Spatial-temporal oscillations in boundary problems of quantum mechanics

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Abstract

We consider the Schrődinger equation with nonlinear boundary conditions and initial conditions. It is shown that attractor of the problem contains periodic piecewise constant function with finite, countable or uncountable points of discontinuities on a period. Solutions exists for a special class of initial data which are small perturbations of invariant solutions of dynamical system. The problem is considered with accuracy $O(h^2)$, where h is a small parameter of the problem. Applications to optical resonators with nonlinear feedback has been considered.

Keywords: The Schrődinger equation • The functional two points boundary conditions • asymptotic periodic piecewise constant distributions of relaxation type

1 Introduction

In last years, in physics studied the nonlinear interaction of light which can mimic the physics at so called an event horizon. As shown in [5] " This analogue arises when a weak probe wave is unable to pass through an intense soliton, despite propagating at a different velocity". These dynamics arises as a soliton-induced refractive index barrier. In all paper this barrier characterize the volume optic properties of a fibre with linear boundary conditions. In this paper, we consider the opposite problem when an optical medium is ideal or linear, but boundaries of the medium have nonlinear optic properties. These surface properties can be modeled by optical transistor or diode [3]. It can be also possible a case a bright soliton is passing through the black soliton. In this case, the intensity of light depends on a fibre refractive index, describing, for example, the well-known Kerr effect. Thus the soliton creates the moving refractive surface perturbations which passage through the another soliton [6, 8, 11]. These nonlinear interactions between such surface waves produce volume waves in a medium. Thus interactions between linear volume perturbations and nonlinear surface perturbations leads to appearing volume waves which represent asymptotic periodic piecewise impulses with finite or infinite points of discontinuities. These distributions are elements of attractors of an initial value boundary problem. From physical point of view, these solutions can be called by asymptotic 'black and white' solitons.

In paper ([5],Figure 2), it has been mimicked two spectral modes of solitons when the mode-locked laser diode generate picosecond solitons. This generation mathematically can be described as functional or differential nonlinear boundary conditions, which will be represented below, and the propagation of light can be described by the Schrödinger type. In this paper, the corresponding mathematical model will be considered. The representation of such modes has the WKB - form $u(z,t) = A(z,t)e^{\frac{i}{h}S(z,t)}$, where $A(z,t)$ is the amplitude of electric field [12], $S(x,t)$ is a phase and h is a small dimensionless parameter.

The problem of the coherent interaction of light impulses in nonlinear mediums is well-known: as noted in [13], "an interaction may be utilized for the transmission of information, for frequency conversion, and for the description of processes which proceed in more intensive fields and at times shorter than to relaxation time" (see, [15]).

Thus we consider an initial boundary value problem for the linear Schrödinger equation with nonlinear functional 'two-point' boundary conditions or the so-called integrable boundary conditions and some initial conditions. The Hamilton formalism will be applied to the study of structure of attractor of the boundary problem. Solutions will be find in the the WKB - form (8) with accuracy $O(h^2)$. We use the well-known methods of the analytical mechanics in order to define a phase and an amplitude of a wave function from the Hamilton-Jacobi equation and a transport equation. Indeed, for equations of quantum mechanics, this method of reduction of the quantum equation to a connected system of two equations for functions $A(z,t)$ and $S(z,t)$ has been developed by Maslov (see, [23]. But it has been applied only for the study of initial boundary value problems. Of course in this case, the selection correct boundary conditions is the main mathematical and physical problem. It will be shown that, for some class of boundary conditions and a special class of initial conditions, the problem is solvable.

The paper is constructed as follows. In section 2 the problem will be formulated, and it will be shown how the original problem can be reduced to the Hamilton-Jacoby equation with nonlinear two-point boundary conditions for the phase. Further we obtain, for the phase, a difference equation with continuous time. It must be noted that we consider only boundary conditions, for the Schrödinger equation, which can be decomposed on corresponding boundary conditions for solutions of the Hamilton-Jacoby equation and solutions of the transport equation, correspondingly. Such solutions describes asymptotic periodic 'white and black' solitons in optical resonator with feedback or likewise constant spatial temporal impulses in electrical circuits with nonlinear filter and amplifier [1, 2].

