Quantum matter as a showcase for quantum gravity: analysis and implications

P. A. Mandrin

Glarnerstrasse 53a, 8854 Siebnen, Switzerland

E-mail: mandrin@hispeed.ch

Abstract. By generically constraining the boundary term of the action of gravity, the formal structure of the observed types of matter fields (scalar, fermion/Dirac and spin 1) is obtained in the weak gravity limit, including their gauge behaviour, covering the standard model. By gravity, we mean any theory having the Gibbons-Hawking-York boundary term as its torsion-free weak gravity limit. The constraining term is assumed to be local, not explicitly coordinate-dependent and to be the boundary term of a bulk function (Lagrangian). In this way, the latter is fixed to a large extent, admitting couplings and mass terms. The formal matching with observed fields suggests that matter should be the consequence of gravity constraining, and quantum matter would result from constrained quantum gravity. This implies that it is possible to compute the value of $6.564 \cdot 10^{-69}$ m^2 for the fundamental quantum constant of gravity – the smallest possible change of the boundary term. Also, the freedom to construct a fundamental quantum concept of gravity is strongly reduced, and the weak gravity limit is completely determined. For strong gravity, the boundary term – rather than the Hamiltonian – yields a key quantum counting operator.

1. Introduction

A common view is that experimental evidence for quantum gravity is still out of reach – unless we already possess experimental data without having noticed it. The goal of this article is to investigate such a possibility of a fundamental relation between quantum gravity and quantum field theory (QFT) for matter and, in the positive case, to take advantage from the experimentally well-established QFT in the weak gravity regime to derive quantum properties for gravity. This investigation not only provides a guide line for a quantisation restriction but also leads to the experimentally falsifiable computation of the fundamental quantum constant for gravity.

The idea that the quantisation of matter fields might reflect quantum space-time is not new, see [1] for a series of arguments. In such a case, if space-time has a discrete microscopic structure, it must cause matter to behave non-continuously as well, eventually preventing matter from being classical at the microscopic level. Nevertheless, no concrete proposal on the precise mechanism of the emergence of quantum matter from quantum space-time has been formulated yet.

Since it is not clear at this time what kind of theory of quantum gravity will survive future observations, we prefer to avoid any restricting a priori assumption on the quantisation procedure, rather concentrating on the relation between gravity and matter. In Section 2, we work out the generating mechanism for matter fields at the "classical" level by constraining gravity on its boundary term. Our perturbative analysis yields that there exist precisely only three possible types of non-trivial, local and not explicitly coordinate-dependent matter fields up to first order in the coordinate derivative and second order in the field power on the boundary (the perturbative analysis being justified because the experimentally best accessible parameter range for significant data is in the weak gravity regime). The emerging fields are of the scalar, the fermion (Dirac) and the spin 1 type, i.e. the observed field types emerge (standard model). In our analysis, we consider two matter field types as being distinct if the form of their matter field equations of motion differ. The detailed proofs are provided in Subsections 2.1–2.5, but can also be skipped by the busy reader. The proposed mechanism only works when considering the boundary term; we do not know of any alternative straight-forward mechanism capable of restricting the types of fields to these same three categories including their gauge behaviour. We conclude that a mere coincidence of such a precise field type matching is unlikely and propose to raise the gravitational constraint concept to a fundamental principle.

We argue in the second part of this article how the quanta of gravity (whatever they are) can force the emergence of quantum matter fields. On the other hand, based on the constraint mechanism, we exclude that matter quantisation could be caused by a mechanism which does not simultaneously affect gravity. As a consequence, the fundamental quantum constant of gravity is related to Planck's constant (via the Planck area). The predicted value for the quantum constant should allow a straight-forward future experimental test by investigating the motion or phase behaviour of a small number of particles propagating "within" their own quantised gravitational field. Moreover, since the quanta live on the 3d-boundary of a region of a general relativistic space-time manifold with no a-priori symmetries, the resulting joint gravitational and matter theory is not based on the concept of a Hamiltonian. This also means that the viability of the theory cannot be guessed from conventional renormalizability arguments.

The constrained gravity concept of matter has additional interesting features:

(i) This is the first proposal for which matter keeps a status different from gravity without being treated as an ad-hoc term. This is also in contrast with theories unifying non-gravitational and gravitational interactions on equal footing while inertia is supposed to emerge as a gravitational phenomenon. In the constrained gravity concept, matter emerges generically, thereby reducing at once the initial assumptions needed. These features are remarkable.

(ii) The constraint is applied to the boundary term of gravity, i.e. in a reduced dimension (3d or 2+1) space which is expected to be the dimension for quantum gravity [2].

(iii) Finally, gauge (or internal) symmetries directly follow from the residual freedom to modify the constraint "parameters" while leaving the gravitational configuration unchanged.

One may wonder why we should propose that nature causes matter to emerge from a gravitational constraint mechanism. To some extent, the motivations may be obtained from some thermodynamic or statistical mechanical interpretations of gravity, e.g.:

- The *path integral formulation* [3] of gravity provides an analogy with statistical mechanics for a canonical ensemble. A constraint field can then be seen as a kind of thermodynamic potential (much like e.g. the chemical potential changes the statistical probabilities of occurrence of certain thermodynamic states of a gas).
- *Thermodynamic formulations* are inspired by black hole thermodynamics [4, 5, 6] or, equivalently, by accelerated observers, via the Unruh temperature [7] or the First Law of thermodynamics interpretation of General Relativity (GR) by [8, 9], from which 2d-surface

areas acquire the status of an entropy as seen by the accelerated observer. In this context, the entropy is related to the heat crossing the horizon (radiation of matter). It is therefore natural to interpret the radiated matter as a manifestation of gravity.

- *Multiple observer entropy*: To go even further, when foliating the boundary of a space-time region into multiple 2d-surfaces, the Gibbons-Hawking-York boundary term acquires the interpretation of a sum of entropies perceived by multiple Rindler observers. In fact, only multiple observers would have a realistic chance to measure a horizon temperature [10]. Interestingly, the boundary term of the gravitational action (up to a non-dynamical term) is precisely proportional to the multiple observer entropy which is maximized in order to find the macroscopic state with highest probability (namely the boundary geometry of GR). The next step is to supply a metric-independent constraint field to the multiple observer entropy which reduces the degrees of freedom. If we then maximize the multiple observer entropy, we obtain a non-vacuum geometry with matter content [11].

2. Constraining the boundary term of gravity via a generic function

The considerations of this article are not restricted to one theory of gravity but allows for any models having GR as their torsion-free weak gravity limit. Considering GR as the simplest example, when computing the action by integrating over a space-time region V with boundary ∂V, the Gibbons-Hawking-York boundary term $S_{\partial V}$ (GHY-term) must be added to the Hilbert-action in order for the metric to be fixed on ∂V [3, 12]:

$$S_{\partial V} = \frac{c^4}{8\pi G} \int_{\partial V} d^3x \ \sqrt{|\gamma|} \ K + C, \tag{1}$$

where K is the trace of the second fundamental form K_{ab} on ∂V and can be written in terms of the induced metric γ_{ab} (with determinant γ) and the Lie derivative $\mathcal{L}_\perp$ with respect to the unit normal vector n^a on ∂V,

$$K = -\frac{1}{2}\gamma^{ab}\mathcal{L}_\perp \gamma_{ab}, \tag{2}$$

and ∂V (assumed spatial in (1)) can be generalised to any piece-wise smooth non-null boundary, C is a free term depending on γ_{ab} only and defines the asymptotic behaviour of γ_{ab} but has otherwise no impact on the gravitational setting. The non-dynamical term of type C is added independently of the chosen theory. While (1) is particularly useful in the context of the weak gravity regime, more general theories are included in the general concept, as for example $f(R)$ gravity (replacing R by $f(R)$ in the action) for which

$$S_{\partial V} = \frac{c^4}{8\pi G} \int_{\partial V} d^3x \ \sqrt{|\gamma|} \ \frac{df}{dR} \ K + C \tag{3}$$

is found [13, 14], while the metric and the scalar curvature R are fixed on ∂V.

Since the boundary term reflects part of the action, it cannot behave arbitrarily but must satisfy its own variation condition. Consider that the bulk action S_{bulk} of a given theory satisfies $\delta S_{bulk} = 0$ under the Neumann condition (fixed derivative of γ_{ab} or fixed torsion-less part of the spin connection) and $\delta(S_{bulk} + S_{\partial V}) = 0$ under the Dirichlet condition (fixed γ_{ab} or fixed frame field e^I_a, and optionally torsion), with $S_{\partial V} \neq 0$. Then, the boundary term $S_{\partial V}$ satisfies the following variation principle with respect to e.g. γ_{ab} (or e^I_a) and the connection:

$$\delta_b S_{\partial V} = 0, \tag{4}$$

where we define the boundary variation of the boundary term $\int_{\partial V} f([X])$ of an action with Lagrangian $F([X])$, with $\delta f = [\delta v(Y)]u(X) + v(Y)[\delta u(X)]$ and $Y = Y([X]) \nsim X$, by

$$\delta_b \int_{\partial V} f([X]) = \int_{\partial V} \delta_b f([X]), \qquad \delta_b f([X]) = v(Y)\frac{\partial u(X)}{\partial X}(\delta_1 X) + \frac{\partial v(Y)}{\partial Y}(\delta_2 Y)u(X), \quad (5)$$

with $\delta_i Z([X]) = Z([X + \delta_i X]) - Z([X])$, with "$[X]$" denoting the dependence on X and its derivative $X_{:\mu}$ and with variations $\delta_1 X, \delta_2 X$ which are related to each other by

$$\delta_1\left(\sqrt{-g}F\right) + \delta_2\left(\sqrt{-g}\left[F - \left(\frac{\partial F}{\partial X_{:\mu}}X\right)_{;\mu}\right]\right) = 0 \quad (6)$$

but otherwise arbitrary, and the colon in $X_{:\mu}$ denotes the partial derivative if X is a gravitational object, e.g. the metric, and the covariant derivative otherwise. The second term of (6) contains a divergence term which does not contribute to the equation of motion. In the case of GR, for example, we use: $X \to g^{\alpha\beta}$ or γ^{ab}, $u = X$, and $Y \to N_{ab} = \sqrt{\gamma}(K\gamma_{ab} - K_{ab})$, $v = Y$. (4) generalises correspondingly if higher derivatives are to be included. In (6), X is fixed and can be off-shell, $\delta_1 X$ and $\delta_2 X$ depend on each other, and (6) should not be confused with a "variation principle". By working out the variations in (6) and integrating over V, we immediately obtain (4) as soon as $g^{\alpha\beta}$ is on-shell, i.e. (4) yields the same result as the usual variation principle $\delta(\sqrt{-g}F) = 0$ with the appropriately fixed boundary condition, if (4) is applied to all possible volumes V within the region with non-vanishing divergence term. Therefore, (4) can be used to solve for physical configurations. This fact is needed in what follows.

Boundary terms like (1) or (3) only describe a vacuum geometry. In general, we are interested in off-vacuum geometries. Stationary off-vacuum geometries can be obtained by constraining the boundary term. Correspondingly, the bulk action must be constrained depending on how we are constraining the boundary term. Constraints can be imposed by an extra-term containing an "independent parameter $\underline{\varphi}$" (similarly to the Lagrange multiplier method): $S_{\partial V} \to S_{\partial V} + S^{constr}_{\partial V}(\underline{\varphi}; \ldots)$. For the desired new, off-vacuum geometry, the gravitational terms will come along with a net "gradient" with respect to γ_{ab} which must be compensated by the net "gradient" of the constraint term. The modified stationarity requirement reads:

$$\delta_b S^{constr}_{\partial V} = -\delta_b S_{\partial V}. \quad (7)$$

Unlike the case of vacuum gravity, we have (at least) one additional variable $\underline{\varphi}$ to vary, so that (7) implicitly contains (at least) 2 boundary variation restrictions in the form of (6), one for gravity (e.g. $\delta_1 g^{\alpha\beta}, \delta_2 g^{\alpha\beta}$) and the remaining one for $\delta_1\underline{\varphi}, \delta_2\underline{\varphi}$. Because the form of the constraint term is not given before-hand, we start from (7) and write the following generic ansatz for $S^{constr}_{\partial V}$ which satisfies locality and non-explicit coordinate dependence (a natural assumption since these two latter conditions already apply to $S_{\partial V}$):

$$S^{constr}_{\partial V} = \int_{\partial V} d^3x\, \sqrt{-g}\; 2\sigma([\underline{\varphi}], [\gamma_{ab}]) + C_m, \quad (8)$$

where we exemplarily consider metric-dependent gravity to simplify the formalism, and C_m is, again, a function of the metric alone and can be absorbed into C. The "parameter" $\underline{\varphi}$ is a formal object of type still to be determined. Since $\underline{\varphi}$ can be a (generalised) function of the coordinates and δ_b also involves the derivative, we use square brackets to denote dependence on $\underline{\varphi}$ and its first derivative $\nabla_\mu\underline{\varphi}$ ($\underline{\varphi}$ may also be tensorial). We shall justify later on by a perturbation argument why we assume no higher than the first order derivative to be involved. As for the derivatives of γ_{ab}, they are excluded below.

One of our assumptions is that a bulk constraint term S^{constr}_{bulk} needs to exist which has $S^{constr}_{\partial V}$ as its boundary compensation term. In fact, whenever S^{constr}_{bulk} depends on $\nabla_\alpha \underline{\varphi}$, a boundary term appears. We start with the bulk ansatz

$$S^{constr}_{bulk} = \int_V d^4x\ \sqrt{-g}\ 2\mathcal{L}([\underline{\varphi}], g_{\alpha\beta}). \tag{9}$$

We identify $\mathcal{L}$ as a Lagrangian candidate for matter and σ as the density of its boundary term. Like σ, the function $\mathcal{L}([\underline{\varphi}], g_{\alpha\beta})$ is not explicitly coordinate-dependent since σ is invariant under the local coordinate transformations leaving the metric unaltered and, therefore, $\mathcal{L}$ must be invariant and thus not explicitly coordinate-dependent in order to preserve the metric. Moreover, since σ is local, so is $\mathcal{L}$. To see this, one can inspect σ via a non-local ansatz (containing expressions $f(x, y)$ integrated over y). However, since σ is local, any local transformation $y \to y'(y)$, $f(x, y) \to f(x, y'(y))$ leaves σ invariant, and $\mathcal{L}$ must also be invariant, so that y can be chosen arbitraryly and $\mathcal{L}$ becomes local. Notice that $\mathcal{L}$ may depend on the metric, but not on its derivatives, otherwise this would yield additional gravitational boundary term contributions, i.e. the gravitational boundary term would depend on $\underline{\varphi}$, in contradiction to the definition of $S_{\partial V}$. Hence, σ does not depend on derivatives of γ_{ab} either. Variation of (9) yields:

$$\frac{\delta S^{constr}_{bulk}}{2} = \int_V d^4x\ \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} + \int_V d^4x\ \sqrt{-g}\ [EL]\delta\underline{\varphi} + \int_{\partial V} d^3x\ \sqrt{-g}\ n_\mu\ \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \underline{\varphi})} \delta\underline{\varphi}, \tag{10}$$

where

$$[EL] = \frac{\partial \mathcal{L}}{\partial \underline{\varphi}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \underline{\varphi})} \tag{11}$$

yields the "Euler-Lagrange Equation" (ELE) for the field $\underline{\varphi}$ when set to zero. There are two ways for obtaining the ELE: we either have to fix $\underline{\varphi}$ in the boundary term in (10) or to add the compensation term

$$S^{constr}_{\partial V} = -2 \int_{\partial V} d^3x\ \sqrt{-g}\ n_\mu\ \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \underline{\varphi})} \varphi, \tag{12}$$

to S^{constr}_{bulk} and then fix $\nabla_\alpha \underline{\varphi}$ on the boundary, in (opposite) equivalence to the boundary compensation procedure of gravity. In the case $\sigma = (\partial\sigma/\partial\underline{\varphi})\underline{\varphi}$ which will be the only relevant one for the analysis in this article, it follows:

$$\frac{\partial \sigma}{\partial \underline{\varphi}} = -n_\mu\ \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \underline{\varphi})}. \tag{13}$$

$\mathcal{L}$ has, therefore, the form

$$\mathcal{L} = -\frac{\partial \sigma}{\partial \underline{\varphi}}_{[\perp \to \mu]} \nabla_\mu \underline{\varphi} - V(\underline{\varphi}, g_{\alpha\beta}), \tag{14}$$

if we consider one single field $\underline{\varphi}$, where ${}_{[\perp\to\mu]}$ means replacement of the index ${}_\perp$ by ${}_\mu$, and $\perp$ incorporates arbitrary orientations since the boundary "shape" can be chosen arbitrarily from the set of all possible boundaries, i.e. the normal vector n^μ varies arbitrarily (piecewise smoothly) along the boundary.

