h_1(t)$ with help of the map Ψ_μ. If Γ is finite, then limit solutions are periodic piecewise constant distributions with finite points of discontinuities on a period. We call such solutions by solutions of relaxation type.

3 Nontrivial ordering structures

We consider now the hamiltonian ODE with hamiltonian $H(x,p) = \frac{1}{2}(p^2 + x^2)$ where

$$\dot{x} = H_p(x,p,t) = p, \quad \dot{p} = -H_x = -x \tag{50}$$

with initial conditions

$$x(0) = x_0(\alpha), \quad p(0) = \frac{\partial S}{\partial x}(\alpha). \tag{51}$$

For each fixed α, we call solutions $(x(\alpha,t), p(\alpha,t))$ of these ODE by double characteristics. Projection of this curve on R^n_x (here $n = 1$). Its projection $x = x(\alpha,t)$ is called by the trajectory of dynamical system. Thus for each fixed x_0 we have the straight line rotates around of initial coordinates with constant angle velocity, so that vectors $x(x_0,t), p(x_0,t)$ can be obtained from the vector (x_0, p_0) by orthogonal transformation with matrix

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \tag{52}$$

For example, for the hamiltonian $H(x,p) = \frac{1}{2}(p^2 + x^2)$, because

$$x(x_0,t) = x_0 \cos t + \alpha \sin t, \quad p(x_0,t) = \alpha \cos t - x_0 \sin t. \tag{53}$$

Consider a set of points (x,p) with $x_0 \in R$. It is a manifold L_0 at plane (x,p) (a straight line parallel to Ox - axes. Then $L_0(t)$ ia helical surface at (x,p,t) [**?**],

$$S(x,t) = S(x_0,t_0) + \int_{t_0}^{t} (p\,dx - H\,dt) \tag{54}$$

where integration is along $x_0 = x_0(x,t)$.

Interpretation of formula (54) is following. Since (x_0,t) are global coordinates on L, we must consider part of L which is homeomorphic on a plane (x,t). It is possible if $t \in [0, \pi/2)$. Here function $x := x(p,t)$ must be a solution of equation

$$p - \frac{\partial S}{\partial x}[x(p,t)] = 0. \tag{55}$$

Thus on L we can introduce coordinate $\tau = (\tau_1, ..., \tau_k) \in U$, where $U = R^k_\tau$ (here $k = 2$). Then a surface L can be given by equations $Q_0(\tau_1, ..., \tau_k)$, where $Q_0(\tau)$ is a smooth n - dimensional function. Let $p(\tau, t), q(\tau, t)$ be solutions of the Hamiltonian system

$$\dot{p} = -H_q, \quad \dot{q} = H_p, \quad p_{|t=0} = \frac{\partial S_0}{\partial x}(q_0(\tau)), \quad q_{|t=0} = q_0(\tau), \, t \in [0,T], \tag{56}$$

and let $b(\tau, t), c(\tau, t)$ be matrix solutions of the system in variations:

$$\dot{b} = -H_{qp}b - H_{qq}c, \quad \dot{c} = H_{pp}b + H_{pq}c \tag{57}$$

with initial conditions

$$b_{|t=0} = \frac{\partial^2 S_0}{\partial x^2}(q_0(\tau)), \quad c_{|t=0} = E, \tag{58}$$

where E is unit matrix of dimension $n \times n$ and functions H_{pp}, H_{qq}, H_{pq} have arguments $p(\tau, t), q(\tau, t), t)$.

Now we consider projection $\pi_q L^k_t$ of manifold L^k_t on q - plane of phase space $R^{2n}_{p,q}$, so that

$$\pi_q L^k_t := \delta^k_{x,t} := \{x \in R^n : |x = q(\tau, t) = k\}. \tag{59}$$

It means that $\pi_q : R^{2n}_{p,q} \to R^n_x$ is diffeomorphism. Then at a closed neighbourhood of $\Delta_{x,t}$ of surface $\delta^k_{x,t}$ the system of equations

$$< x - q(\tau, t), x - q_{\tau_j}(\tau, t) >= 0, \quad j = 1, ..., k \tag{60}$$

is smooth solvable on interval $t \in [0, T]$. There is an approximate solution wich is determined on a set $\Delta_{x,t}$ by the formula

$$S(x,t) = S_0(q_0(\tau)) + \int_0^t (< p(\tau, t_1), \dot{q}(\tau, t_1) > -H(p(\tau, t_1), q(\tau, t_1), t_1)) + O(x - q(\tau, t)_{|\tau=\tau(x,t)} \tag{61}$$

where $\tau(x,t)$ is a solution of system (60). For example, in (54) $\tau(x,t) = x_0(x,t)$.

Next, in the transport equation we use the transformation

$$\varphi(x,t) = \frac{1}{\sqrt{dx/dx_0}} \varphi_1(x,t). \tag{62}$$

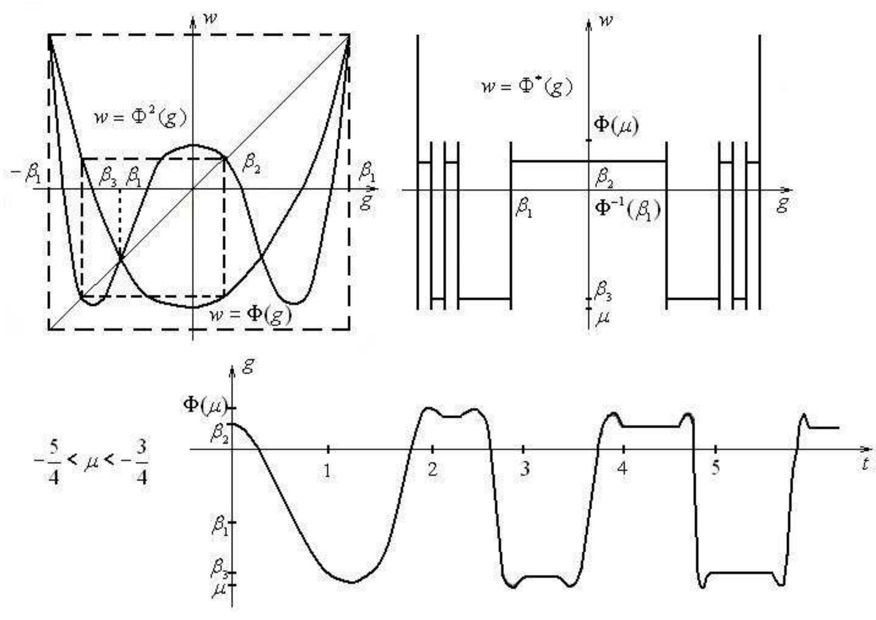

Figure 1: Limit distributions of relaxation type.

Then this equation can be rewritten as

$$\frac{\partial\varphi}{\partial t} + \frac{\partial S}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{1}{2}\frac{\partial^2 S}{\partial x^2}\varphi = \frac{d\varphi}{dt} + \frac{1}{2}\frac{\partial^2 S}{\partial x^2}\varphi = 0 \tag{63}$$

where we us formulas $\frac{\partial S}{\partial x} = p$ and $\dot{x} = p$. Now using the formula

$$\frac{d\varphi}{dt} = \left(\frac{\partial x}{\partial x_0}\right)\frac{d\varphi_1}{dt} \tag{64}$$

we obtain the transport equation [39]

$$\frac{d\varphi_1}{dt} = 0 \tag{65}$$

on L, where d/dt is a derivative along the vector field

$$V(H) := p\frac{\partial}{\partial x} - x\frac{\partial}{\partial p} + \frac{\partial}{\partial t} \tag{66}$$

on L in phase space (x, p, t).