In section 3, we consider a boundary problem for the transport equation with known the phase, which can be find as a solution of the boundary problem the Hamilton-Jacoby equation. Since the phase $S(x,t) \to p_1(t - x/p) \in P^+$ as $t \to \infty$, where p_1 is a periodic piecewise constant function with finite or infinite points of discontinuities Γ on a period, the transport equation has simple asymptotic as $t \to \infty$. Indeed, it is shown that this transport boundary problem can be reduced to an integro-difference equation with known phase. Then, at a neighbourhood of the set P^+ - invariant solutions - the equation can be linearised. Is a result, the problem admits reduction to an autonomic difference equation. This equation has the 'Poincare' map $\Phi : R^3 \to R^3$. We assume that this map is hyperbolic and structural stable. It means that the map has finite number of fixed points $\mathfrak{P}$ which can be organised into circles. For the hyperbolic map, $\mathfrak{P} = \mathfrak{P}^+ \bigcup \mathfrak{P}^- \bigcup \mathfrak{P}^\pm$, where $\mathfrak{P}^+$ is a set of attractive fixed points of the map, $\mathfrak{P}^-$ is a set of repelling fixed points, and $\mathfrak{P}^\pm$ is a set of saddle points. As a result, there is a separatrix, which divides a regions of attraction to fixed points of trajectories. The separatrix plays role a set Γ in $1D$ case. A set of such separatrix can be of form of parabola for non-autonomous logistic map and their number can be countable (see, [9], p.183). Additionally, a structure of these limit distributions depends on structure of initial data of the original problem.

To be more concrete, we consider only 'quasi-invariant' solutions of the original problem, which are small perturbations of solutions $\psi(x,t) = p_1(t - x/p)$, $p_2(t - x/p)$, where $H(p) = \frac{1}{2}p^2$ is the hamiltonian of the system, and $\psi := (p_1(t - x/p), p_2(t - x/p))$ is the limit periodic solution for phase and amplitude in the distribution $\psi := e^{iS}|\psi|$. It will be shown that all perturbation of the function $\psi := \psi(p_1, p_2)$ at the point $(p_1, p_2) \in R^2$ are asymptotically stable. The prove is following: the integro-difference equation is reduced to ordinary difference equation (ODE) with delay argument p/l, where l is a size of the system. Further, the ODE is integrable and, hence, can be reduced to a family of non-autonomous difference equations as we discussed above.

In section 4-6 we consider the asymptotic properties of nonlinear difference equations in the Hausdorff and Schorohod metric. The Hausdorff metric describes deterministic solutions at a neighbourhood of points Γ - characteristic of the Hamilton equation of the dynamical system. The Schorohod metric describes random solutions which produce the so called deterministic chaos in difference equation (see, [24]). In section 7 we consider application of the mathematical method for optical resonators with feedback.

2 Formulation of IVBP for Schr$\ddot{o}$dinger equation

Let us consider the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \tag{1}$$

where ψ is a wave function, $\hbar$ is the Planck constant. Let us reduce this equation to a dimensionless form. Let us divide the two parts of equation on mv^2, where m is the mass of particles and v is their velocity. We obtain

$$i\frac{\hbar}{mv^2}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\hbar^2}{m^2v^2}\frac{\partial^2\psi}{\partial x^2}. \tag{2}$$