We shall now analyse how $\sigma([\underline{\varphi}], \gamma_{ab})$ depends on $\underline{\varphi}$, by considering an expansion in terms of $\underline{\varphi}$. Such an expansion is reasonable in the weak gravity limit, assuming a nearly vanishing curvature, i.e. a stress tensor $T_{\alpha\beta} \sim \delta(\sqrt{-g}\mathcal{L})/\delta g^{\alpha\beta} \approx 0$ in cartesian coordinates, and thus small amplitudes of $\underline{\varphi}$, as follows. We define an amplitude of $\underline{\varphi}$ to have a critical value if a higher order term $\sim \mathcal{O}(\underline{\varphi}^{n_h})$ reaches a magnitude comparable to a lower order term $\sim \mathcal{O}(\underline{\varphi}^{n_l})$. For a much lower than critical amplitude of $\underline{\varphi}$ (weak gravity), the term $\sim \mathcal{O}(\underline{\varphi}^{n_h})$ contributes much less than the term $\sim \mathcal{O}(\underline{\varphi}^{n_l})$, i.e. higher order terms become small corrections for weak gravity. To work out the expansion, we think of the boundary ∂V as the union of small elements $d\Sigma_j$ centered at 3d-positions x^a_j, and the object $\underline{\varphi} = \underline{\varphi}(x^a)$ is a set of variables $\underline{\varphi}_j = \underline{\varphi}(x^a_j)$ (and we do the same for $\underline{\varphi}_{;b}$). Therefore, for every position labeled j, we can expand $\sigma(\underline{\varphi}_j, \underline{\varphi}_{;bj}, \gamma_{ab}(x^a_j))$ in terms of $\underline{\varphi}_j$ and $\underline{\varphi}_{;bj}$, and the continuum limit is finally recovered using $x^a_{j+1} - x^a_j \to 0$. Since all types of solutions of the ELE resulting from (14) are of concern, it may be to restrictive to treat $\underline{\varphi}_j$ as a real-valued scalar or tensor variable; rather, $\underline{\varphi}_j$ may also possess some number N of inner degrees of freedom. We must therefore take into account N different subvariables $\overset{i}{\underline{\varphi}}_j$, $i = 1 \dots N$, instead of just $\underline{\varphi}_j$ (a special case are complex values with $N = 2$), while (14) is extended to contain up to N terms $\sim \nabla_\mu \overset{i}{\underline{\varphi}}$. Furthermore, there may be more than one variable involved. With labels $[l]$, we write $\overset{i}{\underline{\varphi}}{}^{[l]}$ for every species. Keeping all this in mind, we obtain:

$$\sigma = \sum_{r \geq 0} \left[\sum_{k_1=0}^{1} \sum_{[l]i} A^{R}_{1k_1[l]i} \overset{i}{\underline{\varphi}}{}^{[l](k_1)}_{R} + \sum_{k_{21},k_{22}=0}^{1} \sum_{[l]i[m]j} A^{RS}_{2k_{21}k_{22}[l]i[m]j} \overset{i}{\underline{\varphi}}{}^{[l](k_{21})}_{R} \overset{j}{\underline{\varphi}}{}^{[m](k_{22})}_{S} + \dots \right], \tag{15}$$

where $\overset{i}{\underline{\varphi}}{}^{[l](k)}_{R}$ stands for the kth derivative of $\overset{i}{\underline{\varphi}}{}^{[l]}_{R} = \overset{i}{\underline{\varphi}}{}^{[l]}_{a_1 \dots a_r}$ with explicitly written tensor order r (the indices can be boundary space indices or $\perp$), and the coefficients $A^{R\dots}_{nk_{nm_1\dots m_n}[l]i\dots} = A^{R\dots}_{nk_{nm_1\dots m_n}[l]i\dots}(\gamma_{ab})$ do not explicitly depend on the coordinates. One of our assumptions is to consider contributions up to second order in $\overset{i}{\underline{\varphi}}{}^{[l](k)}_{R}$. One would expect a lot of independent types of expressions out of Expansion (15) – roughly given by the dimension of the vector space of the coefficients. In fact, only very few realisations truly give us new types of fields as will come out. To investigate this, we consider finite sums $\sigma_{[n,k,r]}$, where the square bracket indicates the maximum n, k, r occurring in each sum. We shall start with the simplest form – both in terms of $\overset{i}{\underline{\varphi}}{}^{[l]}_{R}$ and with respect to the tensor order r of $\overset{i}{\underline{\varphi}}{}^{[l]}_{R}$ – and progressively scan expressions and objects of increasingly complex form. Every $\sigma_{[n,k,r]}$ may give rise to a new form of Lagrangian and therefore to a new type of field $\underline{\varphi}$.

As specified previously, this procedure is executed up to first order in the derivative of $\underline{\varphi}$ on ∂V. This restriction is reasonable in the weak gravity limit, assuming a nearly vanishing curvature and thus a stress tensor $T_{\alpha\beta} \approx 0$ in cartesian coordinates, with the following argumentation. Every derivative ∇_μ applied to a Fourier mode $\sim \exp(ik_\mu x^\mu)$ of $\underline{\varphi}$ produces a factor ik_μ. Therefore, the kinetic term of $\mathcal{L}$ leads to at least a contribution $\sim k_\beta$ to $T_{\alpha\beta}$, so that we expect a growth of at least some part of $T_{\alpha\beta}$ with increasing mode frequency (or energy). At some critical frequency, a higher (k_h-th) derivative term may have a value comparable to a lower (k_l-th) derivative term. But for the much lower frequencies relevant for weak gravity, the k_h-th derivative term contributes much less than the k_l-th derivative term, i.e. higher derivative terms are small corrections for weak gravity. For this reason, we retain no higher than the first derivative. Despite the perturbative nature of our severe restrictions (up to first derivative and second order in the field power), our main goal to explore the relation between quasi-flat space QFT and constrained gravity remains unaffected.

Internal symmetries: Even after having found every $\sigma_{[n,k,r]}$ including the algebraic structure of the $\overset{i}{\underline{\varphi}}{}^{[l]}_{R}$, and after having solved the ELE up to a free choice of the potential terms, there remains some freedom for how to choose $\overset{i}{\underline{\varphi}}{}^{[l]}_{R}$ without affecting the gravitational configuration. This freedom causes restrictions on the admissible form of $\mathcal{L}$: If a global or local transformation of $\overset{i}{\underline{\varphi}}{}^{[l]}_{R}$ preserves σ, it must preserve $\mathcal{L}$. The (infinitesimal) local transformations are the restrictive ones. For instance, terms with products $(\overset{1}{\underline{\varphi}}{}^{[l_1]}\overset{2}{\underline{\varphi}}{}^{[l_2]})$ are preserved if the $\overset{i}{\underline{\varphi}}{}^{[l_i]}_{R}$ are multiplied by properly matched phase factors $\exp[i\delta\underline{\Omega}_i(x^\mu)]$. The underlying issue is: Due to the freedom of the potential terms, even if σ is preserved by a given transformation, $\mathcal{L}$ may change in a non-trivial way, causing the metric itself to change and eventually modifying the constraint condition on the boundary, in contradiction with the invariance of σ. We therefore have to require that $\mathcal{L}$ remains unaffected by all transformations under which σ is preserved. The transformations mentioned here are precisely gauge transformations of QFT (see the specific cases below). The gauge invariance of $\mathcal{L}$ is known to be crucial for renormalizability of a quantum field theory. It is interesting that matter fields need to have a Lagrangian if gauge invariance shall hold [15]. This also reinforces our assumption about the existence of an expression S^{constr}_{bulk} having $S^{constr}_{\partial V}$ as its boundary term.

In the weak gravity limit, it is the equations of motion for matter which can best be distinguished by the observations; equations of motion of gravity are harder to distinguish. For this reason, if two distinct Lagrangians $\mathcal{L}_1, \mathcal{L}_2$ have the same ELE, we assume that only $\mathcal{L}_1$ or $\mathcal{L}_2$ is of relevance.

Our next task is to reduce as much as possible the number of relevant combinations of n, k and $[l]$ numbers occurring within the same expression $\sigma_{[n,k,r]}$. Consider:

- Fixed $n = 1$: For $n = 1, k = 1$, we immediately have $\mathcal{L} = -V$. For $n = 1, k = 0$ (possibly mixed with $k = 1$), an additional σ-contribution $\sim \alpha_R \varphi^S$ appears, where α_R is constant, the index $\perp$ is either within R or within S, so that $\nabla_\mu \partial\mathcal{L}/\partial(\nabla_\mu \underline{\varphi}) = 0$. Therefore, for $n = 1$, the ELE yield $\partial V/\partial\underline{\varphi} = 0$ and thus V does not depend on $\underline{\varphi}$ and can be ignored. In summary, the ELE for $n = 1$ has physically trivial content.
- $n = 2$ terms with two different species $[l]$, $[m \neq l]$: For $k = 1$, σ contains terms of the form $\sim \overset{1}{\underline{\varphi}}{}^{[l]}\overset{2}{\underline{\varphi}}{}^{[m](1)}$ which leads to a Lagangian with kinetic term $K \sim \overset{1}{\underline{\varphi}}{}^{[l](1)}\overset{2}{\underline{\varphi}}{}^{[m](1)}$, and K has a second boundary term $\sim \overset{1}{\underline{\varphi}}{}^{[l](1)}\overset{2}{\underline{\varphi}}{}^{[m]}$. Expression K leads to ELEs with kinetic terms $\sim \nabla_\mu\nabla^\mu\overset{2}{\underline{\varphi}}{}^{[m]}, \nabla_\mu\nabla^\mu\overset{2}{\underline{\varphi}}{}^{[l]}$, and σ can be substituted by terms $\sim \overset{1}{\underline{\varphi}}{}^{[m]}\overset{2}{\underline{\varphi}}{}^{[m](1)}, \overset{1}{\underline{\varphi}}{}^{[l](1)}\overset{2}{\underline{\varphi}}{}^{[l]}$. For $k = 0$, we have terms of the form $\overset{1}{\underline{\varphi}}{}^{[l]R}\alpha_{R\perp S}\overset{2}{\underline{\varphi}}{}^{[m]S}$ with generalised constant factor $\alpha_{R\perp S}$, i.e. every component can be represented as a matrix. The reality condition for σ implies that we can group such expressions to pairs $\overset{1}{\underline{\varphi}}{}^{[l]R}\alpha_{R\perp S}\overset{2}{\underline{\varphi}}{}^{[m]} + \overset{2}{\underline{\varphi}}{}^{\dagger[m]S}\alpha^\dagger_{S\perp R}\overset{1}{\underline{\varphi}}{}^{\dagger[l]R}$ using adjoint conjugation †. Introducing $\overset{2}{\underline{\varphi}}{}^{[l]R} = \overset{1}{\underline{\varphi}}{}^{\dagger[l]R}$ and $\overset{1}{\underline{\varphi}}{}^{[m]S} = \overset{2}{\underline{\varphi}}{}^{\dagger[m]S}$, we obtain the kinetic term K of the Lagrangian, following the same procedure as outlined in Subsection 2.2,

$$\begin{aligned} K &\sim \lambda_{lm}\overset{1}{\underline{\varphi}}{}^{[l]R}\alpha_{R\mu S}\nabla^\mu\overset{2}{\underline{\varphi}}{}^{[m]S} + \mu_{lm}(\nabla^\mu\overset{1}{\underline{\varphi}}{}^{[l]R})\alpha_{R\mu S}\overset{2}{\underline{\varphi}}{}^{[m]S} \\ &+ \lambda_{ml}\overset{1}{\underline{\varphi}}{}^{[m]S}\alpha^\dagger_{S\mu R}\nabla^\mu\overset{2}{\underline{\varphi}}{}^{[l]R} + \mu_{ml}(\nabla^\mu\overset{1}{\underline{\varphi}}{}^{[m]S})\alpha^\dagger_{S\mu R}\nabla^\mu\overset{2}{\underline{\varphi}}{}^{[l]R} \end{aligned} \tag{16}$$

and therefore two ELEs with kinetic term $\sim \nabla_\mu\overset{i}{\underline{\varphi}}{}^{[l]}$ and two ELEs with kinetic term $\sim \nabla_\mu\overset{i}{\underline{\varphi}}{}^{[m]}$. Each of the obtained kinetic terms can be multiplied by further matrix factors

$\beta^{(\dagger)[l]S}_{T}, \beta^{(\dagger)[m]S}_{T}$ with appropriate dimensions in such a way that we can substitute the original σ-term by (i) $\underset{\sim}{\overset{1}{\varphi}}{}^{[l]R}\alpha_{R\perp T}\beta^{[l]T}_{S}\underset{\sim}{\overset{2}{\varphi}}{}^{[l]S}$ to obtain the first pair of ELEs and by (ii) $\underset{\sim}{\overset{1}{\varphi}}{}^{[m]R}\beta^{[m]T}_{R}\alpha_{T\perp S}\underset{\sim}{\overset{2}{\varphi}}{}^{[m]S}$ to obtain the second pair of ELEs. The equivalence holds whenever $(\alpha_{R\perp S}\beta^{[l]S}_{T})^\dagger = \alpha_{R\perp S}\beta^{[l]S}_{T}$ and $(\beta^{[m]R}_{T}\alpha_{R\perp S})^\dagger = \beta^{[m]R}_{T}\alpha_{R\perp S}$ (which also ensures the reality of the new σ-terms). We can achieve this by splitting the products $A = \alpha^{[l]}_{R\perp S}\beta^{[l]}$ or $\beta^{[m]}\alpha^{[m]}_{R\perp S}$ into hermitian and anti-hermitian parts, $A = A_+ + A_-$, with $A_\pm = (A \pm A^\dagger)/2$, thus doubling the terms of σ. Transforming e.g. $\beta^{[m]} \to i\beta^{[m]}$ allows to construct hermitian matrices for all terms. In summary, terms with different species can be substituted by terms with one single species each. As for the potential terms, they are not yet fixed. We will henceforth consider σ-expressions with pure species and omit the label $[l]$ for easier reading.