As a result, we obtain the Liouvill equation

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\dot{x}_i + \frac{\partial u}{\partial p_i}\dot{p}_i\right) = 0 \tag{67}$$

where a function $u(x,p,t)$ determines a probability to find particles in a volume $dx^n\,dp^n$ at a time t. The Liouvill equation follows from the equation

$$\frac{\partial u}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{68}$$

where $J = vu$ is the flow of particles and v is a velocity of particles. If $\dot{x}$ is independent from x and $\dot{p}$ is independent from p, Liouvill equation (67) is identical to transport equation (68). For the Liouvill equation, a corresponding nonlinear boundary problem in x, p, t) in quadrat has been considered in [**?**]. To the similar boundary problem in can be reduced the Cahn-Hilliard equation in R^3 [**?**].

4 Reduction of transport equations to difference equations

Now we assume that Φ_1, Φ_2 have the form

$$\Psi_1 := e^{i\hat{\Psi}_1[S_2]}\hat{G}_1[\varphi_2], \quad \Psi_2 := e^{i\hat{\Psi}_2[S_1]}\hat{G}_2[\varphi_1], \tag{69}$$

where $S \to S/h$. It means that Φ_1, Φ_2 are maps from one circle on another circle in complex plane which 'stretches' amplitudes and 'rotate' phases. Then the boundary conditions can be decomposed on independent boundary conditions

$$\hat{S}_1 = \hat{\Phi}_1[S_2] \quad \text{as} \quad x = 0, \quad \hat{\Phi}_2 = \hat{G}_2[S_1] \quad \text{as} \quad x = l, \tag{70}$$

$$\varphi_1 = \hat{G}_1\hat{\Phi}_1[\varphi_2] \quad \text{as} \quad x = 0, \quad \varphi_2 = \hat{\Phi}_2[\varphi_1] \quad \text{as} \quad x = l. \tag{71}$$

Thus, the quantum problem is reduced to two 'independent' classic boundary boundary problems: to the Hamilton-Jacobi equation and to the transport equation. In the next section, it will be obtained asymptotic for functions $S_k := S_k(x,t,h)$ with accuracy $O(h^2)$ as $t \to +\infty$.

Now, as noted in Introduction, we can use the formula of analytic mechanics

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\frac{dx}{dt} = -H(p) + p\,dx \tag{72}$$

where $H(p)$ is the Hamiltonian, the first term obtained from the Hamilton-Jacobi equation, and the second term from the relation $p := \partial S/\partial x$, so that equation (72) is ODE along vector-field $dx(t)/dt = p$. Integrating (72) along $dx/dt = p$, and using with functional boundary conditions for phases, we arrive at

$$S_1(l,t) = \hat{\Phi}_1\{\hat{\Phi}_2[S_1(l, t - l/p)] + \mu\} + \mu \tag{73}$$

where $\mu = -pl/2 + p$. Define by Ψ_μ the right part of (73). Then

$$S_1(l,t) = \Psi_\mu[S_1(l, t - l/p)]. \tag{74}$$

We obtained the difference equation with continuous time. Solutions of (74) represent asymptotically piecewise constant periodic impulses (see,[11]). Limit distributions depend on initial functions on $[-l/p, 0)$ and ones may be find, step by step, iterating initial data with help of the map $S_1 : I \to I$. As a result, the study of asymptotic of quantum boundary problem is reduced to the study of one-dimensional maps in some topological space. Such approach has been introduced by Sagdeev et al [33, 34, 35, 36, 37]. One of the applications, quantum-mechanical effects on electron–electron scattering investigated in high-temperature plasmas. An effective pseudo-potential model has been considered, for example, in [32] to describe the total spin states in Born approximation. Note that we can introduce in (74) the temperature as a parameter. Then this parameter admits the existence the series of Sharkovsky's type ordering bifurcations of the phases: for example, the well-known period doubling bifurcations. Particularly, this statement will be proved below.

4.1 Typical examples of discrete maps

In other words, the problem is reduced to the map $\Upsilon : \mathcal{C} \to \mathcal{C}$ at complex plane. The theory of such maps has been developed (see, for example,[25, 27]) in case when we have deal with a map, which acts from circle into circle. If we add to equation (74) sinusoidal periodic function, then we obtain an equation, which has been proposed by Kolmogorov as a simple model that describes the motion of mechanical rotator without or with periodic perturbations. For another physical applications, we may consider the motion of charge particles in sinusoidal magnetic field [17]. As a result, for a phase of free charge particle in periodic magnetic field (see physical example below) we obtain difference equation of Kolmogorov type, which is called by Arnold mapping:

$$\theta_n = \theta_{n-1} + \Omega - \frac{k}{2\pi} \sin 2\pi\theta_{n-1}, \tag{75}$$

where the right part plays the role of periodic perturbation or, for charged particles, the amplitude of magnetic field depending on the phase.

The next typical example is the well-known Chiricov map or standard map:

$$\theta_n = \theta_{n-1} + k \sin u_{n-1}, \tag{76}$$

$$u_n = u_{n-1} + p_n, \tag{77}$$

where θ is a frequency of particle, u is the amplitude of a wave function, and k is the strength of magnetic field. Then if $k \neq 0$, then there is a periodic motion of a charged particle in each finite region, which represent a neighbourhood of some rational point Ω. For $\Omega \in [0, 1]$, there arise regions $(\Pi_n := (\Omega, k_n)$ such that in any of these regions exists a periodic motion. The corresponding regions are called by Arnold tongues. At $k = 0$, the Arnold tongue is isolates set of measure zero, which forms Cantor set of dimension $d = 0.8700 \pm 3.7 \times 10^{-4}$ (see, [43]). Arnold tongues may be interpreted as resonance zone, which emanate from rational numbers.

Further, we assume that $\Psi : I \to I$, the map is structural stable, and $\Psi \in C^2(I, I)$. Then the set of non-wandering points is equal $\Upsilon(\Phi) := Per\,\Psi = Fix\Phi^N$, where $Per\,\Psi$ is a set of periodic, and $Fix\Phi^N$ is a set of fixed points of the map Φ, and $N \in Z^+$ is some integer number. $\Upsilon(\Phi) := A^+ \bigcup A^- \bigcup A^\pm$, where $A^+, A^-, A^\pm$ are sets of corresponding attractive, repelling and saddle fixed points of the map Φ^N. For structural stable map, $Per\,\Psi = P^+ \bigcup P^-$, where P^- is finite or countable and a set P^+ is always finite. Let P^- be finite and $D = \bigcup_{n\geq 0} \Psi_\mu^{-n} \bar{P^-}$ is a separator of the map Ψ. Then D is countable or uncountable nowhere dense closed set of measure zero. D is uncountable if and only if Ψ_μ has circles of periods $\neq 2^i, i = 0, 1, 2, ...$ D is uncountable if and only if Ψ_μ has circles with periods $\neq 2^i, i = 0, 1, 2, ...$ [11].

Let $h_1(t)$ be an initial function given on $[-l/p, 0)$. Define $\Gamma := \Psi_\mu^{-1}$ and assume that $\dot{h}_1(t) \neq 0,\ t \in \Gamma$. Then Γ is closed nowhere dense set. Hence, solutions of equation (74) can be found by iterations of function $h_1(t)$ with help of iterations of Ψ_μ. If Γ is finite, then limit solutions belong to a set $\mathcal{P}\nabla$, that is we have finite points of discontinuities on a period.

Note that this approach has been used to find $3D$ - dimensional distributions spatial-temporal clusters for Cahn-Hilliard $3D$ - equation with functional boundary conditions and double-Neumann homogeneous conditions at surface of $3D$ - dimensional unit cube (see, [15]. The spatial-temporal ordering can be obtained also for non-relativistic equations of quantum field theory (see, [40, 18]). Next, besides applications to charged particle in box with nonlinear surface potentials, the functional boundary conditions has been used in [29, 30] to describe surface nonlinear pairing of white and black solitons [29] that is fundamental problem of modern physics, and on the description of N - soliton interaction in optical fibers that is more simple problem.