- Fixed $n = 2$: Different orders $k = 0, 1$ of derivatives of $\underset{\sim}{\overset{i}{\varphi}}{}_R$ occurring within the same expression, e.g. $\sigma \sim \underset{\sim}{\overset{1}{\varphi}}\, \partial_\perp \underset{\sim}{\overset{2}{\varphi}} + \underset{\sim}{\overset{1}{\varphi}}\, \alpha_\perp \underset{\sim}{\overset{2}{\varphi}}$, can be reduced to one single pure k. To achieve this, one can substitute $\underset{\sim}{\overset{2}{\varphi}} \to e^{-\alpha_\mu x^\mu}\underset{\sim}{\overset{2}{\varphi}}$, where the general exponential is defined as $e^A = \sum_{m\geq 0} A^m/m!$ and the objects α_μ are possibly matrix-valued. The term $\sim \alpha_\perp \underset{\sim}{\overset{2}{\varphi}}$ is compensated via the derivative of the exponential, and the exponential can be cancelled by $\underset{\sim}{\overset{1}{\varphi}} \to \underset{\sim}{\overset{1}{\varphi}} e^{\alpha_\mu x^\mu}$, so that we obtain a pure k expression for σ.
- Fixed k numbers: For $\sigma_{[2kr]}$ with $k_{21}, k_{22} = 0$ or 1, and subterms $\sim \underset{\sim}{\overset{i}{\varphi}}{}^{k_1}$ with $k_1 = k_{2i}$, we can complete the product, i.e. write $\sigma_{[2kr]}$ in the form $\sim [(\underset{\sim}{\overset{1}{\varphi}}{}^{(k_{21})} + \underset{\sim}{\overset{1}{a}})(\underset{\sim}{\overset{2}{\varphi}}{}^{(k_{22})} + \underset{\sim}{\overset{2}{a}})]$, where the extra term $\underset{\sim}{\overset{1}{a}}\underset{\sim}{\overset{2}{a}}$ merely affects the constant C, and then transform $\underset{\sim}{\overset{i}{\varphi}}{}^{(k_{2i})} \to \underset{\sim}{\overset{i}{\varphi}}{}^{(k_{2i})} - \underset{\sim}{\overset{i}{a}}$ $(i = 1, 2)$; this leaves us with a pure $n = 2$ expression.
- General case: With the fixed $n = 2$ trick, we only retain the largest k in the $n = 2$ terms. Any $n = 1$ terms with the same k as in at least one $n = 2$ term can be eliminated with the fixed k number trick. We only need to consider $n = 1$ terms for which k is not the same as in anyone of the $n = 2$ terms. However, $n = 1$ terms with $k = 1$ are trivial, we only need to consider terms $\sim \underset{\sim}{\overset{i}{\varphi}}$ with no $n = 2$ counterpart, i.e. only $n = 2$ terms $\sim \underset{\sim}{\overset{i}{\varphi}}{}^{(1)}\underset{\sim}{\overset{j\neq i}{\varphi}}{}^{(k_{2j})}$ are of concern in σ, whence $k_{2j} = 1$, otherwise a term $\sim \underset{\sim}{\overset{i}{\varphi}}{}^{(1)}\underset{\sim}{\overset{j\neq i}{\varphi}}{}^{(1)}$ would appear in the Lagrangian which would produce a term $\sim \underset{\sim}{\overset{i}{\varphi}}\underset{\sim}{\overset{j\neq i}{\varphi}}{}^{(1)}$ in σ. Therefore, terms $\sim \underset{\sim}{\overset{i}{\varphi}}$ with no $n = 2$ counterpart are trivial for the same reason as for the $n = 1$ case.

As we have seen, n can be restricted to $n = 2$ as the only case of interest. We are lead to retain expressions with single numbers n, k, r. However, this guide line does not prevent us from making exceptions below when gauge theoretical reasons suggest it. Table 1 shows an overview of all pure n, k, r expressions which are relevant in this sense.

As an example, the second line of Table 1 typically describes a term $\sim A_{201}(\gamma_{ab})\ \underset{\sim}{\varphi}^\dagger\ \partial_\perp \underset{\sim}{\varphi}$ and $\underset{\sim}{\varphi}$ is e.g. a scalar function. Or the object $\underset{\sim}{\varphi}$ on the fourth line is of the form $\underset{\sim}{\varphi}_a$. In our investigations, we concentrate on the weak gravity limit (with $\gamma_{ab} \approx \eta_{ab}$ using Euclidean coordinates, assuming a negligible cosmological constant Λ). In this limit, the detailed dependence on γ_{ab} is not essential. In case of ambiguities, we only retain the simplest expressions in terms of γ_{ab}. For later convenience, factors $\sqrt{-g}$ are kept throughout inside the integrals.

The expressions leading to the lines of Table 1 (except the first one) are analyzed in detail in the following subsections. In summary, it is shown that the second line yields the scalar type field, the third line yields the fermion (Dirac) type field, and the fourth line yields the spin 1

Table 1. Overview of possibly relevant formal expressions for σ.

Number of factors $\underline{\varphi}$	**Order of coordinate derivative**	**Tensor order of** $\underline{\varphi}$	**Resulting boundary term**
$n = 1$	$k = 0, 1$		Trivial ($\sigma \sim \underline{\varphi}_R$ or $\nabla_\perp \underline{\varphi}_R, \nabla_a \underline{\varphi}_R$)
$n = 2$	$k = 1$	$r = 0$	Scalar field term
$n = 2$	$k = 0$	$r = 0$	Fermion field term (Dirac)
$n = 2$	$k = 1$	$r = 1$	Spin 1 field term
$n = 2$	$k = 0$	$r = 1$	Gauge invariance fails
$n = 2$	$k = 0, 1$	$r \geq 2$	No degrees of freedom / Gauge invariance fails

type field (including Yang-Mills), while the remaining lines violate the assumptions. The busy reader interested in the conclusions can readily skip Subsections 2.1–2.5.

2.1. The scalar field term

On the second line of Table 1, we consider $n = 2$, $k = 1$ and $r = 0$. We start with $\sigma = -\kappa \overset{1}{\underline{\varphi}} \partial_\perp \overset{2}{\underline{\varphi}}$ ($\kappa =$ constant; the sign is only a convention) as our first ansatz. In the event that $\overset{1}{\underline{\varphi}}$ coincides $\overset{2}{\underline{\varphi}}$, the number of independent objects is reduced and a factor 2 appears in the following application of the functional derivative. However, the analysis of such special cases does not change the main outcome but merely affects field attributes like charge multiplicity; these shall not be discussed further in the context of this article for simplicity. Inserting σ into (14) and varying with respect to $\overset{1}{\underline{\varphi}}$ yields:

$$\mathcal{L} = \kappa \partial_\mu \overset{1}{\underline{\varphi}} \partial^\mu \overset{2}{\underline{\varphi}} - V, \tag{17}$$

$$\kappa \nabla_\mu \partial^\mu \overset{2}{\underline{\varphi}} + \frac{\partial V}{\partial \overset{1}{\underline{\varphi}}} = 0. \tag{18}$$

The first term of (17) is of the form of the kinetic term of the Klein-Gordon field for which it is most commonly assumed that $\overset{2}{\underline{\varphi}} = \underline{\varphi}$, $\overset{1}{\underline{\varphi}} = \underline{\varphi}^\dagger$. We shall, however, carry on with the general case with arbitrary $\overset{i}{\underline{\varphi}}$. The second term V can be interpreted as the "potential term" of $\mathcal{L}$. (18) is the corresponding ELE. The form of the kinetic term of (17) implies that an additional term should be incorporated in σ to compensate both divergence terms in the variation procedure:

$$\sigma = -\kappa \overset{1}{\underline{\varphi}} \partial_\perp \overset{2}{\underline{\varphi}} - \kappa (\partial_\perp \overset{1}{\underline{\varphi}}) \overset{2}{\underline{\varphi}}. \tag{19}$$

Because of locality, V can be expanded in terms of $\overset{i}{\underline{\varphi}}$:

$$V = V_0 + \underline{V}_1 \overset{1}{\underline{\varphi}} + \underline{V}_{1-} \overset{2}{\underline{\varphi}} + V_2 \overset{1}{\underline{\varphi}} \overset{2}{\underline{\varphi}} + \underline{V}_{2+} \overset{1}{\underline{\varphi}} \overset{1}{\underline{\varphi}}^T + \underline{V}_{2-} \overset{2}{\underline{\varphi}}^T \overset{2}{\underline{\varphi}} + \mathcal{O}(\overset{i}{\underline{\varphi}}^{(T)3}). \tag{20}$$

To obtain a direct interpretation of the coefficients of V, we take the limit $g_{\alpha\beta} \to \eta_{\alpha\beta}$ and write $\overset{2}{\underline{\varphi}}(x^\alpha)$ as a Fourier-back-transform in (18), $\overset{2}{\underline{\varphi}}(x^\alpha) = \int d^4k \, \exp(ik_\mu x^\mu) \overset{2}{\underline{\tilde{\varphi}}}(k_\beta)$. Ignoring higher than first order terms (in any fields), we rewrite:

$$\int d^4k\ e^{ik_\mu x^\mu}\ [\kappa k_\mu k^\mu \overset{2}{\underline{\tilde{\varphi}}}(k_\beta) - \underline{\tilde{V}}_1 - V_2\overset{2}{\underline{\tilde{\varphi}}}(k_\beta) - 2V_{2+}\overset{1}{\underline{\tilde{\varphi}}}(k_\beta) - \mathcal{O}(\overset{i}{\underline{\tilde{\varphi}}}^{(T)2})] = 0. \tag{21}$$

Therefore, the bracket expression of (21) must vanish for any k_β (this is the ELE in momentum space). By writing $\underline{\tilde{V}}_{12+} = \underline{\tilde{V}}_1 + 2V_{2+}\overset{1}{\underline{\tilde{\varphi}}}$, we immediately see that either $\underline{\tilde{V}}_{12+} = 0$ or $\underline{\tilde{V}}_{12+}$ is an expression of fields which are dynamical only if chosen from Table 1, and $\underline{\tilde{V}}_{12+}$ is of first order in any fields. However, $k_\mu k^\mu$ is invariant under Lorentz-transformations $k_\beta \to k'_\beta$, and V_2 does not depend on k_β. Expanding $\underline{\tilde{V}}_{12+} = \underline{\tilde{V}}_{10} + \underline{\tilde{V}}^\mu_{11} k_\mu + \ldots$, we see that all coefficients $\underline{\tilde{V}}^\mu_{1i}$ must be the same multiple of the respective coefficients of $\overset{2}{\underline{\tilde{\varphi}}}$, order by order, and thus $\underline{\tilde{V}}_{12+}(k_\beta) \sim \overset{2}{\underline{\tilde{\varphi}}}(k_\beta)$, and $\underline{\tilde{V}}_{12+}$ can be absorbed into $V_2\overset{2}{\underline{\tilde{\varphi}}}(k_\beta)$. The only remaining first order potential term in any fields in (21) is, therefore, the third one in the bracket, with $V_2 \sim m^2$ constant. Hence, up to second order in $\overset{i}{\underline{\varphi}}$, the expression V (20) also is Klein-Gordon type compatible. Higher orders in any fields allow for more general potentials (including e.g. Yukawa-type couplings (see below) and a φ^4-term). As would be expected, (18) can already be satisfied by a complex scalar function, $\overset{1}{\underline{\varphi}} = \varphi^*$, $\overset{2}{\underline{\varphi}} = \varphi$, but higher internal degrees of freedom are admissible as well.

Finally, we have to examine the gauge conditions. First of all, σ (19) is locally $U(1)$-invariant. Moreover, we may have local invariance of σ with respect to (infinitesimal) transformations $\overset{2}{\underline{\varphi}} \to \exp[i\delta\chi^m(x^\mu)T_m]\overset{2}{\underline{\varphi}}$ and $\overset{1}{\underline{\varphi}} \to \overset{1}{\underline{\varphi}}\exp[-i\delta\chi^m(x^\mu)T_m]$, where T_m are the generators of $SU(N/2)$, if the total number of degrees of freedom of $\overset{i}{\underline{\varphi}}$ is $N > 2$ (including real / imaginary parts), or $SU(N_s)$ if $N_s > 1$ distinct (interchangeable) scalar complex fields are comprised. All these invariances must hold for $\mathcal{L}$ as well. The kinetic term of $\mathcal{L}$ is already invariant, and we thus obtain a restricting condition for admissible potential terms, including potential terms with coupled fields from Table 1. At first order, the $SU(N/2)$- and $SU(N_s)$-invariances require that V_2 be diagonal in the inner degrees of freedom and does not cross-couple within the N_s fields, i.e. $V_2 \sim m^2$. With the procedure described in this article, the obtained invariances are in line with QFT (e.g. [16]).

To describe the most general case, we also should consider the ansatz $\sigma = -\kappa\overset{1}{\underline{\varphi}}\alpha\partial_\perp\overset{2}{\underline{\varphi}}$ for fields with $N > 2$ inner degrees of freedom, where α can be described in a matrix representation, and the symmetric order of the factors offers full generality. If α is regular, the ELE for this ansatz is equivalent to (21) with the condition of diagonal V_2 restricted or dropped depending on the structure of α. Indeed, if α is non-diagonal, we must skip (or reduce) the $SU(N/2)$ symmetry requirement for $\mathcal{L}$ from σ.

2.2. *The fermion field term (Dirac type)*

The third line expression of Table 1 differs from the second line by the missing derivative $\partial_\perp$:

$$\sigma = -\kappa\overset{1}{\underline{\varphi}}\alpha_\perp\overset{2}{\underline{\varphi}} \tag{22}$$

We have anticipated the relevant order of the factors (which can be non-commutative due to the internal degrees of freedom of $\overset{i}{\underline{\varphi}}$; the symmetric order allows full generality), and the object $\alpha_\perp$ in-between has a structure yet to be determined. Inserting $-\partial\sigma/\partial\overset{2}{\underline{\varphi}} = \kappa\overset{1}{\underline{\varphi}}\alpha_\perp$ and $-\partial\sigma/\partial\overset{1}{\underline{\varphi}} = \kappa\alpha_\perp\overset{2}{\underline{\varphi}}$ into (14) in the extended form and considering the limit $g_{\alpha\beta} \to \eta_{\alpha\beta}$ yields two different expressions for $\mathcal{L}$:

$$\mathcal{L} = \kappa\lambda_1(\partial_\mu \overset{1}{\underline{\varphi}})\alpha^\mu \overset{2}{\underline{\varphi}} + K_1^\mu([\overset{i}{\underline{\varphi}}])\partial_\mu \overset{2}{\underline{\varphi}} - V(\overset{1}{\underline{\varphi}}, \overset{2}{\underline{\varphi}}, g_{\alpha\beta}), \tag{23}$$

$$\mathcal{L} = (\partial_\mu \overset{1}{\underline{\varphi}})K_2^\mu([\overset{i}{\underline{\varphi}}]) + \kappa\lambda_2 \overset{1}{\underline{\varphi}}\alpha^\mu \partial_\mu \overset{2}{\underline{\varphi}} - V(\overset{2}{\underline{\varphi}}, \overset{1}{\underline{\varphi}}, g_{\alpha\beta}). \tag{24}$$

Identifying $\kappa\lambda_1\alpha^\mu \overset{2}{\underline{\varphi}} = K_2^\mu([\overset{i}{\underline{\varphi}}])$ and $\kappa\lambda_2 \overset{1}{\underline{\varphi}}\alpha^\mu = K_1^\mu([\overset{i}{\underline{\varphi}}])$ and defining

$$K_\pm = \kappa[\overset{1}{\underline{\varphi}}\alpha^\mu \partial_\mu \overset{2}{\underline{\varphi}} \pm (\partial_\mu \overset{1}{\underline{\varphi}})\alpha^\mu \overset{2}{\underline{\varphi}}] \tag{25}$$

gives us the general Lagrangian ansatz

$$\mathcal{L} = \lambda_+ K_+ + \lambda_- K_- - V(\overset{i}{\underline{\varphi}}, g_{\alpha\beta}). \tag{26}$$

Only the contribution K_+ gives rise to a non-vanishing boundary term when varied with respect to $\overset{1}{\underline{\varphi}}$ and $\overset{2}{\underline{\varphi}}$, this fixes $\lambda_+ = 1/2$, while λ_- remains free. On the other hand, in the limit $g_{\alpha\beta} = \eta_{\alpha\beta}$, $e^I_\alpha = \delta^I_\alpha$, $\mathcal{L}$ describes dynamical fields only if $\lambda_- \neq 0$, since the variation of the kinetic term must be non-vanishing and K_+ does not contribute to the ELE. While we derive the latter, the choice of λ_+ in turn is irrelevant, we may even set $\lambda_+ = 0$ (standard anti-symmetric form). We shall choose $\lambda_- = \pm 1/2$, these will be the suitable values in order to couple $\underline{\varphi}$ to other fields without unnecessary extra factors (see below):

$$\mathcal{L}_\pm = \pm\frac{\kappa}{2}[\overset{1}{\underline{\varphi}}\alpha^\mu \partial_\mu \overset{2}{\underline{\varphi}} - (\partial_\mu \overset{1}{\underline{\varphi}})\alpha^\mu \overset{2}{\underline{\varphi}}] - V(\overset{i}{\underline{\varphi}}, g_{\alpha\beta}). \tag{27}$$

E.g. $\mathcal{L}_+$ yields the ELE

$$\kappa\alpha^\mu \partial_\mu \overset{2}{\underline{\varphi}} - \frac{\partial V}{\partial \overset{1}{\underline{\varphi}}} = 0, \tag{28}$$

$$\kappa\partial_\mu \overset{1}{\underline{\varphi}}\alpha^\mu + \frac{\partial V}{\partial \overset{2}{\underline{\varphi}}} = 0. \tag{29}$$

Again, V can be expanded using (20) and we write the ELE (28) to first order in any fields in momentum space:

$$i\kappa\ \alpha^\mu k_\mu \overset{2}{\underline{\tilde\varphi}}(k_\beta) - \underline{\tilde V}_{12+}(k_\beta) - V_2 \overset{2}{\underline{\tilde\varphi}}(k_\beta) - \mathcal{O}(\overset{i}{\underline{\tilde\varphi}}{}^2) = 0. \tag{30}$$

Again, we immediately see that either $\underline{\tilde V}_{12+} = 0$ or $\underline{\tilde V}_{12+}$ is a first order expression with any of the Table 1 fields. The Lorentz invariance of $\mathcal{L}$ determines how the transformation

$$\overset{2}{\underline{\tilde\varphi}} \to S\overset{2}{\underline{\tilde\varphi}} \tag{31}$$

is non-trivially related to the transformation of k_μ. Correspondingly,

$$\overset{1}{\underline{\tilde\varphi}} \to \overset{1}{\underline{\tilde\varphi}}S^{-1} \tag{32}$$

ensures Lorentz-invariance of σ. On the other hand, the Lorentz invariance of $k_\mu k^\mu$ truly yields an additional condition and can thus be used to restrict (30) further. To this end, we multiply (30) from the left by $[i\kappa\ \alpha_\nu k^\nu + V_2]$, where $\alpha_\beta \approx \alpha^\alpha \eta_{\alpha\beta}$, and we simplify this with $\beta^\beta = -i\kappa\alpha^\beta$:

$$[\kappa^2\beta_\mu\beta^\nu k^\mu k_\nu + \kappa\beta_\mu k^\mu V_2 - \kappa V_2 \beta^\mu k_\mu - V_2^2]\overset{2}{\underline{\tilde\varphi}} + \kappa\beta_\mu k^\mu \underline{\tilde V}_{12+} - V_2 \underline{\tilde V}_{12+} = 0. \tag{33}$$

(33) considerably constrains the characteristics of β^β. Since the Lorentz transformation behaviour of $\overset{i}{\underline{\tilde{\varphi}}}$ is already fixed and (33) must hold independently of the Lorentz frame, we can split (33) into parts which transform differently from each other and therefore must vanish independently. In this process, the term $\underline{\tilde{V}}_{12+}$ progressively disappears from (33). The contribution $\underline{\tilde{V}}_1$ (first order in any fields) can be written as a sum, $\underline{\tilde{V}}_{1s}+\underline{\tilde{V}}_{1-}+\underline{\tilde{V}}_{1+}$, where $\underline{\tilde{V}}_{1s}$ is proportional to a field of type $[n=2, k=1, r=0, \overset{i}{\underline{\tilde{\varphi}}}']$ and $\underline{\tilde{V}}_{1\pm}$ of type $[n=2, k=0, r=0, \overset{1/2}{\underline{\tilde{\varphi}}}']$ ($\overset{i}{\underline{\tilde{\varphi}}}'$ denotes the coupled field; other types of fields are irrelevant as shown in the other subsections). We can immediately split off the terms transforming like (32) ($\underline{\tilde{V}}_{1+} + 2V_{2+}\overset{1}{\underline{\tilde{\varphi}}} = 0$) and also eliminate the only terms transforming like a scalar function ($\underline{\tilde{V}}_{1s} = 0$) and like a vector ($\kappa\beta_\mu k^\mu \underline{\tilde{V}}_{1s} = 0$). We next split off the terms which transform like (31), i.e. the term $\sim \underline{\tilde{V}}_{1-}$ as well as $\overset{2}{\underline{\tilde{\varphi}}}$ preceded by the extractable Lorentz-invariant part from the square bracket of (33), which can be either:

(i) $-V_2^2$,
(ii) $\kappa^2\beta_0\beta^0 k_\mu k^\mu - V_2^2$,
(iii) $\kappa^2\beta_i\beta^i k_\mu k^\mu - V_2^2$ for an arbitrary $i = 1, 2, 3$,
(iv) or combinations of (ii–iii).

Variant (i) is trivial (${\beta^0}^2 = 0 = {\beta^i}^2$), while the remaining variants lead to mutually equivalent results. It is therefore sufficient to pick out variant (ii) and to test its viability:

$$[\kappa^2\beta_0\beta^0 k_\mu k^\mu - V_2^2]\overset{2}{\underline{\tilde{\varphi}}} - V_2\underline{\tilde{V}}_{1-} = 0. \tag{34}$$

Using the same arguments as in Subsection 2.1, we can absorb $\underline{\tilde{V}}_{1-}$ into $V_2\overset{2}{\underline{\tilde{\varphi}}}$ in (30) and (34), i.e. ignore $\underline{\tilde{V}}_{1-}$. At this point, we introduce local coordinates with propagation along an arbitrarily selected i-direction: $k^j = 0$ for $j \neq i, 0$. The residual part of (33) then becomes

$$[\kappa^2(\beta_0\beta^0 - \beta_i\beta^i){k^i}^2 + \kappa^2(\beta_i\beta^0 - \beta_0\beta^i)k^0k^i + \kappa(\beta_0V_2 - V_2\beta^0)k^0 + \kappa(\beta_iV_2 - V_2\beta^i)k^i]\overset{2}{\underline{\tilde{\varphi}}} = 0. \tag{35}$$

Since (35) must hold for arbitrary k^0 and i (notice that β^ik^i depends on the arbitrary choice of i), every round bracket expression must vanish separately. We thus obtain the conditions

$$(\beta^0)^2 = -(\beta^i)^2; \quad [\beta^0, \beta^i]_+ = 0; \quad [\beta^0, V_2]_- = 0 = [\beta^i, V_2]_+. \tag{36}$$

For the residual part with propagation parallel to the ij-plane ($k^k = 0$ for $k \neq i, j, 0$), we obtain

$$[\beta^i, \beta^j]_+ = 0. \tag{37}$$

Using (34) and since ${\beta^0}^2 \neq 0$, we can find an explicit representation in which the number of components of $\overset{i}{\underline{\tilde{\varphi}}}$ is minimized but positive, i.e. ${\beta^0}^2$ is regular. Since ${\beta^0}^2$ is diagonalizable, there is even a representation so that $V_2^2 \sim 1$ ($\sim$ identity). With this, (34-37) can be summarised as

$$[\beta^\mu, \beta^\nu]_+ = 2\eta^{\mu\nu}(\beta^0)^2; \quad \kappa^2(\beta^0)^2 k_\mu k^\mu = V_2^2 \sim 1; \quad [\beta^0, V_2]_- = 0 = [\beta^i, V_2]_+, \tag{38}$$

i.e. $\beta^\beta = \kappa\gamma^\beta$ satisfies a Clifford algebra in 3 dimensions. The simplest realisation of (28) with (38) is of the form of a 2-component field (Majorana). The next realisation is the 4-component Dirac field, with $V_2 \sim \beta^0$. Realisations with $M \times 4 > 4$ internal degrees of freedom also must be examined, where $V_2 = \epsilon\beta^0$, ϵ acts on the M-fold and γ^μ on the 4-fold components. We distinguish fermions of

- Majorana type (2 components),
- (generalised) Dirac type: ϵ is diagonal,
- non-Dirac type: ϵ is non-diagonal, e.g. antisymmetric with $M = 2$ (ELKO) [17, 18].

The Majorana type must be excluded for the following reason. The Majorana field is identical to its anti-field and therefore neutral. On the other hand, the $U(1)$ invariance is satisfied by σ but not by the kinetic term, thus causing an excess term which requires a $U(1)$ coupling (see also examples below); but this coupling is suppressed since the field is neutral.

We shall next examine in more detail the $M = 1$ Dirac field. In order to properly couple $\mathcal{L}_\pm$ to gravity ($e^I_\alpha \neq \delta^I_\alpha$), (27) must be brought into a general covariant and thereby Lorentz gauge invariant form – this is best illustrated in the well-known case $\overset{2}{\underline{\varphi}} = \underline{\varphi}$, $\overset{1}{\underline{\varphi}} = \underline{\varphi}^\dagger$, using the covariant derivative for spinors, $\partial_\mu\underline{\varphi} \to D_\mu\underline{\varphi} = \partial_\mu - \frac{i}{4}\Gamma^{IJ}_\mu\sigma_{IJ}$, with $\Gamma^I_{J\alpha}e^J_\beta = e^I_\gamma\Gamma^\gamma_{\beta\alpha} - e^I_{\beta,\alpha}$, $\sigma_{IJ} = \frac{i}{2}[\gamma^I, \gamma^J]$ and $\gamma^\mu = \gamma^I e^\mu_I$, and we set $\lambda_+ = 0$ ($\mathcal{L}$ real and anti-symmetrtic) [19]:

$$\mathcal{L}_+ = \frac{\kappa}{2}[\bar{\underline{\varphi}}i\gamma^\mu D_\mu\underline{\varphi} - (D_\mu\bar{\underline{\varphi}})i\gamma^\mu\underline{\varphi}] - v_2\bar{\underline{\varphi}}\underline{\varphi}, \quad v_2 = \kappa\sqrt{k_\mu k^\mu}. \tag{39}$$

We then have to examine the internal gauge invariances with local infinitesimal transformations $\overset{1}{\underline{\varphi}} \to \overset{1}{\underline{\varphi}}\exp(i\delta\overset{1}{\chi})$, $\overset{2}{\underline{\varphi}} \to \exp(i\delta\overset{2}{\chi})\overset{2}{\underline{\varphi}}$ – we consider $g_{\alpha\beta} \approx \eta_{\alpha\beta}$. We find that σ is invariant with respect to $U(1)$ and to $SU(2)$ via a decomposition of $\underline{\varphi}$ into postive and negative helicity components $(1 \mp \gamma^5)\underline{\varphi}/2$, but not with respect to a larger group, e.g. the one generated by $\delta\overset{i}{\chi} = \delta\overset{i}{\chi}{}^\mu\alpha_\mu$. This is due to the fact that the direction of the normal vector on a boundary varies, i.e. σ fails to be invariant since $\alpha_\mu\alpha_\perp = \pm\alpha_\perp\alpha_\mu$ with upper sign for a component $\perp = \mu$ and lower sign for $\perp \neq \mu$. With respect to $U(1)$ and $SU(2)$, it is well-known that K_- is not invariant due to the excess (first order) terms $-i\kappa\overset{1}{\underline{\varphi}}\alpha^\mu(\partial_\mu\delta\overset{1}{\chi})\overset{2}{\underline{\varphi}}$ and $-i\kappa\overset{1}{\underline{\varphi}}\sigma_k\alpha^\mu(\partial_\mu\delta\overset{1}{\chi}{}^k)\overset{2}{\underline{\varphi}}/2$, where σ_k can be represented by the Pauli matrices and $\overset{i}{\underline{\varphi}}$ becomes a $SU(2)$ doublet field, but both terms can be compensated by a potential term of $\mathcal{L}_\pm$ which couples $\overset{i}{\underline{\varphi}}$ at third order to vector fields $\xi^0_\alpha, \xi^k_\alpha$ occurring on the fourth line of Table 1,

$$i\kappa\overset{1}{\underline{\varphi}}\alpha^\mu g\xi^0_\mu\overset{2}{\underline{\varphi}}, \qquad i\frac{\kappa}{2}\overset{1}{\underline{\varphi}}\alpha^\mu\bar{g}\sigma_k\xi^k_\mu\overset{2}{\underline{\varphi}}, \tag{40}$$

where the coupling constants $g, \bar{g}$ are introduced to enable free rescaling of $\xi^0_\alpha, \xi^k_\alpha$. (40) are the well-known "gauge connection" terms which are most commonly transferred to D_μ. The invariance of e.g. $\mathcal{L}_+$ requires that $g\xi^0_\alpha \to g\xi^0_\alpha + \partial_\alpha\delta\overset{1}{\chi}{}^0$, under which the Lagrangian for ξ^0_α can be kept locally invariant (see Subsection 2.3), and a corresponding transformation behaviour for ξ^k_α with an additional non-linear term. This allows for e.g. electro-weak gauge theory. There is also the possibility to have $n_f > 1$ different fermion fields for which σ is locally invariant under $SU(n_f)$ transformations, so that "colour couplings" are correctly supported, using Yang-Mills theory (see Subsection 2.3). Finally, as is well known, electro-weak gauge invariance of massive fermions can be satisfied if a Yukava-type coupling to a scalar field is included (e.g. Higgs); this is realised via the second line of Table 1. Although our gauge analysis is in the approximation $g_{\alpha\beta} \approx \eta_{\alpha\beta}$, an extension to the generally covariant formalism is possible by adapting the potential terms, or, equivalently, adapting the derivative D_μ.

In the presence of torsion (Einstein-Cartan theory [19]), we need to ensure that $\mathcal{L}$ remains compatible with σ. In a Riemann-Cartan manifold, the connection is $\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} - K_{\alpha\beta}{}^{\gamma}$, where $\Gamma^{\gamma}_{\alpha\beta}$ is the Christoffel symbol computed from $g_{\alpha\beta}$, and $K_{\alpha\beta}{}^{\gamma}$ is the contorsion tensor. We can split the covariant divergence term in the following way:

$$\int_V d^4x\ \sqrt{-g}\tilde{\nabla}_\mu(\underline{\varphi}^\dagger\alpha^\mu\delta\underline{\varphi}) = \int_V d^4x\ (\partial_\mu\ [\sqrt{-g}\ (\underline{\varphi}^\dagger\alpha^\mu\delta\underline{\varphi})] - \sqrt{-g}\ K_{\nu\mu}{}^{\mu}\underline{\varphi}^\dagger\alpha^\nu\delta\underline{\varphi})$$
$$= \int_{\partial V} d^3x\ \sqrt{-g}n_\mu(\underline{\varphi}^\dagger\alpha^\mu\delta\underline{\varphi}) - \int_V d^4x\ \sqrt{-g}K_{\nu\mu}{}^{\mu}\underline{\varphi}^\dagger\alpha^\nu\delta\underline{\varphi}, \qquad (41)$$

where $\tilde{\nabla}_\mu$ is the covariant derivative with torsion. The covariant divergence term splits into

- a divergence term without torsion which is compensated by our original boundary term and
- an extra term without derivative, coupling to the torsion at first order – it can be absorbed into the potential term of the bulk action.