5 Asymptotic of amplitudes

Now we consider the similar asymptotic of solutions of the transport equation with functional boundary conditions. Initially, we have deal with the equation

$$\frac{\partial S_k}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \varphi_k}{\partial t} + \frac{1}{2}\varphi_k \frac{\partial S_k^2}{\partial x^2} = 0, \tag{78}$$

where $\partial S_k/\partial x = \pm p$, $k = 1, 2$. Then from (78) we arrive at

$$\frac{\partial \varphi_1}{\partial t} + p\frac{\partial \varphi_1}{\partial x} + \frac{1}{2}\varphi_1 \frac{\partial^2 S_1}{\partial x^2} = 0, \tag{79}$$

$$\frac{\partial \varphi_2}{\partial t} - p\frac{\partial \varphi_2}{\partial x} + \frac{1}{2}\varphi_1 \frac{\partial^2 S_2}{\partial x^2} = 0 \tag{80}$$

with boundary conditions (70). As before, along $dx(t)/dt = \pm p$ equations (79),(80) are

$$\frac{d\varphi_1(x(t), t)}{dt} = -\frac{1}{2}\varphi_1 \frac{\partial^2 S_1}{\partial x^2} \quad \text{as} \quad dx(t)/dt = p, \tag{81}$$

$$\frac{d\varphi_2(x(t), t)}{dt} = -\frac{1}{2}\varphi_2 \frac{\partial^2 S_2}{\partial x^2} \quad \text{as} \quad dx(t)/dt = -p. \tag{82}$$

Integrating (81), (82) and using the boundary conditions, we get the system of integro-difference equations:

$$\varphi_1(l,t) = \hat{\Phi_1}\big[\hat{\Phi_2}(\varphi_1(l,t-2l/p) - \frac{1}{2}\int_{t-2l/p}^{t-l/p} \varphi_2(p(t-s)+l,s)\frac{\partial^2 S_2}{\partial x^2}(p(t-s)+l,s)ds\big] \quad (83)$$
$$-\frac{1}{2}\int_{t-l/p}^{t} \varphi_1(p(s-t)+l,s)\frac{\partial^2 S_1}{\partial x^2}(p(s-t)+l,s)ds,$$

$$\varphi_2(0,t) = \hat{\Phi_2}\big[\hat{\Phi_1}(\varphi_2(0,t-2l/p) - \frac{1}{2}\int_{t-2l/p}^{t-l/p} \varphi_1(p(s-t)+2l,s)\frac{\partial^2 S_1}{\partial x^2}(p(s-t)+2l,s)ds\big] \quad (84)$$
$$-\frac{1}{2}\int_{t-l/p}^{t} \varphi_2(p(t-s),s)\frac{\partial^2 S_2}{\partial x^2}(p(t-s),s)ds,$$

where S_1, S_2 are known functions, which can be find from difference equations

$$S_1(l,t) = \hat{G_1} \circ \hat{G_2}[S_1(l,t-l/p)] + \mu\} + \mu, \quad (85)$$

$$S_2(0,t) = \hat{G_2} \circ \hat{G_1}[S_2(0,t-l/p)] + \mu\} + \mu, \quad (86)$$

For example, if $G_2 := Id$, where Id is identical map, then we obtain two identical difference equations. Asymptotic behaviour of solutions depends on initial phases, which are given on $[-l/p, 0)$. For instance, $S_1(l,t)$ is oscillating periodic piecewise constant function as $t \to +\infty$. But $S_2(0,t)$ tends to a constant that can be interpreted as localisation of the phase. Next, since $S_1(x,t), S_2(x,t)$ are known, solutions of integro-difference equations (83),(84) can be fined by iterations if initial data $\varphi_1(l,t), \varphi_2(0,t)$ on $[-l/p, 0)$.

The main observation is that there is convergence of solutions of integro-difference equations (83),(84) to solutions of difference equations

$$\varphi_1(l,t) = \hat{\Phi_1} \circ \hat{\Phi_2}(\varphi_1(l,t-2l/p), \quad (87)$$

$$\varphi_2(0,t) = \hat{\Phi_2} \circ \hat{\Phi_1}(\varphi_2(0,t-2l/p), \quad (88)$$

where $(\circ)$ is superposition of maps. Thus, we obtained the two independent difference equations. Each of these equations has asymptotic periodic piecewise constant periodic solutions with finite or infinite points of discontinuities on a period.

5.1 The structure of attractors

The special boundary condition allow us to find a phase and farther to substitute the known function in integro-difference equations. Next, a problem is to find in these equations asymptotic of integral terms $I_1(t), I_2(t))$ in (83),(84). But it is easy to prove that asymptotic of these integrals is $I_1(t) \sim e^{-k_1 t}$, $I_2(t) \sim e^{-k_2 t}$, where numbers k_1, k_2 are positive. The prove follows from observation that

$$\frac{\partial^2 S_1}{\partial x^2}(p(s-t)+2l, s) \to 0, \quad \frac{\partial^2 S_2}{\partial x^2}(p(t-s), s) \to 0 \tag{89}$$

for almost all points $t \in R^+$ as $t \to +\infty$, excluding a set Γ of points of discontinuities of limit functions for phases $S_1(t - x/p), S_2(t + x/p)$.

Indeed, it is easy to see that

$$\frac{\partial^2 S_1}{\partial x^2}(x,t) = \frac{\partial^2 S_1}{\partial x^2}(t - x/p), \quad \frac{\partial^2 S_2}{\partial x^2}(x,t) = \frac{\partial^2 S_2}{\partial x^2}(t + x/p). \tag{90}$$

Then from (90) it follows that integro-difference equations can be rewritten as

$$\begin{aligned}\varphi_1(l,t) = \hat{\Phi}_1\big[\hat{\Phi}_2(\varphi_1(l, t-2l/p) - \frac{1}{2}\frac{\partial^2 S_2}{\partial x^2}(t + l/p)\int_{t-2l/p}^{t-l/p}\varphi_2(p(t-s)-l, s)ds\big] \\ -\frac{1}{2}\frac{\partial^2 S_1}{\partial x^2}(t - l/p, s)\int_{t-l/p}^{t}\varphi_1(p(s-t)+l, s)ds,\end{aligned} \tag{91}$$

$$\begin{aligned}\varphi_2(0,t) = \hat{\Phi}_2\big[\hat{\Phi}_1(\varphi_2(0, t-2l/p) - \frac{1}{2}\frac{\partial^2 S_1}{\partial x^2}(t - 2l/p)\int_{t-2l/p}^{t-l/p}\varphi_1(p(s-t)+2l, s)ds\big] \\ -\frac{1}{2}\frac{\partial^2 S_2}{\partial x^2}(t)\int_{t-l/p}^{t}\varphi_2(p(t-s), s)ds,\end{aligned} \tag{92}$$

But

$$\frac{\partial^2 S_1}{\partial x^2}(\zeta) = \frac{1}{p^2}\frac{\partial^2 S_1}{\partial \zeta^2}(\zeta), \quad \frac{\partial^2 S_2}{\partial x^2}(\eta) = \frac{1}{p^2}\frac{\partial^2 S_2}{\partial \zeta^2}(\eta). \tag{93}$$

From (93) it follows that equations (91),(92) can be rewritten as

$$\begin{aligned}\varphi_1(l,t) = \hat{\Phi}_1\big[\hat{\Phi}_2(\varphi_1(l, t-2l/p) - \frac{1}{2p^2}\frac{\partial^2 S_2}{\partial \eta^2}(t + l/p)\int_{t-2l/p}^{t-l/p}\varphi_2(p(t-s)-l, s)ds\big] \\ -\frac{1}{2p^2}\frac{\partial^2 S_1}{\partial \zeta^2}(t - l/p)\int_{t-l/p}^{t}\varphi_1(p(s-t)+l, s)ds,\end{aligned} \tag{94}$$

$$\begin{aligned}\varphi_2(0,t) = \hat{\Phi}_2\big[\hat{\Phi}_1(\varphi_2(0, t-2l/p) - \frac{1}{2p^2}\frac{\partial^2 S_1}{\partial \zeta^2}(t - 2l/p)\int_{t-2l/p}^{t-l/p}\varphi_1(p(s-t)+2l, s)ds\big] \\ -\frac{1}{2p^2}\frac{\partial^2 S_2}{\partial \eta^2}(t)\int_{t-l/p}^{t}\varphi_2(p(t-s), s)ds.\end{aligned} \tag{95}$$