The torsion field can be solved for using the equation of motion for the torsion which is part of the gravitational field equations (e.g. Einstein-Cartan).

To see whether an $M > 1$ theory could be viable, we first investigate additional infinitesimal local transformations which mix inner degrees of freedom and under which σ is invariant; these are $SU(M)$ transformations $\underline{\varphi}^a_A \to [\exp(-i\delta\chi^m T_m)]^B_A\underline{\varphi}^a_B$ and they mix the M-fold components only (not the 4-fold ones). Again, they do not leave K_- invariant, but the excess term can be compensated by a Yang-Mills coupling. Moreover, while the mass term is invariant for a Dirac type field, it is not for a non-Dirac type field, due to the non-diagonal $\epsilon = \epsilon^l T_l$. Transforming the non-Dirac "mass" term produces one more extra term $-\kappa v_2 \epsilon^l \underline{\bar{\varphi}}^A_a [T_l, T_m]^B_A \delta\chi^m \underline{\varphi}^a_B$ which must be compensated. One option is to accommodate this compensation via the above-mentioned Yang-Mills coupling by properly adapting its transformation behaviour. This is demonstrated in Subsection 2.3. As for the remaining symmetries, $U(1)$ and $SU(2)$ both apply to the Dirac-like fields, while the non-Dirac-like fields cause restrictions on the admissible couplings. As an example, $M = 2$ with totally antisymmetric tensor ϵ (ELKO [18]) yields a "charge conjugation eigenspinor" field; it must be neutral with respect to the charge it refers to and this prevents the $U(1)$ gauge coupling with respect to this charge. In the same way, for each non-Dirac field $\underline{\varphi}^l$ specified by a choice $\epsilon^l \neq 0$, one can find a "conjugation" operator $\mathcal{C}^l$ for which $\underline{\varphi}^l$ is an eigenspinor. Indeed, ϵ^l causes a crossing of spinor components. At the same time, there is a transformation $U \in SU(M)$ which causes precisely a crossing of the same spinor components. Although the generator of U would need to appear in the couplings, it is prevented by the eigenspinor property. This matter of fact requires us to exclude the non-Dirac field types.

To summarise, in conformity to the standard model, we have obtained Dirac-type fields with coupling to spin 1 type fields (e.g. electromagnetic, $W^\pm$, Z^0 or gluon), to a scalar field (e.g. Higgs) and the possibility to couple it to torsion.

2.3. The spin 1 field term

On the fourth line of Table 1, we consider a vector field $\overset{i}{\underline{\varphi}}_a$ or $\overset{i}{\underline{\varphi}}_\mu$ ($r = 1$) and a first order (covariant) derivative ($k = 1$). Three types of expressions,

$$\sigma_1 = -\kappa(\nabla_\perp \overset{1}{\underline{\varphi}}_a)\overset{2}{\underline{\varphi}}^a, \tag{42}$$

$$\sigma_2 = -\kappa(\nabla_a \overset{1}{\underline{\varphi}}_\perp)\overset{2}{\underline{\varphi}}^a, \tag{43}$$

$$\sigma_3 = -\kappa(\nabla_a \overset{1}{\underline{\varphi}}^a)\overset{2}{\underline{\varphi}}_\perp, \tag{44}$$

must be linearly combined. (42-44) lead to the following kinetic terms for the Lagrangian:

$$K_1 = \kappa\nabla_\alpha \overset{1}{\underline{\varphi}}_\beta \nabla^\alpha \overset{2}{\underline{\varphi}}^\beta, \tag{45}$$

$$K_2 = \kappa\nabla_\alpha \overset{1}{\underline{\varphi}}_\beta \nabla^\beta \overset{2}{\underline{\varphi}}^\alpha, \tag{46}$$

$$K_3 = \kappa\nabla_\alpha \overset{1}{\underline{\varphi}}^\alpha \nabla_\beta \overset{2}{\underline{\varphi}}^\beta. \tag{47}$$

These terms depend on the connection and therefore on the derivative of the metric which has to cancel out of $\mathcal{L}$. This is achieved by choosing the linear combination to be $\sim K_1 - K_2$, i.e.

$$\mathcal{L} = -\frac{\kappa}{2}\overset{1}{F}_{\alpha\beta}\overset{2}{F}^{\alpha\beta} - V \tag{48}$$

with the field tensor defined as

$$\overset{i}{F}_{\alpha\beta} = \nabla_\alpha \overset{i}{\underline{\varphi}}_\beta - \nabla_\beta \overset{i}{\underline{\varphi}}_\alpha, \tag{49}$$

so that we retain $\sigma = \sigma_2 - \sigma_1$. This boundary term only accounts for the variation of $\mathcal{L}$ with respect to $\overset{2}{\underline{\varphi}}^a$ and must be paired with an expression for the variation with respect to $\overset{1}{\underline{\varphi}}^a$:

$$\sigma = \frac{\kappa}{2}[\overset{1}{F}_{\perp a}\overset{2}{\underline{\varphi}}^a + \overset{1}{\underline{\varphi}}^a \overset{2}{F}_{\perp a}]. \tag{50}$$

Expanding V in (48) and writing the ELE $\delta\mathcal{L}/\delta\overset{1}{\underline{\varphi}}^a = 0$ in momentum space yields (with $g_{\alpha\beta} \to \eta_{\alpha\beta}$):

$$\kappa k_\beta k^\beta \overset{2}{\underline{\tilde{\varphi}}}_\alpha(k_\mu) - \kappa k_\alpha k^\beta \overset{2}{\underline{\tilde{\varphi}}}_\beta(k_\mu) - \underline{\tilde{V}}_{12+\alpha} - V_2 \overset{2}{\underline{\tilde{\varphi}}}_\alpha(k_\mu) - \mathcal{O}(\overset{i}{\underline{\tilde{\varphi}}}{}_\mu^{(T)2}) = 0. \tag{51}$$

We split (51) into a longitudinal component by multiplying it by k^α, and a transverse component by computing its 4d-vector product with k_γ, using the operation $v_\alpha \to v_\alpha k_\gamma - v_\gamma k_\alpha$:

$$k^\alpha \underline{\tilde{V}}_{12+\alpha} = k^\alpha V_2 \overset{2}{\underline{\tilde{\varphi}}}_\alpha, \tag{52}$$

$$\kappa k_\beta k^\beta (\overset{2}{\underline{\tilde{\varphi}}}_\alpha k_\gamma - \overset{2}{\underline{\tilde{\varphi}}}_\gamma k_\alpha) = \underline{\tilde{V}}_\alpha k_\gamma - \underline{\tilde{V}}_\gamma k_\alpha, \tag{53}$$

where $\underline{\tilde{V}}_\alpha = \underline{\tilde{V}}_{12+\alpha} + V_2 \overset{2}{\underline{\tilde{\varphi}}}_\alpha$. From (52), we can ignore the longitudinal components of $\underline{\tilde{V}}_\alpha$. With (53), we find $\underline{V}_\alpha \sim \overset{2}{\underline{\varphi}}_\alpha + \partial_\alpha \underline{\chi}$, where $\underline{\chi}$ is an arbitrary function. We are free to fix $\underline{\chi}$ so that $\nabla^\beta \overset{2}{\underline{\varphi}}_\beta = 0$ (Lorenz gauge), and the second term of (51) vanishes. Therefore, to first order in any fields, since $k_\beta k^\beta$ is Lorentz-invariant and V_2 is k_β-independent, we can skip the $\tilde{V}_{12+\alpha}$-term for the same reason as in Subsection 2.1. To first order in any fields, $\overset{2}{\underline{\varphi}}_\alpha$ obeys the same equation of motion as the spin 1 field (for the Yang-Mills form of $\mathcal{L}$, see below). The simplest form of $\overset{i}{\underline{\varphi}}_\alpha$ is a vector field $\overset{i}{\xi}_\alpha$ which is (at most) complex-valued. There is also the option of additional internal degrees of freedom or additional species of spin 1 fields, which allows for Yang-Mills type fields.

We shall first analyse the abelian theory, writing $\overset{1}{\underline{\varphi}}_\alpha = \underline{\varphi}^\dagger_\alpha$, $\overset{2}{\underline{\varphi}}_\alpha = \underline{\varphi}_\alpha$ for simplicity although all steps generalise to $\overset{i}{\underline{\varphi}}_\alpha$ fields. For σ, we find local invariances only if $\underline{\varphi}^\dagger_\alpha = \underline{\varphi}_\alpha$. Then we have $U(1)$-invariance and invariance under (infinitesimal) transformations $\underline{\varphi}_\alpha \to [\exp(i\delta\chi)]^\beta_\alpha \underline{\varphi}_\beta$, where the $\delta\chi^b_a(x^\mu)$ are real. These invariances must hold for $\mathcal{L}$ as well. In all mentioned cases, the kinetic term of $\mathcal{L}$ is already invariant, and we thus obtain a restricting condition for admissible potential terms. Moreover, if $\mathcal{L}$ is the Lagrangian of a gauge field which couples to a fermion field (Subsection 2.2 plus 2.3), we have to ensure that $\mathcal{L}$ be invariant under infinitesimal gauge transformations

$$g\varphi_\alpha \to g\varphi_\alpha + \partial_\alpha \delta\chi, \tag{54}$$

as announced in Subsection 2.2, simultaneously with the abelian fermion transformation $\underline{\varphi} \to \exp(-i\delta\chi)\underline{\varphi}$. This is already the case for the kinetic term of $\mathcal{L}$, and the potential term can be restricted so that the sum $\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_f + \mathcal{L}_i$ satisfies the gauge condition, where $\mathcal{L}_f$ is the uncoupled fermion Lagrangian and $\mathcal{L}_i$ is the fermion-spin-1 coupling term, see Subsection 2.2. In the case $\underline{\varphi}^\dagger_\alpha = \underline{\varphi}_\alpha$, in order for $\mathcal{L}_i$ to satisfy the remaining, above-mentioned invariances of σ, it suffices to adapt the first $\mathcal{L}_i$ term in (40) to

$$i\kappa \overset{1}{\underline{\varphi}}_f \alpha^\mu g(\xi^0_\mu + \xi^{\dagger 0}_\mu) \overset{2}{\underline{\varphi}}_f / 2. \tag{55}$$

In case of torsion, any contribution $\sim K^\rho_{\alpha\beta}\underline{\varphi}_\rho$ to $F_{\alpha\beta}$ can be split away from the divergence term in $\delta\mathcal{L}/\delta\underline{\varphi}^{\dagger\mu}$ (similarly to Subsection 2.2), so that $\sigma \sim \underline{\varphi}^{\dagger a} F_{\perp a} + F^\dagger_{\perp a}\underline{\varphi}^a$ with

$$F_{\alpha\beta} = \nabla_\alpha \underline{\varphi}_\beta - \nabla_\beta \underline{\varphi}_\alpha + 2K^\rho_{\alpha\beta}\underline{\varphi}_\rho \tag{56}$$

(which is the torsion-less spin 1 field tensor). With (56), we immediately obtain a torsion-less kinetic term in a Lagrangian of the form of (48) and a coupling to torsion in the potential term.

Non-abelian theory in the limit $g_{\alpha\beta} \to \eta_{\alpha\beta}$: If more than one field or inner degree of freedom is related to the gauge mechanism of more than one fermion (of common structure), we label the coupled spin 1 fields by an extra index k: $\underline{\varphi}_\alpha = \xi^k_\alpha$. We shall start with the fermion transformation $\underline{\varphi}_i \to [exp(-i\delta\chi^k T_k)]^j_i \underline{\varphi}_j$ from Subsection 2.2 which represents a non-abelian gauge group. For Dirac type fields, imposing gauge invariance on the Lagrangian with spin 1 coupling leads to the well-known transformation law for Yang-Mills fields:

$$g\xi^k_\alpha \to g\xi^k_\alpha + \partial_\alpha \delta\chi^k + g f^k_{lm} \xi^l_\alpha \delta\chi^m, \tag{57}$$

where f^k_{lm} are the (completely antisymmetric) structure constants satisfying $[T_l, T_m] = i f^k_{lm} T_k$. For the gauge mechanism to work, we must bring the spin 1 Lagrangian into an invariant form. Indeed, this can be achieved by replacing (49) by the well-known Yang-Mills field tensor

$$F^k_{\alpha\beta} = \partial_\alpha \xi^k_\beta - \partial_\beta \xi^k_\alpha + g f^k_{lm} \xi^l_\alpha \xi^m_\beta \tag{58}$$

and using the Jacobi identity for f_{klm}. For non-Dirac type fermion fields, to take into account the "mass" term with non-diagonal ϵ, we can modify the transformation law (57) to

$$g\bar{\xi}^k_\alpha \to g\bar{\xi}^k_\alpha + \partial_\alpha \delta\chi^k + f^k_{lm} g \bar{\xi}^l_\alpha \delta\chi^m, \tag{59}$$

together with the substitution

$$\bar{\xi}^k_\alpha = \xi^k_\alpha + \frac{v_2}{4g}\gamma_\alpha \epsilon^k. \tag{60}$$

Also replacing $\xi^k_\alpha \to \bar{\xi}^k_\alpha$ in (58), we immediately see that the resulting new kinetic term for the spin 1 Lagrangian is invariant, and the potential term can be restricted to be invariant as well. In the case $\underline{\varphi}^\dagger_\alpha = \varphi_\alpha$, in order for $\mathcal{L}_i$ to satisfy the remaining invariances of σ, it suffices to adapt the $\mathcal{L}_i$ terms in (40) to

$$i\kappa \overset{1}{\underline{\varphi}}_f \alpha^\mu g(\xi^0_\mu + \xi^{\dagger 0}_\mu)\overset{2}{\underline{\varphi}}_f/2, \qquad i\kappa \overset{1}{\underline{\varphi}}_f \alpha^\mu \bar{g}\sigma_k(\xi^k_\mu + \xi^{\dagger k}_\mu)\overset{2}{\underline{\varphi}}_f/2. \tag{61}$$

2.4. No derivative-free vector field

In contrast to the fourth line of Table 1, the fifth line does not provide a derivative of the vector field in the expression for σ, and we must consider a linear combination of three expressions:

$$\sigma_1 \sim \overset{1}{\underline{\varphi}}^a \alpha^1_{a\perp b}\overset{2}{\underline{\varphi}}^b, \tag{62}$$

$$\sigma_2 \sim \delta^b_\perp \overset{1}{\underline{\varphi}}_b \alpha^2_a \overset{2}{\underline{\varphi}}^a, \tag{63}$$

$$\sigma_3 \sim \overset{1}{\underline{\varphi}}^a \alpha^3_a \overset{2}{\underline{\varphi}}_b \delta^b_\perp. \tag{64}$$

A special case of (62–64) can be obtained by setting $\alpha^1_{a\perp b} = \alpha^1_\perp \eta_{ab}$. For this case, the same procedure as in Subsection 2.2 leads to the kinetic terms

$$K_1/\kappa = \lambda_1 \overset{1}{\underline{\varphi}}_\alpha \alpha^1_\beta \nabla^\beta \overset{2}{\underline{\varphi}}^\alpha + \mu_1(\nabla^\beta \overset{1}{\underline{\varphi}}_\alpha)\alpha^1_\beta \overset{2}{\underline{\varphi}}^\alpha, \tag{65}$$

$$K_2/\kappa = \lambda_2 \overset{1}{\underline{\varphi}}_\beta \alpha^2_\alpha \nabla^\beta \overset{2}{\underline{\varphi}}^\alpha + \mu_2(\nabla^\beta \overset{1}{\underline{\varphi}}_\beta)\alpha^2_\alpha \overset{2}{\underline{\varphi}}^\alpha, \tag{66}$$

$$K_3/\kappa = \lambda_3 \overset{1}{\underline{\varphi}}^\alpha \alpha^3_\alpha \nabla_\beta \overset{2}{\underline{\varphi}}^\beta + \mu_3(\nabla_\beta \overset{1}{\underline{\varphi}}^\alpha)\alpha^3_\alpha \overset{2}{\underline{\varphi}}^\beta. \tag{67}$$