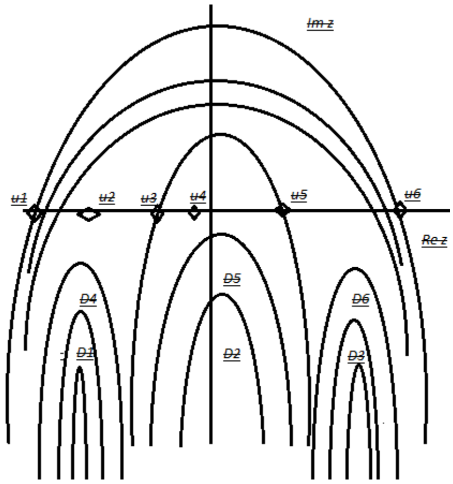

Figure 2: Hyperbolic attractor in polar coordinates. u_1, u_6 are saddle points and u_2, u_3 form circle of period 2

.

Further, linearizing equations (85),(86) for phases at some neighbourhoods of attractive fixed points P_1, P_2, we get the linear equations

$$S_1(\zeta) = \lambda_1 S_1(\zeta - l/p), \quad S_2(\eta) = \lambda_2 S_2(\eta - l/p), \tag{96}$$

where $\lambda_1 = [\hat{G}_1 \circ \hat{G}_2]'(P_1)$, $\lambda_2 = [\hat{G}_2 \circ \hat{G}_1]'(P_2)$ and $|\lambda_1| < 1$, $|\lambda_2| < 1$. Solutions of these equations are

$$S_1(\zeta) = e^{k_1 \zeta}, \quad S_2(\eta) = e^{k_2 \eta}, \quad k_1 = \frac{p}{l} \ln|\lambda_1|, \quad k_2 = \frac{p}{l} \ln|\lambda_2| \tag{97}$$

with accuracy to a constant factor.

Then phases equations (94),(95) can be written as

$$\begin{aligned}\varphi_1(l,t) = \hat{\Phi_1}\big[\hat{\Phi_2}(\varphi_1(l, t-2l/p) - \frac{k_2^2 e^{-k_2/p}}{2p^2} e^{k_2 t} \int_{t-2l/p}^{t-l/p} \varphi_2(p(t-s) - l, s) ds\big] \\ - \frac{k_1^2 e^{-k_1 l/p}}{2p^2} e^{k_1 t} \int_{t-l/p}^{t} \varphi_1(p(s-t) + l, s) ds,\end{aligned} \tag{98}$$

$$\begin{aligned}\varphi_2(0,t) = \hat{\Phi_2}\big[\hat{\Phi_1}(\varphi_2(0, t-2l/p) - \frac{k_1^2 e^{-k_1 l/p}}{2p^2} e^{k_1 t} \int_{t-2l/p}^{t-l/p} \varphi_1(p(s-t) + 2l, s) ds\big] \\ - \frac{k_2^2 e^{-k_2 l/p}}{2p^2} e^{k_2 t} \int_{t-l/p}^{t} \varphi_2(p(t-s), s) ds.\end{aligned} \tag{99}$$

If $|p| \to \infty$, then equations (98),(99) can be reduced to the equations

$$\varphi_1(l,t) = \hat{\Phi_1} \circ \hat{\Phi_2}(\varphi_1(l,t)), \quad \varphi_2(0,t) = \hat{\Phi_2} \circ \hat{\Phi_1}(\varphi_2(0,t)). \tag{100}$$

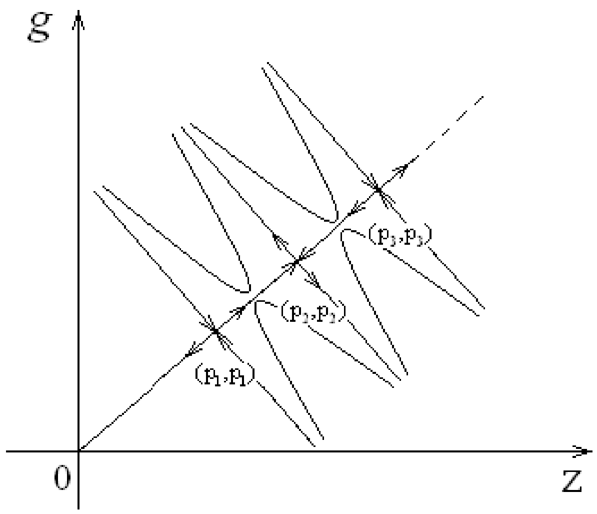

Figure 3: Typical distributions of trajectories for a hyperbolic map

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Solutions of equations (100) 'placed' on fixed points of mapping $\hat{\Phi_1} \circ \hat{\Phi_2}$, $\varphi_2(0,t) = \hat{\Phi_2} \circ \hat{\Phi_1}(\varphi_2(0,t)) \in C^2(I,I)$. The fixed point correspond to a constant solution. If $|p| < \infty$, then solutions stick together into oscillating solutions of equations (98),(99).

Next, let us denote by $\alpha_1, \alpha_2, \beta_1, \beta_2$ coefficients in equations (98),(99), respectively. Then

$$\varphi_1(l,t) = \hat{\Phi_1} \circ \big[\hat{\Phi_2}(\varphi_1(l,t-2l/p) - \alpha_1 e^{k_2 t} \int_{t-2l/p}^{t-l/p} \varphi_2(p(t-s)-l,s)ds\big] \qquad (101)$$
$$-\alpha_2 e^{k_1 t} \int_{t-l/p}^{t} \varphi_1(p(s-t)+l,s)ds,$$

$$\varphi_2(0,t) = \hat{\Phi_2} \circ \big[\hat{\Phi_1}(\varphi_2(0,t-2l/p) - \beta_1 e^{k_1 t} \int_{t-2l/p}^{t-l/p} \varphi_1(p(s-t)+2l,s)ds\big] \qquad (102)$$
$$-\beta_2 e^{k_2 t} \int_{t-l/p}^{t} \varphi_2(p(t-s),s)ds.$$

Now, we assume that $\hat{\Phi_2} := Id$. Then from (98),(99) it follows that

$$\varphi_1(l,t) = \hat{\Phi_1}\big[\varphi_1(l,t-2l/p) - \alpha_1 e^{k_2 t} \int_{t-2l/p}^{t-l/p} \varphi_2(p(t-s)-l,s)ds\big] \qquad (103)$$
$$-\alpha_2 e^{k_1 t} \int_{t-l/p}^{t} \varphi_1(p(s-t)+l,s)ds,$$

$$\varphi_2(0,t) = \hat{\Phi_1}[\varphi_2(0,t-2l/p)] - \beta_1 e^{k_1 t}\int_{t-2l/p}^{t-l/p} \varphi_1(p(s-t)+2l,s)ds \qquad (104)$$
$$-\beta_2 e^{k_2 t}\int_{t-l/p}^{t} \varphi_2(p(t-s),s)ds.$$

In the next subsection, it will be proved that asymptotically solutions of these integro-difference equations tend to corresponding difference equations.