The covariant derivative $\nabla_\alpha \overset{2}{\underline{\varphi}}_\beta$ contains the connection $\Gamma^\rho_{\alpha\beta}$, and the symmetric part $\Gamma^\rho_{(\alpha,\beta)}$ (the Christoffel symbol) is determined by derivatives of the metric which must cancel. Therefore, each term containing e.g. $\nabla_\alpha \overset{2}{\underline{\varphi}}_\beta$ should have $\Gamma^\rho_{(\alpha,\beta)}$ cancelled by $-\Gamma^\rho_{(\beta,\alpha)}$ from a term containing $-\nabla_\beta \overset{2}{\underline{\varphi}}_\alpha$. In this manner, no space-time connection remains, except for a possible contorsion term which is not a kinetic gravitational contribution. We obtain $\alpha^1_\alpha = \alpha^2_\alpha = \alpha^3_\alpha = \alpha_\alpha$ together with $\lambda_2 = -\lambda_1 = -\lambda$, $\mu_3 = -\mu_1 = -\mu$, and $\mu_2 = \lambda_3 = 0$ since the second term of (66) and the first term of (67) cannot be paired for cancelling. Therefore, (65–67) becomes

$$\mathcal{L} = \kappa\lambda \overset{1}{\underline{\varphi}}_\beta \alpha_\alpha(\nabla^\alpha \overset{2}{\underline{\varphi}}^\beta - \nabla^\beta \overset{2}{\underline{\varphi}}^\alpha) + \kappa\mu(\nabla^\alpha \overset{1}{\underline{\varphi}}_\beta - \nabla_\beta \overset{1}{\underline{\varphi}}^\alpha)\alpha_\alpha \overset{2}{\underline{\varphi}}^\beta - V. \tag{68}$$

We now repeat the procedure starting from (62) which can be used as the general ansatz (without the restriction $\alpha^1_{a\perp b} = \alpha^1_\perp \eta_{ab}$). The kinetic term reads (with term labels skipped):

$$K/\kappa = \lambda \overset{1}{\underline{\varphi}}^\alpha \alpha_{\alpha\gamma\beta} \nabla^\gamma \overset{2}{\underline{\varphi}}^\beta + \mu(\nabla^\gamma \overset{1}{\underline{\varphi}}^\alpha)\alpha_{\alpha\gamma\beta}\overset{2}{\underline{\varphi}}^\beta. \tag{69}$$

In order to suppress all Christoffel symbols again, we need to antisymmetrize the differentials in (69). Or, equivalently, $\alpha_{\alpha\gamma\beta}$ must be totally antisymmetric in α, γ, β. If we again examine the special case $\alpha_{a\perp b} = \alpha_\perp \eta_{ab}$ and antisymmetrize $\alpha_{a\perp b}$, it vanishes identically, therefore only the full expression (69) may be non-trivial. Again, we define $K_\pm = K[\mu = \pm\lambda]$. Evaluating the ELE in momentum space in analogy to Subsection 2.2, one finds that $\alpha_{\alpha\gamma\beta}$ satisfies a Clifford algebra with anti-commutator $[\alpha_\alpha{}^{\mu\delta}, \alpha_\delta{}^\nu{}_\beta]_+ \sim \eta^{\mu\nu} V_{2\alpha}{}^\delta V_{2\delta\beta}$.

Examining the internal invariances of σ, we find (i) $U(1)$ invariance and (ii) one more invariance with respect to local infinitesimal transformations $\overset{1}{\underline{\varphi}}{}^{\alpha} \to \overset{1}{\underline{\varphi}}{}^{\beta}[\exp(i\delta\overset{1}{\chi})]^{\alpha}_{\beta}$, $\overset{2}{\underline{\varphi}}{}^{\alpha} \to [\exp(i\delta\overset{2}{\chi})]^{\alpha}_{\beta}\overset{2}{\underline{\varphi}}{}^{\beta}$, where the $\delta\overset{i}{\chi}{}^{\alpha}_{\beta}$ are real, with $\delta\overset{1}{\chi}{}^{\delta}_{\alpha}\alpha_{\delta\gamma\beta} + \alpha_{\alpha\gamma\delta}\delta\overset{2}{\chi}{}^{\delta}_{\beta} = 0$ or, equivalently, $\delta\overset{1}{\chi}{}^{\delta}_{\alpha} = \delta\overset{2}{\chi}{}^{\delta}_{\alpha} = \delta\chi^{\delta}_{\alpha}$ (obtained for $\alpha = \beta$) and $\delta\chi^{\delta}_{\alpha}\alpha_{\delta\gamma\beta} = \delta\chi^{\delta}_{\beta}\alpha_{\delta\gamma\alpha}$. This represents 16 elements of $\delta\chi^{\delta}_{\alpha}$ and 12 equations (i.e. combinations of component index triples $\gamma < \alpha < \beta$) with Clifford-algebra valued coefficients. In every equation, both coefficients have the same γ-index, i.e. the same algebra element. This non-singular system of equations is underdetermined by 4 excess degrees of freedom. Therefore, (ii) must be taken into account in our analysis. K_- is not invariant with respect to any of (i) or (ii). While the excess term from (i) can easily be compensated by third order term coupling to a vector field ξ_α occurring on the fourth line of Table 1, (ii) would require a third order term coupling to a tensor field $\xi_{\alpha\beta\gamma}$ (i.e. $r = 3$) which is not available according to the sixth line of Table 1. Therefore, $\mathcal{L}$ fails to satisfy invariance (ii). For this reason, we must exclude the case $n = 2$, $k = 0$, $r = 1$.

2.5. Higher tensor order fields

Throughout this subsection, although we write $\overset{2}{\underline{\varphi}}_R = \underline{\varphi}_R$, $\overset{1}{\underline{\varphi}}_R = \underline{\varphi}^{\dagger}_R$ for simplicity, this restriction is not necessary for the following steps to be valid.

In the $n = 2$, $k = 1$, $r \geq 2$ case, we have to repeat the procedure of Subsection 2.3. In order for the Christoffel symbols to cancel, we must antisymmetrize as follows:

$$\mathcal{L} = -\frac{\kappa}{(r+1)!} F^{\dagger}_{\alpha_1\ldots\alpha_{r+1}} F^{\alpha_1\ldots\alpha_{r+1}} - V(\underline{\varphi}_\mu, g_{\mu\nu}) \tag{70}$$

with

$$F_{\alpha_1\ldots\alpha_{r+1}} = \sum_{\pi} (-1)^{\mathrm{sgn}(\pi)} \tilde{\nabla}_{\pi_1} \underline{\varphi}_{\pi_2\ldots\pi_{r+1}}, \tag{71}$$

where π: $(\alpha_i) \mapsto (\pi_i)$, $i = 1 \ldots r+1$ are the permutations of the indices and $\tilde{\nabla}$ is the torsionless covariant derivative. Neglecting torsion effects, expanding V in (70) and writing the ELE $\delta\mathcal{L}/\delta\underline{\varphi}^{\dagger\alpha_2\ldots\alpha_{r+1}} = 0$ in momentum space yields (with $g_{\alpha\beta} \to \eta_{\alpha\beta}$):

$$\kappa k^{\alpha_1}[\sum_{\pi} (-1)^{\mathrm{sgn}(\pi)} k_{\pi_1} \underline{\tilde{\varphi}}_{\pi_2\ldots\pi_{r+1}}] - \underline{\tilde{V}}_{12+\ \alpha_2\ldots\alpha_{r+1}} - V^{\beta_2\ldots\beta_{r+1}}_{2\alpha_2\ldots\alpha_{r+1}} \underline{\tilde{\varphi}}_{\beta_2\ldots\beta_{r+1}} - \mathcal{O}(\underline{\tilde{\varphi}}^2) = 0. \tag{72}$$

This can also be written as

$$\kappa k^{\alpha_1}[\sum_{i=1}^{r+1} k_{\alpha_i} \underline{\tilde{\varphi}}^{A}_{\alpha_{i+1}\ldots\alpha_{r+1}\alpha_1\ldots\alpha_{i-1}}] - \underline{\tilde{V}}_{12+\ \alpha_2\ldots\alpha_{r+1}} - V^{\beta_2\ldots\beta_{r+1}}_{2\alpha_2\ldots\alpha_{r+1}} \underline{\tilde{\varphi}}_{\beta_2\ldots\beta_{r+1}} - \mathcal{O}(\underline{\tilde{\varphi}}^2) = 0, \tag{73}$$

where

$$\underline{\tilde{\varphi}}^{A}_{\alpha_2\ldots\alpha_{r+1}} = \sum_{\bar{\pi}} (-1)^{\mathrm{sgn}(\bar{\pi})} \underline{\tilde{\varphi}}_{\bar{\pi}_2\ldots\bar{\pi}_{r+1}} \tag{74}$$

and $\bar{\pi}$: $(\alpha_i) \mapsto (\bar{\pi}_i)$, $i = 2 \ldots r+1$ are the permutations restricted to the field indices. We shall now compute the r partial longitudinal components of (73) by multiplying it by k^{α_i} with $i = 2 \ldots r+1$, and a totally transverse component by computing its r-fold antisymmetric product with vectors k_{γ_i}, using the following operation and convention: $v_{\dots\alpha_i\dots} \to v_{\dots\alpha_i\dots}k_{\gamma_i} - v_{\dots\gamma_i\dots}k_{\alpha_i} = 2v_{\dots[\alpha_i\dots}k_{\gamma_i]}$, with $i = 2\dots r+1$. We have:

$$
\begin{aligned}
k^{\alpha_i}\underline{\tilde{V}}_{12+\alpha_2\dots\alpha_{r+1}} &= k^{\alpha_i}V^{\beta_2\dots\beta_{r+1}}_{2\alpha_2\dots\alpha_{r+1}}\underline{\tilde{\varphi}}_{\beta_2\dots\beta_{r+1}}, &(75)\\
\kappa k^{\alpha_1}k_{\alpha_1}\underline{\tilde{\varphi}}^A_{[\alpha_2\dots[\alpha_{r+1}}k_{\gamma_2]}\dots k_{\gamma_{r+1}]} &= \underline{\tilde{V}}_{[\alpha_2\dots[\alpha_{r+1}}k_{\gamma_2]}\dots k_{\gamma_{r+1}]}, &(76)
\end{aligned}
$$

where $\underline{\tilde{V}}_{\alpha_2\dots\alpha_{r+1}} = \underline{\tilde{V}}_{12+\alpha_2\dots\alpha_{r+1}} + V^{\beta_2\dots\beta_{r+1}}_{2\alpha_2\dots\alpha_{r+1}}\underline{\tilde{\varphi}}_{\beta_2\dots\beta_{r+1}}$. From (75), we can ignore the longitudinal components of $\underline{\tilde{V}}_\alpha$ in (72). We can successively integrate (76) and thereby choose the Lorenz gauge for each step as in Subsection 2.3. E.g. the first step yields

$$
\kappa k^{\alpha_1}k_{\alpha_1}\underline{\tilde{\varphi}}^A_{[\alpha_2\dots[\alpha_r\alpha_{r+1}}k_{\gamma_2]}\dots k_{\gamma_r]} = \underline{\tilde{V}}_{[\alpha_2\dots[\alpha_r\alpha_{r+1}}k_{\gamma_2]}\dots k_{\gamma_r]}, \tag{77}
$$

together with the outermost Lorenz gauge condition $\nabla^{\alpha_{r+1}}\nabla_{[\gamma_2}\dots\nabla_{[\gamma_r}\underline{\tilde{\varphi}}^A_{\alpha_2]\dots\alpha_r]\alpha_{r+1}} = 0$. At the end, we obtain $\underline{\tilde{V}}_{\alpha_2\dots\alpha_{r+1}} \sim \underline{\tilde{\varphi}}^A_{\alpha_2\dots\alpha_{r+1}}$. From this and from the innermost Lorenz gauge conditions $\nabla^{\alpha_i}\underline{\varphi}^A_{\alpha_1\dots\alpha_r} = 0$ with $i \le r$, (72) becomes

$$
\kappa k_{\alpha_1}k^{\alpha_1}\underline{\tilde{\varphi}}^A_{\alpha_2\dots\alpha_{r+1}} - V_2\underline{\tilde{\varphi}}^A_{\alpha_2\dots\alpha_{r+1}} = 0. \tag{78}
$$

However, the number of physically significant components of the field $\underline{\varphi}^A_{\alpha_1\dots\alpha_r}$ are highly restricted by the antisymmetry and by the Lorenz gauges. The antisymmetry imposes the restriction $r \le 4$.

- $r = 4$: The only degree of freedom is lost by any of the Lorenz gauge conditions.
- $r = 3$: We start with 4 degrees of freedom. From the antisymmetry, the 3 innermost Lorenz gauge conditions are equivalent and are covered by $\nabla^\alpha\underline{\varphi}^A_{\alpha\beta\gamma} = 0$ which at once removes all degrees of freedom (the system is overdetermined).
- $r = 2$: We start with 6 degrees of freedom. The innermost Lorenz gauge condition $\nabla^\alpha\underline{\varphi}^A_{\alpha\beta} = 0$ removes 4 degrees of freedom. The next outer Lorenz gauge condition $\nabla^\gamma\nabla_{[\alpha}\underline{\varphi}^A_{\beta]\gamma} = 0$ removes the remaining degrees of freedom (the system is overdetermined).
- $r = 1$ for comparison: In the well-known vector field case, we start with 4 degrees of freedom. There is only 1 level of Lorenz condition which removes 1 degree of freedom, and 3 degrees of freedom (polarisations) remain, which can be further restricted in the case $V_2 = 0$.

Since no degree of freedom is available for the polarisation tensors of the field types with $r \ge 2$, the case $k = 1$, $r \ge 2$ must be excluded.

The $n = 2$, $k = 0$, $r \ge 2$ case can be treated in a similar way as in Subsection 2.4:

$$
\sigma = -\kappa\overset{1}{\underline{\varphi}}{}^{a_1\dots a_r}\alpha_{a_1\dots a_r\perp b_1\dots b_r}\overset{2}{\underline{\varphi}}{}^{b_1\dots b_r}. \tag{79}
$$

To obtain the Lagrangian, we must let all Christoffel symbols cancel; this requires total antisymmetrization of any differentiated fields or, equivalently, that $\alpha_{\alpha_1\dots\alpha_r\gamma\beta_1\dots\beta_r}$ be totally antisymmetric, and $\mathcal{L}$ has the form

$$
\mathcal{L} = \kappa[\lambda\overset{1}{\underline{\varphi}}{}^{\alpha_1\dots\alpha_r}\alpha_{\alpha_1\dots\alpha_r\gamma\beta_1\dots\beta_r}\nabla^\gamma\overset{2}{\underline{\varphi}}{}^{\beta_1\dots\beta_r} + \mu(\nabla^\gamma\overset{1}{\underline{\varphi}}{}^{\alpha_1\dots\alpha_r})\alpha_{\alpha_1\dots\alpha_r\gamma\beta_1\dots\beta_r}\overset{2}{\underline{\varphi}}{}^{\beta_1\dots\beta_r}] - V. \tag{80}
$$

Varying (80) with respect to $\overset{1}{\underline{\varphi}}{}^{\alpha_1\dots\alpha_r}$ and following the same procedure as in Subsection 2.4 leads to the ELE in momentum space ($g_{\alpha\beta} \to \eta_{\alpha\beta}$) and to the corresponding Clifford algebra with bracket $[\alpha_{\alpha_1\dots\alpha_r}{}^{\mu\beta_1\dots\beta_r}, \alpha_{\beta_1\dots\beta_r}{}^{\nu}{}_{\gamma_1\dots\gamma_r}]$. As in Subsection 2.4, σ is invariant with respect to (i) $U(1)$ and (ii) a group of transformations $\overset{1}{\underline{\varphi}}{}^{\alpha_1\dots\alpha_r} \to \overset{1}{\underline{\varphi}}{}^{\beta_1\dots\beta_r}[\exp(i\delta\overset{1}{\chi})]^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_r}$, $\overset{2}{\underline{\varphi}}{}^{\alpha_1\dots\alpha_r} \to [\exp(i\delta\overset{2}{\chi})]^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_r}\overset{2}{\underline{\varphi}}{}^{\beta_1\dots\beta_r}$, satisfying $\delta\overset{1}{\chi}{}^{\delta_1\dots\delta_r}_{\alpha_1\dots\alpha_r}\alpha_{\delta_1\dots\delta_r\gamma\beta_1\dots\beta_r} + \alpha_{\alpha_1\dots\alpha_r\gamma\delta_1\dots\delta_r}\delta\overset{2}{\chi}{}^{\delta_1\dots\delta_r}_{\beta_1\dots\beta_r} = 0$. As in Subsection 2.4, (ii) would require a third order term coupling to a tensor field $\xi_{\alpha_1\dots\alpha_r\beta_1\dots\beta_r\gamma}$ (i.e. $r \geq 5$) which is not available according to the previous paragraph. Therefore, $\mathcal{L}$ fails to satisfy invariance (ii), and we must exclude the case $n = 2$, $k = 0$, $r \geq 2$.

2.6. Consequences of our investigations

Following the above investigations, when constraining the boundary term of gravity via an extra term with arbitrary dependence on field functions $\underline{\varphi}$ as "parameters", it is only possible to reproduce three types of particle field functions (i.e. distinct forms of matter equations of motion), namely scalar, fermion (Dirac) and spin 1 type fields, in conformity to the observations, in the low gravity regime, to first order in the derivative of fields and to second order in the field power. By starting from the boundary, the form of dependence on the derivative of $\underline{\varphi}$ in the kinetic term of $\mathcal{L}$ is dramatically restricted. This is in contrast to the larger freedom in the choice of the field dependence of the potential term where restrictions only come from gauge conditions (and renormalizability when being quantised). The restriction for the kinetic term and its allowed field structures could hardly be explained from a bulk concept. Moreover, constraining the boundary term of gravity naturally causes the variation principles (boundary and bulk) to be extended to the matter fields: the variation also must be performed with respect to the emerging "parameters" $\underline{\varphi}$. Finally, our procedure automatically leads to the concept of gauge invariance for particle fields. The form of admissible fields and the form of the potential terms are restricted by the requirement that the bulk gauge symmetries properly match the symmetries of the boundary term. We do not know about any alternative mechanism capable of reproducing each one of these results with comparable thoroughness. Taking this matter of fact seriously has crucial implications, not only on the suggested viable fields, but also on the properties of gravity on the quantum level as shown below.

3. Implications of quantised gravity on emergent matter and vice versa

We are now able to investigate immediate implications of the constraint mechanism described in Section 2:

(i) If the gravitational field within some region of space is primarily determined by a source of quantum matter, it is not possible to apply a theory of gravity of classical type, e.g. GR. The reason is: If a given quantisation mechanism is required for matter, the constraint mechanism will necessarily force gravity to be quantised as well.

(ii) The properties of quantum matter in the low gravity regime drastically restrict what the properties of quantum gravity are allowed to be.

For simplicity, we shall examine the implication (ii) for the field $\underline{\varphi}$ of a single species, primarily focusing on metric-dependent gravity, for weak gravity, $g_{\alpha\beta} \approx \eta_{\alpha\beta}$ in cartesian coordinates. We assume $|\underline{\varphi}\rangle$ to be some superposition of number eigenstates $|n_{\boldsymbol{p}}\rangle$ of $\underline{\varphi}$-particles with momentum $\boldsymbol{p}$. Incrementing the particle number, $|n_{\mathbf{p}}\rangle \to |n_{\mathbf{p}} + 1\rangle$, causes an increase of the Hamiltonian ΔH and of the 3-momentum ΔP_i, i.e. an increase of the 4-momentum ΔP_α:

$$\Delta P_\alpha = \Delta \int d^3x\, \sqrt{-g}\, T^p_{0\alpha} = \hbar c k_\alpha \tag{81}$$

with the stress tensor computed according to QFT (from the translation invariance of $\mathcal{L}$):

$$T^p_{\alpha\beta} = \sum_i \frac{\partial \mathcal{L}}{\partial \nabla^\alpha \underline{\overset{i}{\varphi}}} \nabla_\beta \underline{\overset{i}{\varphi}} - \mathcal{L} g_{\alpha\beta}, \tag{82}$$

where $\underline{\overset{i}{\varphi}}$ is a scalar ($\overset{i}{\varphi}$) or spin 1 ($\overset{i}{\varphi}{}^\nu$) type field. (81) reflects the eigenvalues of the operator $\hat{P}_\alpha$ which is diagonal in the number eigenstate basis. A first idea would be to identify $T^p_{\alpha\beta} \to T_{\alpha\beta} = (2/\sqrt{-g})\delta(\sqrt{-g}\mathcal{L})/\delta g^{\alpha\beta}$ and to relate (81) to the gravitational field equations

$$\frac{\delta(\sqrt{-g}\mathcal{L}_g)}{\delta g^{\alpha\beta}} = -\sqrt{-g} T_{\alpha\beta}, \tag{83}$$

where $\mathcal{L}_g$ is the gravitational Lagrangian and (83) tends, for torsionless weak gravity, to Einstein's Equations coupled to $\underline{\overset{i}{\varphi}}$:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^4} T_{\alpha\beta}. \tag{84}$$

We shall keep in mind that $T^p_{\alpha\beta}$ and $T_{\alpha\beta}$ are computed from different concepts. Merely approximating $\Lambda \approx 0$ and linearizing (84) using $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ or $q_{\alpha\beta} = h_{\alpha\beta} - \frac{h^\mu_\mu}{2}\eta_{\alpha\beta}$, and the gauge condition

$$2h^\alpha_{\beta,\alpha} = h^\alpha_{\alpha,\beta}, \tag{85}$$

we have:

$$\Box q_{\alpha\beta} = -\frac{8\pi G}{c^4} T_{\alpha\beta}. \tag{86}$$

(86) reminds us how, for weak gravity, metric fluctuations around vacuum are linearly related to matter density fluctuations (81). Assuming $T_{\alpha\beta} \approx T^p_{\alpha\beta}$ and setting the vacuum state momentum P_α to zero (which corresponds to normal ordering of the fields in $\hat{P}_\alpha$), we can describe the increase from the vacuum to the one-particle state as (dropping the "Δ")

$$-\int d^3x\ \sqrt{-g}\Box q_{\alpha\beta} \approx \frac{8\pi G}{c^4}\int d^3x\ \sqrt{-g}\ T_{0\alpha} \approx 8\pi L_p^2 k_\alpha, \qquad L_p^2 = \frac{\hbar G}{c^3}. \tag{87}$$

Expression (87) is not an invariant and therefore no longer can be used as a quantum constant for a non-perturbative general relativistic treatment. To obtain an invariant, it is necessary to multiply (87) by $k^\alpha/(k_\mu k^\mu)$. On the other hand, (87) represents the quanta of gravity in the low gravity limit. To avoid the frame dependence problematic on this level, we strive to express the quantum counter $8\pi L_p^2 k_\alpha$ in terms of $S_{\partial V}$. To achieve this, we integrate (82) over space, assuming $\underline{\overset{1}{\varphi}} \neq \underline{\overset{2}{\varphi}}$ and that the potential term satisfies

$$V = (\partial V/\partial \underline{\overset{i}{\varphi}})\underline{\overset{i}{\varphi}}. \tag{88}$$

Even for cases for which (88) is not strictly satisfied, e.g. φ^4-theory, the correction is small in the weak gravity limit and hardly observable. Writing $\mathcal{L} = K - V$ with the kinetic term K for the scalar or spin 1 type field, we use the ELE to eliminate V,

$$\frac{\partial V}{\partial \underline{\overset{i}{\varphi}}} = -\nabla_\mu \frac{\partial K}{\partial \nabla_\mu \underline{\overset{i}{\varphi}}}, \tag{89}$$

and K can be written as

$$K = \frac{\partial K}{\partial \nabla_\mu \overset{i}{\underline{\varphi}}} \nabla_\mu \overset{i}{\underline{\varphi}}, \tag{90}$$

so that

$$\mathcal{L} = \nabla_\mu \Big(\frac{\partial K}{\partial \nabla_\mu \overset{i}{\underline{\varphi}}} \overset{i}{\underline{\varphi}} \Big). \tag{91}$$

We are free to choose $g_{0\alpha} = \eta_{0\alpha}$ as our preferred gauge ((85) is not used here) and obtain:

$$\begin{aligned} &\frac{8\pi G}{c^4} \int d^3x \, \sqrt{-g} \, T^p_{0\alpha} \\ &= \frac{8\pi G}{c^4} \int d^3x \, \sqrt{-g} \, [(\sum_i \frac{\partial \mathcal{L}}{\partial \nabla^0 \overset{i}{\underline{\varphi}}} \nabla_\alpha \overset{i}{\underline{\varphi}}) - \nabla_\mu (\frac{\partial K}{\partial \nabla_\mu \overset{i}{\underline{\varphi}}} \overset{i}{\underline{\varphi}}) \eta_{0\alpha}] = 8\pi L_p^2 k_\alpha. \end{aligned} \tag{92}$$

Considering the transition from vacuum to a single particle state, $\underline{\varphi}$ shall be the plane wave approximation of the one particle field, $\underline{\varphi} = \underline{\varphi}_0 \exp(i k_\mu x^\mu)$. The second term of (92) is a divergence term and can be converted into a boundary term when integrated in 4 dimensions. Consider a space-time section V delimited by ∂V consisting of the flat time-slices Σ_- at time t_- and Σ_+ at $t_+ = t_- + (2\pi)/(k_0 c)$ (Σ_- translated by one period). When (92) is integrated over V, the resulting boundary term vanishes. Taking $\underline{\varphi}, \underline{\varphi}^\dagger$ as independent field variables, we have

$$\begin{aligned} &\frac{8\pi G}{c^4} \int d^4x \, \sqrt{-g} \, [\frac{\partial \mathcal{L}}{\partial \nabla^0 \underline{\varphi}} \nabla_\alpha \underline{\varphi} + \frac{\partial \mathcal{L}}{\partial \nabla^0 \underline{\varphi}^\dagger} \nabla_\alpha \underline{\varphi}^\dagger] \\ &= -\frac{8\pi G}{c^4} \int d^3x \, \sqrt{-g} \, \sigma k_\alpha \frac{2\pi}{k_0 c} = 8\pi L_p^2 k_\alpha \frac{2\pi}{k_0 c}, \end{aligned} \tag{93}$$

where $\nabla_\alpha \underline{\varphi} = k_\alpha \underline{\varphi}$ and ∂V has a time-like normal vector, $n^a = \delta^a_\perp = \delta^a_0$. Multiplying (93) by $k_0 c k^\alpha / (2\pi k_\mu k^\mu)$ finally leads to the desired invariant result for one quantum unit:

$$-\frac{8\pi G}{c^4} \int d^3x \, \sqrt{-g} \, 2\sigma = 8\pi L_p^2 = \not{A} = 6.564 \cdot 10^{-69} \text{ m}^2. \tag{94}$$

From (7), we conclude that, for on-shell values, any increase of the number of quanta of matter occurs at the expense of the number of gravitational quanta, i.e. quanta of gravity can be "booked" for matter. The sum $(8\pi G/c^4) \int d^3x \, \sqrt{-g} \, (\sigma_g + 2\sigma)$ must thus represent the total amount of available quanta (up to an irrelevant offset), part of which is assigned to matter fields.

For a fermion field, the same computations can be performed while e^I_α replaces $g_{\alpha\beta}$ and the stress tensor is replaced by the one-form $T^I_\alpha = T_{\alpha\beta} g^{\beta\gamma} e^I_\gamma$. In case of torsion, a corresponding expression for the quanta of spin would have to be supplemented.

(94) fixes the quantum constant as measured "in units of boundary term". At this stage, we observe that

(i) the quantum constant for gravity is also the quantum constant for general relativistic matter,

(ii) it is given in units of boundary term for gravity or matter,

(iii) the quantum constant should be of the dimension of an area as suggested from Bekenstein entropy, this is already satisfied by $\not{A}$,

(iv) integrating σ and σ_g over the boundary of any space-time region V yields a suitable quantity to count quanta in a general relativistic frame-work, at least in the low-gravity regime.

From the Gibbons-Hawking-York term as a suitable expression in the torsion-less low gravity regime, it is the appropriately contracted connection integrated over the 3-volume which provides the quantum constant of gravity, and $\not{A}$ is equal to the area of a sphere of radius $\sqrt{2}L_p$.

To investigate how the quantum properties of matter reflect those of gravity, we consider $|\underline{\varphi}\rangle$ as a true mixture of number eigenstates $|n_{\mathbf{p}}\rangle$ for weak gravity and in the GR limit, while ignoring here the role of the cosmological constant, $\Lambda \approx 0$. Integrating Einstein's Equations with $U^{\alpha\beta} = \int d^3x\, G^{\alpha\beta}$, we have

$$U^{0\alpha} = -\frac{8\pi G}{c^4} P^\alpha \tag{95}$$

and see that $U^{\alpha\beta}$ no longer has sharp values since P^α is not sharp for a mixed number state. Therefore, both sides must be replaced by quantum operators,

$$\hat{U}^{0\alpha} = -\frac{8\pi G}{c^4} \hat{P}^\alpha. \tag{96}$$

$\hat{U}^{0\alpha}$ is determined by its eigenvalues,

$$\hat{U}^{0\alpha}|n_{\mathbf{p}}\rangle = U^{0\alpha}|n_{\mathbf{p}}\rangle = -\frac{8\pi G}{c^4} P^\alpha |n_{\mathbf{p}}\rangle. \tag{97}$$

For the vacuum state, (97) suggests $U^{0\alpha} \approx 0$. This can be achieved by normal-ordering of the expression for $\hat{P}^\alpha$, in order to avoid a non-zero vacuum energy. (If we had chosen $\Lambda \neq 0$ and then perturbed $g_{\alpha\beta}$ around vacuum, this would have forced a non-zero vacuum energy.)

(97) only holds for weak gravity, $g_{\alpha\beta} \approx \eta_{\alpha\beta}$. If this condition is dropped, $g_{\alpha\beta}$ must be treated as an operator due to the quantisation of gravity via (94). P^α (which counts the particle number in QFT) cannot be the eigenvalue of a quantum number operator for gravity since a change of frame, determined by $g_{\alpha\beta}$, would cause a change of the quantum number. Only multiplication by $g_{\alpha\beta}k^\beta$ would produce a gravitational invariant, as needed. It is therefore rather $S_{\partial V}$ which is suitable to count the quanta of gravity. In the strong gravity regime, the particle eigenstates $|n_{\mathbf{p}}\rangle$ lose their outstanding role in favour of the eigenstates of $\hat{S}_{\partial V}$, i.e. the concept of Hamiltonian is replaced by the concept of boundary term.