5.1.1 Attractor of integro-difference equations is equal to attractor of difference equations

Here we prove that an attractor of problem (103),((104) coincides with an attractor of the difference equations

$$\varphi_1(l,t) = \hat{\Phi_1}\big[\varphi_1(l,t-2l/p)], \qquad (105)$$

$$\varphi_2(0,t) = \hat{\Phi_1}[\varphi_2(0,t-2l/p)]. \qquad (106)$$

Indeed, solutions of equations (103),((104) can be determined by iterations of initial data on $[-p/l,0)$ step by step. For example, the first iteration for the first equation has the form:

$$\varphi_1(l,t+2l/p) = \hat{\Phi_1}\circ\Big[\hat{\Phi_1}\big[\varphi_1(l,t-2l/p) - \alpha_1 e^{k_2 t}\int_{t-2l/p}^{t-l/p}\varphi_2(p(t-s)-l,s)ds\big] \qquad (107)$$
$$-\alpha_2 e^{k_1 t}\int_{t-l/p}^{t}\varphi_1(p(s-t)+l,s)ds - \alpha_1 e^{k_2(t+2l/p)}\int_{t}^{t+l/p}\varphi_2(p(t+2l/p-s)-l,s)ds\Big]$$
$$-\alpha_2 e^{k_1(t+2l/p)}\int_{t+l/p}^{t+2l/p}\varphi_1(p(s-t-2l/p)+l,s)ds.$$

If $\alpha_1 = \alpha_2 = 0$, then there are invariant solutions of difference equation $(p_1(t), p_2(t))$ such that $p_2(t) = \hat{\Phi_1}^{-1}(p_1(t))$. These solutions are asymptotic periodic piecewise constant 4^N - periodic functions with finite points of discontinuities on a period, where N is leat common multiple of attractive circles of the map $\hat{\Phi_1}$ [11].

The map $\varphi_1 \to \hat{\Phi_1}(\varphi_1)$ can be constructing as a map from C^2 into C^2 for almost all points Γ, excluding a finite set of points of discontinuities Γ. Then there is $\delta > 0$ such that, for all $|\alpha_1 + \alpha_2| < \delta/M$, the corresponding integral operator also is constructing in C^2 - norm for almost all points $t \in R^+$. Similarly, we can prove that the operator for the second integro-difference equation is constructing in C^2 - norm. It means that there is a vector-function $(\hat{\varphi_1}, \hat{\varphi_2})$, which is a C^2 - solution of the integro-difference equation. If for the difference equation a limit function is oscillating and asymptotic periodic with non-decay amplitude, then for integro-difference equation a solution $(\hat{\varphi_1}, \hat{\varphi_2})$ is also oscillating at a small neighbourhood of invariant periodic solution $(p_1(t), p_2(t))$.

We prove that $(\hat{\varphi_1},\, \hat{\varphi_2}) \Rightarrow (p_1(t),\, p_2(t))$ in C^2 - norm as $t \to +\infty$ for almost all $t \in R^+$. Indeed, we assume that

$$|\int_{t-2l/p}^{t-l/p} \varphi_2(p(t-s)-l,s)ds| < M, \quad |\int_{t-l/p}^{t} \varphi_1(p(s-t)+l,s)ds| < M, \tag{108}$$
$$|\int_{t}^{t+l/p} \varphi_2(p(t+2l/p-s)-l,s)ds| < M, \quad |\int_{t+l/p}^{t+2l/p} \varphi_1(p(s-t-2l/p)+l,s)ds| < M.$$

Then the first iteration for the first of the integro-difference equations is described by equation (107). Continuing the iterations, we obtain that the last terms of equation (107) tends to zero as $t \to +\infty$. Indeed,

$$\beta(t) = \alpha_2 e^{k_1(t+2l/p)} \int_{t+l/p}^{t+2l/p} \varphi_1 ds + \alpha_2 e^{k_1(t+4l/p)} \int_{t+2l/p}^{t+4l/p} \varphi_1 ds + ... \tag{109}$$
$$\leq \alpha_2 M e^{k_1(t+2l/p)} \frac{1-(e^{k_1(t+2l/p)})^n}{1-e^{2l/p}}.$$

Then $\beta(t) \to 0$ as $\to +\infty$. Further, from inequalities $e^{k_1 t} < 1$ and $e^{k_2 t} < 1$ it follows that if a solution of equation (107) belongs to a δ - neighbourhood of attractive invariant solution $(p_1(t),\, p_2(t)$ of the difference equations, then this solution $(\hat{\varphi_1},\, \hat{\varphi_2})$ belongs to a δ - neighbourhood of a solution $(p_1(t),\, p_2(t)$ for all $t \in R^+$. Moreover, a solution $(\hat{\varphi_1},\, \hat{\varphi_2})$ tends t

o a solution $(p_1(t),\, p_2(t)$ as $t \to +\infty$. The prove is complete for solutions relaxation type.

As a result, the quantum problem has hyperbolic structural stable global attractor which is described on Fig.1. Elements of the attractor are exact invariant solutions of the quantum boundary problem. For limit solutions of relaxation type, from (35) it follows that

$$\frac{(-ih)^2}{2} \frac{\partial \varphi_k^2(x,t)}{\partial x^2} \to 0 \tag{110}$$

as $t \to +\infty$ for almost all points.

5.2 Applications for classical mechanics

Here, we consider applications to quantum mechanics for free particles in surface scalar potentials. We will focused on the influence of a magnetic field on a charge particle. Classically, we have the Lorentz force law $F = q(E + vB)$, where q is the charge and v is the velocity. The q may be positive or negative. For a quantum particle, the Hamiltonian has the form $H(q,p) := \frac{1}{2}mv^2 + q\psi$, where m is the mass and ψ is a wave function.

For quantum equations, the dynamical boundary conditions describes functional connection between velocities of surface particles on left and right sides of the box in x - direction. Indeed, let $v(t) = (a(t), b(t))$ be a velocity of a charged particle. Let us introduce a complex (a, b) - plane, so that $z(t) = a(t) + ib(t)$, or in polar coordinates $z(t) = |z(t)|e^{i\varphi(t)}$. Next, we assume that on a particle acts a force F on the direction of vector $\vec{e} = (a, b)$. For example, it may be a damping force $F(v) = -\gamma v$, where $\gamma \in R$. It is possible also a case $F(v) = -\gamma v + \beta v^2$, $\beta \in R$, arising from a double-well potential.

We assume that at time t_n the particle gets a kick $c = a + ib$, where a is the kick strength in x_1 - direction , and b is the kick strength in x_2 - direction. Now we define velocities $v_n^{\pm} = (u_n^{\pm}, w_n^{\pm}$ before and after the kick. Then we obtain that $u_n^+ = u_n^- + \hat{a}$ and $w_n^+ = w_n^- + \hat{b}$, or $z_n^+ = z_n^- + c$.

Further, there is a magnetic field $B(t)$, which has on interval $t \in [t_n, t_{n+1}$ a constant value $B = (0, 0, B_n)$. Then, as shown in [17], is

$$B_n = h(v_n^+, \varphi_n^+), \tag{111}$$

$$\tau_n = f(v_n^+, \varphi_n^+), \tag{112}$$

and $\tau_n = t_{n+1} - t_n$ is a time between kicks.

Next, we assume that there is the initial problem

$$g_t' = F(g), \quad g(0) = g_0. \tag{113}$$

Then again from [17] it follows that an evolution of a particle satisfies to the system of discrete difference equations

$$v_{n+1}^- = g(\tau_n, v_n^-), \tag{114}$$

$$\varphi_{n+1}^- = \varphi_n^- + \omega_n \tau_n, \tag{115}$$

where v_n^- and φ_n^- denote module and angle of a velocity of the particle before a next kick. The angle φ rotates with the Larmor frequency.

Here, a simplest form of time difference $\tau_n = t_n - t_{n-1}$ has been described as function on v has been described in ([17], Figury 5), where $\tau \in [0, 6]$ and $v \in [0, 2]$, and the charge of the particle is $q = 2, 3, ..., 10$. The function $\tau := \tau(v)$ has the form of monotone decreasing hyperbole $\tau \sim \mu/v$, where μ is a constant. Then from (114) we obtain the discrete difference equation:

$$v_{n+1}^- = \Phi_2(\mu, v_n^-). \tag{116}$$

From (116) we arrive at

$$v_{t+l/p}^- = \Phi_2(\mu, v_t^-), \tag{117}$$

where $\Phi_2(v) = g[f(v, v)]$. From (117) it follows that an approach in [17] can be used to generalise the homogeneous motion of the particle on the case, depending on space variable x. The reason is because the quantum particle can be reduced on characteristic to the initial problem, but to the difference equation with continues time.

Of course, the such approach can be generalized. For example, we can obtain the relation

$$v_{n+1} e^{i\varphi_{n+1}} = g[f(v_n, \varphi_n), \varphi_n] \times e^{i(\varphi_n + h[v, \varphi_n), f(\varphi_n), \varphi_n)] f(v_n)}, \tag{118}$$

where index (+) has been omitted (see, [17], Eq.(17)). These equations can be considered as 'boundary conditions' for the quantum problem. Indeed, we can to study in similar manner the motion of two type of particles with opposite impulses, assuming that quantities (immediately before (−) or after (+) kick) arise as kicks between particles with opposite impulses at a point $x = 0$ in a moments t_n, t_{n-2} and at a point $x = l$ in a moments $t_{n+1} t_{n-1}$. Then we must solve the two similar initially problem with potentials $A_1(\psi_1, \psi_2)$ at $x = 0$ and $A_2(\psi_1, \psi_2)$ at $x = 0$. Then we obtain the boundary quantum problem which is reduced to the canonical system of difference equations. For problem from [17], such study of iterations of some complex maps complex map $\Phi \in \mathcal{C}$ can be represented as $\Phi : (v, \varphi) \to \Phi_1(v, \varphi), \Phi_2(v, \varphi)$, where Φ_1 and Φ_2 are determined by the boundary conditions of type (118). In equations (118), the function $h := B(v, \varphi)$, where B is a magnetic field.

5.3 Applications to quantum electronic

Next example, where may be considered nonlinear boundary conditions, depending on phases and amplitudes together, is the parametric generator, which is composed by three LC - circuits and by the quadratic nonlinear element on the basis of the operational amplifier. Here, we can consider the boundary equations, which describe the interacting mode amplitudes. These equations represent the mathematical analog of the amplifier. In this case, the problem is reduced to a system of the three differential equations for the amplitudes with an attractor of the Lorenz type [28].

In the absence of the de-tuning ot, Hamilton-Jacobi equations can be omitted, and respectively the quantum problem admits the reduction to one difference equation for amplitudes. In the presence of the de-tuning of, we must consider also the phase dynamics. Assuming quadratic nonlinearity, in [28] has been formulated a system of amplitudes equations, which (in the case of a fixed phase) are reduced to a system of differential equations the first order. The nonlinear element circuit diagram is shown in ([42], Figure 1).

6 Asymptotic behaviour of solutions for system of difference equations

Thus, it has been shown that the boundary quantum problem can be reduced to the study of behaviours of orbits or trajectories of the two-dimensional dynamical system, which is produced by a some map $\Phi : (\varphi, S) \to (f_1(\varphi, S), f_2(\varphi, S)$ that transform plane (φ, S) into plane. Let G be a bounded open subset in R^2 and $\Phi : \bar{G} \to G$ is a continuous map such that: (1) differential $D\Phi_u$ is continuous on G, where $u := (\varphi, S)$; (2) a set $\Phi^{-1}(u)$ is finite for each $u \in G$; (3) a set $\Omega(\Phi)$ of non-wandering points of the map Φ is finite and hyperbolic. It means that if a spectra $\sigma(B)$ of operator B, then the supposition that $\Omega(\Phi)$ is finite is satisfied, for example, if $\Omega(\Phi) = Fix(\Phi)$, where $Fix(\Phi)$ is a set of fixed points of Φ, and $\sigma(D\Phi_u) \subset \{e^z : |\Re z| > \delta > 0\}$ for each $u \in \Omega(\Phi)$. The condition of hypertonicity $\Omega(\Phi)$ implies that if $u \in \Omega(\Phi)$ and $\Phi^k(u) = u$, then $\sigma(D\Phi_u) \bigcap \{z : |z| = 1\} = \phi$, where ϕ is empty set.

Asymptotic behaviour of solutions of system of difference equations in R^2, which are produced by the map Φ with the properties $(1) - (3)$ are known for a hyperbolic structural stable map $\Phi \in C^2(G, \bar{G}$, where $G \subset R^2$ for spacial initial data given on interval $[-p/l, 0)$

(see, for example, [11, 8, 12]). If the map

$$\Phi : (v, \varphi) \to (\Phi_1(v, \varphi), \Phi_2(v, \varphi)) \tag{119}$$

is hyperbolic and structural stable, then there are asymptotic periodic piecewise constant solutions $p(t) = (p_1(t), p_2(t))$, where $p \in P^+$, where P^+ is a set of attractive circles of $\Phi : R^2 \to R^2$. For example, if Φ has two attractive fixed points $p_1, p_2 \in P^+$ and one fixed point of saddle type $P^\pm$ of codimension 1 with unstable manifold $W^u(P^\pm)$ (see, Figure 1). Then a components $v(t), \varphi(t)) \to (p_{1,2}, 0)$, where $(p_{1,2}, 0)$ are two attractive fixed points. A saddle point $(0, 0)$ has a one-dimensional separatrixe $W^u(P^\pm)$, so that each given initial curve $\nu(t), t \in [-1, 0)$ tends to a point p_2 on intervals $[-1, t^*) \bigcup [t^*, 0)$, where a point t^* is determined by an intersection $\nu(t) \bigcap W^u(P^\pm)$ (see, Figure 1).

In this particular case, component of limit solutions for the two difference equations are described on Figure 2. We call such solutions by limit solutions of relaxation type. In application to physics, the graphic on Figure 2 represent the velocity or 'number' of charged free particles in surface potential with three critical points and unique energetic barrier, that is, for the double-well potential. Limit solutions on Figure 2 are 2 - periodic with one point $p^\pm \in \Gamma$ of 'discontinuities' on a period. A set Γ is produced by a set of pre-images $\Gamma = \Phi(W^\pm)^{-n}$, $n = 0, 1, 2, ...$ [11, 8, 14, 15, 14]. A set $W^\pm$ is finite, but a set Γ may be finite, countable, or countable. If Γ is countable with some points of condensations on interval $[t, t+2)$, we call limit solutions by solutions of pre-turbulent type. If Γ is uncountable, then we call limit solutions by solutions of turbulent type [8, 11]. The similar limit distributions can be obtained for an 'angle of rotation' of a particle, that is, for phase of the wave function.

Including a permanent magnetic field we can observe more complex distributions of electron density, which is described by the term $|u(x, t, h)|^2 = |\varphi(x, t)|^2 + O(h)$ that in quantum mechanics has the interpretation as density of probability to find 'electron' in a given bulk. Notice that in classical mechanics $|\varphi(x, t, h)|^2$ has of measure corresponding to the transport equation in the canonical system. If $\Im S \neq 0$, the measure depends on the small parameter h, so that $|u|^2 \approx e^{-2\Im S/h}|\varphi|^2$ (see, [38], p.34).

7 Example 2

Is we assume time $t = t_{n+1}$ a kick of strength c is acting, that is $z_n^+ = z_n^- + c$. Then this relation on $\mathcal{C}$ - plane can be written as [17]

$$v_{n+1}^+ e^{i\varphi_n^+} = v_{n+1}^- e^{i\varphi_n^-} + c. \tag{120}$$

Putting (116),(117) into (120), we arrive at

$$v_{n+1}^+ e^{i\varphi_{n+1}^+} = \Phi_1(v_{n+1}^+, \varphi_n^+) \times e^{i\Phi_1(v_{n+1}^+, \varphi_n^+)} + c \tag{121}$$

(see, [17], Eq. (16)). Difference equations repeat the difference equations, which has been constructed for the quantum problem if we assume that $c = 0$. It corresponds to the case when of the simplism functional boundary conditions used in the canonical system.