To obtain the quantisation prescription of gravity in the weak gravity limit, we start from the canonical quantisation prescription of QFT in flat space-time and use the quantum version of (84). We exemplarily consider the case of a real single-component scalar quantum field,

$$[\hat{\pi}(x^a, t), \hat{\varphi}(y^a, t)] = i\hbar\delta(x^a - y^a), \tag{98}$$

where x^a is space-like on the time-slice at time t, and $\hat{\pi} = (\partial\sigma/\partial\varphi)\hat{}$ is the momentum operator canonically conjugate to $\underline{\hat{\varphi}}$. Expanding $\hat{\varphi} = \varphi_0[\exp(-ik_\mu x^\mu)\hat{a} + \exp(ik_\mu x^\mu)\hat{a}^\dagger]$, with $\varphi_0 = \sqrt{\hbar c^2/(2V\omega_k)}$ (V is the 3-volume for the scalar field), with the bosonic ladder operators $\hat{a}, \hat{a}^\dagger$, $\Lambda = 0$, $\hat{\varphi}$ monochromatic and for one species), yields the lowest order equation:

$$
\begin{aligned}
-\frac{c^4}{8\pi G}\hat{R}_{\alpha\beta} &= \varphi_0^2[(k_\alpha k_\beta + \frac{k_\mu k^\mu \eta_{\alpha\beta}}{2})(e^{-2ik_\mu x^\mu}\hat{a}^2 + e^{2ik_\mu x^\mu}\hat{a}^{\dagger 2}) \\
&\quad -(k_\alpha k_\beta - \frac{k_\mu k^\mu \eta_{\alpha\beta}}{2})[\hat{a}, \hat{a}^\dagger]_+], \qquad (99)
\end{aligned}
$$

where no normal ordering has been performed yet (it should eventually be performed for $\Lambda = 0$), and we anticipate $(8\pi G/c^4)\varphi_0^2 = \not{A}/(2Vk^0)$. For linearized gravity, we have (without gauge)

$$
\hat{R}_{\alpha\beta} \approx \hat{g}_{\alpha\beta,\mu,}{}^\mu + \hat{g}_{\mu\nu,\alpha,\beta}\eta^{\mu\nu} - \hat{g}_{\mu\alpha,\nu,\beta}\eta^{\mu\nu} - \hat{g}_{\mu\beta,\nu,\alpha}\eta^{\mu\nu}. \qquad (100)
$$

The ansatz for the metric operator $\hat{g}_{\alpha\beta}$ must be $\hat{a}, \hat{a}^\dagger$-dependent to be consistent with (99). Besides the leading order term $\eta_{\alpha\beta}$, the $\hat{g}_{\alpha\beta}$ must contain a term $\hat{b}_{\alpha\beta} \sim \hat{a}^2\exp(-2ik_\mu x^\mu) + \hat{a}^{\dagger 2}\exp(2ik_\mu x^\mu)$, while integration of the last term of (99) yields a term quadratic in x^μ which is $\sim [\hat{a}, \hat{a}^\dagger]_+$, and the integration constants yield one term $\hat{h}_{1\alpha\beta} \sim x^\mu$ and one constant $\hat{h}_{0\alpha\beta}$. Since we have approximated $g_{\alpha\beta} \approx \eta_{\alpha\beta}$ in (99) and we can choose the coordinates and frame so that, locally, $x^\mu \approx 0$ and $\hat{g}_{\alpha\beta}$ can be a superposition of $\hat{b}_{\alpha\beta}$ and a local inertial frame, we set $\hat{h}_{1\alpha\beta} = 0 = \hat{h}_{0\alpha\beta}$. We therefore find

$$
\begin{aligned}
\hat{g}_{\alpha\beta}(x^\mu) &= \eta_{\alpha\beta} + \hat{b}_{\alpha\beta}(x^\mu) + f_{\alpha\beta}(x^\mu)[\hat{a}, \hat{a}^\dagger]_+, \\
\hat{g}^{\alpha\beta}(x^\mu) &= \eta^{\alpha\beta} - \hat{\bar{b}}^{\alpha\beta}(x^\mu) - \bar{f}^{\alpha\beta}(x^\mu)[\hat{a}, \hat{a}^\dagger]_+, \qquad (101)
\end{aligned}
$$

with

$$
\begin{aligned}
\hat{b}_{\alpha\beta} &= h_{2\alpha\beta}(k^\nu)[e^{-2ik_\mu x^\mu}\hat{a}^2 + e^{2ik_\mu x^\mu}\hat{a}^{\dagger 2}], \\
\hat{\bar{b}}^{\alpha\beta} &= \bar{h}_2{}^{\alpha\beta}(k^\nu)[e^{-2ik_\mu x^\mu}\hat{a}^2 + e^{2ik_\mu x^\mu}\hat{a}^{\dagger 2}]. \qquad (102)
\end{aligned}
$$

In (101–102), the orders of magnitude are $\hat{b}_{\alpha\beta} \sim n_p^{-1}n_w^{-1}$ and $f_{\alpha\beta} \sim n_p^{-1}n_w$, where $n_p \sim V/(\not{A}x^0) \gg n_w$ is the number of Planck areas contained in a 2d-section of the wave packet of (time-like) spread $\sim x^0$, and $n_w \sim k_0 x^0 \gg 1$ is the wave number of the wave packet. The $\hat{g}_{\alpha\beta}$ and $\hat{g}^{\alpha\beta}$ must satisfy $\hat{g}^{\alpha\beta}\hat{g}_{\beta\gamma} = \delta^\alpha_\gamma = \hat{g}_{\gamma\beta}\hat{g}^{\beta\alpha}$ at the order of the scale $\sim n_p^{-1}n_w^{-1}$, whence:

$$
\bar{h}_2{}^{\alpha\beta} = h_{2\gamma\delta}\eta^{\alpha\gamma}\eta^{\beta\delta}, \qquad \bar{f}^{\alpha\beta} = f_{\gamma\delta}\eta^{\alpha\gamma}\eta^{\beta\delta}, \qquad \bar{f}^{\alpha\beta}f_{\beta\gamma} \approx 0. \qquad (103)
$$

$\hat{g}_{\alpha\beta}$ and hence $h_{2\alpha\beta}$ and $f_{\alpha\beta}$ are symmetric in their indices. Since $\hat{g}_{\alpha\beta} = \hat{g}_{\alpha\beta}(\hat{a}, \hat{a}^\dagger, x^\mu)$, the left- and right-derivatives of $\hat{g}_{\alpha\beta}$ have the same value,

$$
\begin{aligned}
\hat{g}_{\alpha\beta,\rho} &= \hat{b}_{\alpha\beta,\rho} + f_{\alpha\beta,\rho}[\hat{a}, \hat{a}^\dagger]_+, \\
\hat{b}_{\alpha\beta,\rho} &= 2ik_\rho h_{2\alpha\beta}(k^\nu)(-\exp(-2ik_\mu x^\mu)\hat{a}^2 + \exp(2ik_\mu x^\mu)\hat{a}^{\dagger 2}). \qquad (104)
\end{aligned}
$$

Expressions like $\hat{g}_{\alpha\beta,\rho}\hat{g}^{\alpha\beta}$ are insensitive to operator ordering at order $\sim n_p^{-1}n_w^{-1}$. (99) yields:

$$
\begin{aligned}
k^\gamma(k_\beta h_{2\gamma\alpha} + k_\alpha h_{2\gamma\beta}) - k_\gamma k^\gamma h_{2\alpha\beta} - k_\alpha k_\beta h_{2\gamma}^{\ \gamma} &\approx \frac{\not{A}}{8Vk^0}(k_\alpha k_\beta + \frac{k_\gamma k^\gamma \eta_{\alpha\beta}}{2}), \qquad (105) \\
f_{\alpha\beta,\mu,}{}^\mu + f_{\mu\nu,\alpha,\beta}\eta^{\mu\nu} - f_{\mu\alpha,\nu,\beta}\eta^{\mu\nu} - f_{\mu\beta,\nu,\alpha} &\approx \frac{\not{A}}{2Vk^0}(-k_\alpha k_\beta + \frac{k_\mu k^\mu \eta_{\alpha\beta}}{2}). \qquad (106)
\end{aligned}
$$

Due to its ab initio (approximate) independence from where the inertial frame is being located in space, $\hat{b}_{\alpha\beta}$ is of primary physical interest. (105) yields

$$h_{2\alpha\beta} \approx -\frac{\not{A}}{16Vk^0}\eta_{\alpha\beta}. \tag{107}$$

Together with (104), we obtain a monochromatic quantisation prescription for $\hat{g}_{\alpha\beta}$ at the origin:

$$\left[\hat{g}_{\alpha\beta}(k_i^1;x^\mu), \hat{g}_{\gamma\delta,\rho}(k_i^2;x^\mu)\right]_{x^\mu\approx 0} \approx i\frac{\not{A}^2}{16V^2k^{0\,2}}\eta_{\alpha\beta}\eta_{\gamma\delta}k_\rho[\hat{a},\hat{a}^\dagger]_+\delta_{k_i^1k_i^2}. \tag{108}$$

(108) depends on the state it acts on; it would be preferable to find a state-independent prescription and to get rid of the monochromaticity condition. This requires the bilinear notation

$$\hat{g}_{\alpha\beta}(x^\mu) = \eta_{\alpha\beta} + \hat{h}^I_\alpha\eta_{IJ}\hat{h}^J_\beta + \hat{h}_{0\alpha\beta}, \tag{109}$$

where $\hat{h}_{0\alpha\beta} \neq 0$ must be reintroduced, so that

$$\hat{h}^I_\alpha \approx i\sqrt{\frac{\not{A}}{16Vk^0}}\,\delta^I_\alpha\left(\mp e^{-ik_\mu x^\mu}\hat{a} + e^{ik_\mu x^\mu}\hat{a}^\dagger\right), \qquad \hat{h}_{0\alpha\beta} \approx \pm 2i\sqrt{\frac{\not{A}}{16Vk^0}}\,\eta_{\alpha\beta}[\hat{a},\hat{a}^\dagger]_+. \tag{110}$$

We have $[\hat{h}^I_\alpha(k_i^1;x^\mu), \hat{h}^J_\beta(k_i^2;x^\mu)] \sim \delta_{k^i,k'^i}$. Also introducing distinct coordinate variables and summing (or integrating) over all k^i, k'^i finally yields the equal-time quantisation prescription with general real scalar field content:

$$\left[\hat{h}^I_\alpha(x^i,t), \hat{h}^J_{\beta,0}(x'^i,t)\right] \approx i\frac{\not{A}}{16}\delta^I_\alpha\delta^J_\beta\delta(x^i - x'^i), \tag{111}$$

where the δ-function is restricted to 3 dimensions due to the 3-dimensional integration over k^i. (111) can also be formulated as a (space-like) boundary space quantisation prescription:

$$\begin{aligned}
\left[\hat{\gamma}^{ab}(x^c), \hat{K}_{ab}(x'^c)\right] &= -\left[\hat{h}^a_I(x^c), \mathcal{L}_\perp\hat{h}^I_a(x'^c)\right] + \mathcal{O}(\frac{1}{n_p^2n_w^2}) \\
\sim \left[\hat{h}^a_I(x^c), \hat{h}^I_{a,0}(x'^c)\right] &\approx \frac{i\not{A}}{4}\delta(x^c - x'^c),
\end{aligned} \tag{112}$$

This is precesily the kind of prescription we expect from (7) since $K_{ab} \sim \partial\sigma_g/\partial\gamma^{ab}$, i.e. the pairs of generalised "coordinates" and "momenta" ($\underline{\varphi}$,$\underline{\pi}$) and (γ^{ab}, K_{ab}) necessarily satisfy the same kind of uncertainty relation since they reflect the joint gravitational and matter quantisation of the total boundary term.

Prescriptions like (111–112) or the vielbein version thereof can be used as a constraint for quantum gravity model constructions, possibly after having been generalised to include torsion and paired with a quantisation prescription for torsion. In order for a theory to be compatible with the concept of gravity constrained by generic matter, its quantisation prescription must recover each of these field-dependent prescriptions (scalar, fermion-Dirac and spin 1 fields) in the weak gravity limit.

4. Conclusions

Our detailed investigations of the possible generic expressions for constraining the boundary term of the gravitational action have shown that, at least in the weak gravity regime, namely up to the order of the first coordinate derivative and second order in the field power, all non-trivial, not explicitly coordinate-dependent and local realisations with distinct form of matter equations of motion correspond to the formal expressions of the boundary term of the matter field actions for scalar, fermion (Dirac) and spin 1 fields, so as to cover the standard model field types, allowing the fields to have couplings and mass within the potential terms, and automatically providing the expected gauge symmetries. Since, to our knowledge, this is the only mechanism capable of explaining to such a large extent the mathematical form of the observed matter fields including the gauge mechanisms, it seems unlikely that this result is only a coincidence. Rather, it could have to be regarded as a fundamental principle. It indicates that the role of matter is to restrict the locally available space-time configurations. This interpretation does not depend on the specific theory of gravity as long as its torsion-less weak field limit is General Relativity. As a consequence, quantum matter can only exist if gravity itself is quantised, and the quantum operators of gravity are largely determined by those of quantum matter for weak gravity. The fundamental quanta of matter are not independent, they are bound to the quanta of gravity. In the strong gravity regime, the fundamental states are not the eigenstates of the Hamilton operator, but rather the eigenstates of the boundary term operator. Finally, the fundamental gravitational quantum constant $\not{A} = 6.564 \cdot 10^{-69}\ \mathrm{m}^2$ is predicted, in terms of boundary term content, thus making the approach falsifiable via elementary observations.

References

[1] Elze H T 2009 *J. Phys.: Conf. Ser.* **174** 012009

[2] 't Hooft G 1993 Dimensional reduction in quantum gravity *Preprint* arXiv 9310026

[3] Gibbons G W and Hawking S W 1977 *Phys. Rev.* D **15** 2752

[4] Hawking S W 1975 *Commun. math. Phys.* **43** 199

[5] Bekenstein J D 1973 *Phys. Rev.* D **7** 2333

[6] Wald R M 2001 *Liv. Rev. Rel.* **4** 2001-6wald

[7] Unruh WG 1976 *Phys. Rev.* D **14** 870

[8] Jacobson T 1995 *Phys. Rev. Lett.* **75** 1260

[9] Padmanabhan T 2014 *Gen. Rel. Grav.* **46** 1673

[10] Mandrin P A 2017 *J. Phys.: Conf. Ser.* **880** 012028

[11] Mandrin P A 2019 *J. Phys.: Conf. Ser.* **1275** 012060

[12] York J W 1986 *Found. Phys.* **16** 249

[13] Madsen M S and Barrow J D 1989 *Nucl. Phys.* B **323** 242–252

[14] Dyer E and Hinterbichler K 2009 *Phys. Rev.* D **79** 024028

[15] Deser S 2019 *Phys. Lett.* B **790** 408–409

[16] Peskin M E and Schroeder D V 1995 *An introduction to quantum field theory* Westview Press ABP (Boulder)

[17] Ahluwalia D V and Grumiller D 2005 *JCAP* **0507** 012; Ahluwalia D V and Grumiller D 2005 *Phys. Rev.* D **72** 067701

[18] Rodrigues W A Jr and Capelas de Oliveira E 2016 *The many faces of Maxwell, Dirac and Einstein equations – A Clifford bundle approach* Springer Nature (Switzerland) 517–543

[19] Hehl F W, von der Heyde P, Kerlik G D, and Nester J M 1976 *Rev Mod Phys* **48** 393