8 One-dimensional topology

In one-dimensional topology, it is known that if there is a circle of period 3 then there are circles of each period. But in the two-dimensional topology, or on unit circle this statement is incorrect. If unit circle S^1 is parameterized by an angle $\Theta \in [0,1]$, then maps can be written as

$$\Theta_{n+1} = f(\Theta_n \,(mod\, 1). \tag{122}$$

Assume that we have two systems on periodic circle with frequencies γ and γ', respectively. Then $\Theta(t) = \gamma(t)\,(mod\, 1)$. Next, suppose that we need only measure the sample $\Theta_n := \Theta(t_0 + n/\gamma)$ of the first angle, which is given by

$$\Theta_{n+1} = f(\Theta_n + \omega\,(mod\, 1), \tag{123}$$

where $\omega = \gamma/\gamma'$. Then the map in (123) describes a rotation by a fraction ω of a full turn per sampling period. This rotation will be denoted by $R(\omega)$. Then the two different cases are: (1) If ω is a rational, p/q with $p, q \in Z$, then

$$\theta_{n+1} = \theta_n + q\omega\,(mod\, 1). \tag{124}$$

Such maps are called by lifts [27, 43, 26, 23, 25]. Hence, we obtain a periodic orbit, and θ_n has only a finite number of different values; (2) If ω is a irrational, the sequences θ_n is dense on $[0,1]$. Hence, we have a quasi-periodic orbit with a period which is a superposition of the two incommensurate frequencies γ and γ'.

Further, the set of rational numbers is dense on $[0,1]$ with zero measure. Then the relation $\Omega = \gamma/\gamma'$ is irrational with probability of 1, even we can find rational values arbitrary close . As a result, we observe 'frequency locking'. The frequency ratio Ω remains fixed at a neighbourhood of the point p/q, that is Ω is constant on some interval $(p/q - \Delta, p/q + \Delta)$, where a size Δ can be considered as a width of a plateau.

Let us consider the well-known Arnold map

$$\theta_{n+1} = \theta_n + \omega + \frac{k}{2\pi}\sin 2\pi\theta_n\,(mod\, 1). \tag{125}$$

The rotation number is

$$\rho = \lim_{n\to\infty} \frac{1}{n}\sum_{n=0}^{n-1} \Delta\theta_n \tag{126}$$

where

$$\Delta\theta_n = \omega + \frac{k}{2\pi}\sin 2\pi\theta_n. \tag{127}$$

Note that $\rho = \omega$ if $k = 0$.

If $k \leq 1$, then the rotation number $\rho(\omega, k)$ can be describe as following: (i) The ρ does not depend on orbits used to compute this number; (ii) If ρ is irrational, then the map is equivalent to the rotation R, and the motion is quasi-periodic; (iii) If $\rho = p/q$, then asymptotic orbits are periodic with a period q.

For $k \leq 1$, we can determine parameter regions in (ω, k) - plane, where $\rho(\omega, k)$ is rational. As an example, let us consider the region $\rho(\omega, k) = 0$. The corresponding asymptotic regime is $\theta_{n+1} = \theta_n$, which location is given by

$$\omega = -\frac{k}{2\pi}. \tag{128}$$

For $-k/2\pi \leq \omega \leq k/2\pi$, there are two stable fixed points and one unstable fixed point (see, [23], Figure 7.2)). Applying this result to the quantum boundary problem, we obtain 2 - periodic asymptotic functions with one point of discontinuities on a period, since here is only one repelling fixed point, and the main property is that the map is invertible. As a result, we get the example of limit distributions of relaxation type.

For $w = \pm k/2\pi$, the graph of the map is tangent to the diagonal, producing that stable and unstable periodic orbits are created together. These orbits must be destroyed through saddle-node bifurcations. Notice that the width of frequency-locking interval $\rho(\omega, k) = 0$ increases linearly with k that corresponds to $K = 1$ to almost one ω' from third possible values of w. Thus, for $k = \pm 2\pi$ we have saddle-node bifurcation.

9 Asymptotic properties of two dimensional maps

Thus, the circle map $\theta_n = f(\theta_{n-1}$, where θ_n is the angle, normalized to 2π, corresponds to n - th iteration around the circle. This map may be considered as the prototype for more complex behaviour generated by a class of two-dimensional circle maps. For such maps, there are two iterated variables that couple a radial coordinate r_n with an angle coordinate θ_n, so that

$$\theta_n = f(\theta_{n-1}, r_{n-1}), \quad r_n = g(\theta_{n-1}, r_{n-1}, \tag{129}$$

where f and g are arbitrary functions. But f is periodic, so that

$$f(\theta + 1) = f(\theta) + 1. \tag{130}$$

In other words, f is a lift. Moreover, there are functions $\hat{f}$ which are homeomorphic to f [23]. Particularly, sine circle map is the lift.

Remind that average number of rotations per iteration on a circle map, that is a measure of 'periodicity' of the system, can be calculated as

$$W(k, \omega) = \lim_{n \to \infty} \frac{1}{n}(f(\theta_n) - \theta_0). \tag{131}$$

For example, if in the sine map $\omega = 0$, then $W = 0$. The variable θ_n can be converged to a series, which is periodic in the sense that we obtain the lift

$$\theta_{n+p} = \theta_n + p \tag{132}$$

with rational $W = p/q$ for periodic series, and with irrational W for quasi-periodic series. Note that positions θ_{n+1} and θ_n changes monotonic around of perimeter of a torus.

Here, 'chaos' is possible only if a map is non-invertible. For example, it is possible for the sine map if the map has local extremum at the points

$$\theta_{max} = 1 - \frac{1}{2\pi}\cos^{-1} k^{-1}, \quad \theta_{min} = \frac{1}{2\pi}\sin^{-1} k^{-1} \tag{133}$$

This extremum exist only for $k > 1$, so that the line $k = 1$ means the possibility of chaos. More precisely, it means the existence of limit solutions of pre-turbulent and turbulent type for the quantum boundary problem (see, [**?**], Figure 1.B). For $k > 1$, the map is no invertible. Hence, if we have such boundary conditions that the quantum problem can be reduced to Arnold map, there are quasi-turbulent distributions of phases.

Next, the branch of the map can be sketched within the square $0 < \theta_n < 1$, $0 < \theta_{n+1} < 1$, because of its periodicity. For $k < 1$, and ω rational, there are frequency locking within regions, which are known as Arnold tongues. The region of corresponding parameters is $0 < k < 1$, $0 < \omega < 1$. Arnold tongues represent infinite numbers of tongues in this region. These tongues arise at each rational number p/q as $\omega \in [0, 1]$. Correspondingly, for the quantum boundary problem, in this region of parameters we obtain limit piecewise constant periodic solutions of pre-turbulent type.

As k is increasing then the tongues move together. But despite their finite width they do not intersect until $k = 1$. If $k = 1$, then overlap is complete. Above $k = 1$, there are possible (for difference equation with the phase) limit solutions of turbulent type. But ones can coexist with ordering that corresponds to so-called intermittency when limit solutions of pre-turbulent and turbulent type coexist. It is associated with previous frequency-locked regions. In the square $(p/q, \omega) := [0, 1] \times [0, 1]$, a plot against rational p/q generate a 'devil's staircase' if $k = 1$. In this case, each horizontal step for p/q corresponds to a frequency-locked state or limit solutions of relaxation type.

The 'staircase' is self-similar, that is in 'reality' there is overall structure for graphic of 'staircase' with ordering picture, which is fractal. Thus, we have the fractal structure for a phase of a wave function for the quantum boundary problem. The same structures can be construct for amplitudes.

10 Computer modelling

Note that there are parametric models for analysis of synthesis architecture iterations on a circle (see, for example, [26]). Indeed, let r be analysis segment index. We define the general expression for k - th partial parametric model, according to

$$\hat{x}_k(n, r) = \sum_{m=1}^{M_k} A_{m,k}(n, r) \cos\left(\Phi_{m,k}(n, r)\right), \tag{134}$$

where $A_{m,k}$ is the time-varying amplitude, M_k is the modelling order, that is a number of sinusoids of k - th partial model. $\Phi_{m,k}$ is the time-varying phase. Typical multi-pitch signals, modelling with the harmonic model is the following: If $K = 1$, we have

$$A_{m,k}(n) = a_m e^{d_m n}, \quad \Phi_{m,k}(n) = \omega_n n + \phi_m, \tag{135}$$

where $\omega_k, 0 \leq k \leq K - 1$ is a set of angular frequencies. And we have to satisfy $M_k \omega_k \leq \pi$. We denote by $a_{m,k}, \phi_{m,k}$ the m - th amplitude phase and real damping-factor parameterized of partial model. The pole is defined by $z_{m,k} = e^{d_{m,k} + i\omega_{m,k}}$.

Notice that due to the varying amplitude, the modulus of the pole is not limited to the unitary circle. Indeed, we have

$$|z_{m,k}| = e^{2d_{m,k}}, \quad arg\,(z_{m,k}) = \omega_k m. \tag{136}$$

The pole can take any value in the complex plane.

The synthesis of the modelling signal is

$$\hat{x}_l(n) = \hat{x}_{l-1}(n) + a_l d_l n \cos{(\omega_l n + \phi_l)}, \tag{137}$$

where $l = kM_k + m$ and $1 \leq l \leq M$. Typically, for large iterations we may obtain a piecewise constant periodic signal as shown in ([26], Figure 7). Thus, with help of such type algorithm we can construct an attractor of quantum problem.

Application can be found (to addition to the moving of charge particles) in the context of low bit-rate audio coding. The model is deduced to the multi-pitch speech signals, modelling and to the representation of multiple harmonic musical instruments [26].

11 Asymptotic for hyperbolic two dimensional dynamical systems

Now we consider a complex map $\hat{f}_\mu : (x, z) \to (x_z^2 + \mu, bz)$ which is determined by the system of difference equations:

$$x(t+1) = x^2(t) + z(t) + \mu, \tag{138}$$

$$, z(t+1) = bz(t) \tag{139}$$

where $b = e^a < 1$. Then asymptotic of solutions of system (138),(139) is determined by asymptotic properties of trajectories of the map

$$f_\mu : x \to x^2 + \mu \tag{140}$$

in R^1. For example, if $\mu > 1/4$ then the map f_μ has no fixed points, and $z(t) \to 0$, $x(t) \to +\infty$ as $t \to +\infty$, where $z(t)$ and $x(t)$ are imaginary and real parts of a point at a complex plane. If $\mu \leq 1/4$, then f_μ has the two fixed points $(x_1, 0), (x_2, 0)$, where

$$x_{1,2} = \frac{1}{2}(\left(1 \pm \sqrt{1 - 4\mu}\right). \tag{141}$$

There are no periodic points with a period which is larger then 1.

If $-3/4 < \mu < 1/4$, the point $(x_1, 0)$ is attractive of 'node' type, and the point $(x_2, 0)$ is of 'saddle' type (see, [11], Figure 72). A region of attraction of the point $(x_1, 0)$ is open unbounded region W at (x, z) - plane with a boundary

$$\partial W := \bigcup_{n=0}^{\infty} f_\mu^{-n}(P^\pm), \tag{142}$$

where ($P^\pm$ is a saddle point $P^\pm = (x_2, 0)$.

Now, we use the following definition: The filled Julia set of a function f defined as

$$K(f) = \{w \in \mathcal{C} : f^k(w) \to +\infty\}. \tag{143}$$

Then the Julia set of a function f is defined as a boundary of

$$J(f) = \partial K(f). \tag{144}$$

Then from Figure. 3 it follows that $K(f)$ represents all points $w = x + iz$, which lies below separatrices at (x, z) - plane. The set $J(f)$ lies on real axes in $\mathcal{C}$ upper the separatrices.

If $-5/4 < \mu < -3/4$, the two fixed points $(x_1, 0), (x_2, 0)$ are saddle type. The map $\hat{f}_\mu$ has an attractive circle of period 2, which is formed by points $(x_3, 0), (x_4, 0)$, so that

$$x_{3,4} = \frac{1}{2}(\left(-1 \pm \sqrt{-3 - 4\mu}\right) \tag{145}$$

with an attractive area W. The boundary ∂W contains points $a_1 = (x_1, 0)$, $a_2 = (x_2, 0)$ of saddle type and their separatrices $P^{(1)}_{\hat{f}_\mu}(a_1)$, $P^{(2)}_{\hat{f}_\mu}(a_1)$. Each of these separatrices consists from countable set of curves (see, Fig.3).

Let us define these curves by

$$G_k = \bigcup_{n=1}^{3} W_n^{(k)}, \tag{146}$$

where $k = 0, 1, 2, ...$ For example, on Figure 4, we have $k = 0$, or $k = 5$.

Then each set $\hat{G}_k$, which lies below the separatrices G_k, form the filled Julia set $K_k(f)$ which is repelling set because $f^k(w) \to \infty$ as $t \to \infty$. On the other hand, the Fatou set forms attractive set because on this set we obtain a limit function $p(t) \in (x_i, 0)$, $i = \bar{1,4}$ on a set $K(f)$. Then the Julia set

$$J(f) := \partial K(f) := \bigcup_{k=1}^{\infty} G_k \tag{147}$$

determines a set of points of discontinuities Γ of a limit solution, that is, these points correspond to characteristic of canonical system for the quantum problem.

If $J(f)$ is finite, we have limit solutions of relaxation type. If $J(f)$ is countable, we call corresponding limit solutions of the quantum problem by solutions of pre-turbulent type. If $J(f)$ is uncountable, then we obtain limit solutions of turbulent type, respectively.

On Fig.2, points u_3, u_6 are saddle type of codimension 1. Then the two-dimensional map has attractive circle of period $2l/p$, which is formed by points u_2, u_4 with an attractive region W, which contains u_3, u_6 - fixed points and their separatrices. Each of these separatrices is some pre-image of 'saddle-type' main separatrice. They together form finite, countable or countable set of curve $D_1, ..., D_6, ...$ Note that on Fig.2 it is described a case at $(Re\, z, Im\, z)$ - plane, but the same picture take place topologically also at (φ, S) - plane that can be done by the corresponding transform the variables.

11.1 Remark 1

According to the theory of Schr$o$der and Siegel (see, [25]) , certain complex analytic maps possess a family of closed invariant curves in the complex plane. We have made a numerical study of these curves by iterating the map, and have found that the largest curve is a fractal. When the winding number of the map is the golden mean, the fractal curve has universal scaling properties, and the scaling parameter differs from those found for other types of maps. Also, for this winding number, there are universal scaling functions.

12 Conclusion

An initial value boundary problem for the two linear system of the Shrödinger's type equations with nonlinear dynamic boundary conditions is considered. Approximate solutions with accuracy $O(h^2)$ are constructed. It is proved that an attractor of the problem contains periodic piecewise constant function along characteristic $dx/dt = \pm p$, which have finite, countable or uncountable number discontinuous characteristic — limit distributions of wave function of relaxation, pre-turbulent or turbulent type, respectively. The problem is solved by reduction of the origin problem to system of integro-difference equations of the Volterra type. The structure of the approximate attractor depends on initial; data of the quantum problem and topology of the boundary conditions. Physical applications to the motion of free charged particle in bounded box with nonlinear surface potentials in permanent magnetic field has been done. The connection with Mandelbrot set is discussed.

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