Then from $\hat{\lambda} = \hbar/mv$, where λ is a length of the de Broil wave, it follows that (3) can be written as

$$i\frac{\lambda}{v\tau}\frac{\partial\psi}{\partial\bar{t}} = -\frac{1}{2}\lambda^2\frac{\partial^2\psi}{\partial x^2} \tag{3}$$

where $\bar{t} = t/\tau$ and τ is a relaxation time of the wave function to some equilibrium state. Now we introduce the dimensionless constant $h = \frac{\lambda}{v\tau}$ and rewrite the last equation in the form:

$$\frac{\lambda}{v\tau}\frac{\partial\psi}{\partial\bar{t}} = -\frac{1}{2}h^2\left(\frac{v^2\tau^2}{l^2}\right)\frac{\partial^2\psi}{\partial x^2} \tag{4}$$

where l is a size of the system. Thus we choose $\frac{v^2\tau^2}{l^2} = 1$ so that $\tau = L/v$, and substituting this value into the above equation, we obtain the dimensionless equation

$$ih\frac{\partial\psi}{\partial\bar{t}} = -\frac{1}{2}h^2\frac{\partial^2\psi}{\partial x^2} \tag{5}$$

where h is a small dimensionless parameter.

Let us consider liner two points boundary conditions

$$\psi(0,t) = \theta\psi(l,t),\ t > 0, \tag{6}$$

initial conditions

$$\psi(x,0) = h_1(x),\ 0 < x < l \tag{7}$$

where $h_1(x)$ is a given function, and $\theta \in R$ is a parameter. Such problem has been considered in [19] for generalizes Shrödinger equation, where the theorem of existence of solutions has been proved.

Let

$$\psi_{1,2}(x,t) = e^{iS_{1,2}(x,t)/h}\varphi_{1,2}(x,t) \tag{8}$$

where $S_{1,2}$ is a phase and $\varphi_{1,2}$ is an amplitude. Then, substituting (8) in the Shrödinger equation), we obtain that

$$\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2\right)\varphi - ih\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}\varphi\triangle S\right) + \frac{(-ih)^2}{2}\triangle\varphi = 0. \tag{9}$$

We find solutions with accuracy $O(h^2)$ so that

$$\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2\right)\varphi = 0, \tag{10}$$

$$\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}\varphi\triangle S\right) = 0. \tag{11}$$

Let us consider the two-point nonlinear boundary conditions

$$\psi(0,t) = \Phi[\psi(l,t)] \tag{12}$$

or the boundary conditions of special form

$$\psi_{|x=0} = \Psi[|\psi|^2]f\left(\frac{|\psi|}{\psi}\right)_{|x=l} \tag{13}$$

where Φ, Ψ, and f are given function from open bounded interval I into itself.

Now, we go back to equations (10),(11). These equations satisfies to the two-point boundary conditions. For the Hamilton-Jacobi equation with Hamiltonian $H(p) = \frac{1}{2}p^2$, we consider the Hamilton system of ODE

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = \frac{\partial H}{\partial x} = 0 \tag{14}$$

with the initial conditions

$$x(0) = x_0, \quad p(0) = \frac{\partial S}{\partial x}(x_0). \tag{15}$$

System (14 has solutions

$$x - x_0 = p(t - t_0). \tag{16}$$

Then about of trajectories of Hamilton system $dx/dt = p$ solutions of Hamilton-Jacobi equations can be written as

$$\frac{dS(x(t), t)}{dt} = \partial S \partial t + \frac{\partial S}{\partial x}\frac{\partial x(t)}{\partial t} = -H(p) + \frac{\partial S}{\partial x}p. \tag{17}$$

We can find a solution $S(x, t)$ on a surface L so that a function $x(p, t)$ is a solution of equation

$$p - \frac{\partial S(x, t)}{\partial x} = 0. \tag{18}$$

Here, $p(x, t)$ is a solution of this equation. We recall that the function of the action $S(q, t)$ is the integral

$$S_{q_0, t_0}(q, t) = \int_{\varpi} Ldt \tag{19}$$

along the extremal ϖ, connecting points q_0, t_0 and q, t, where $L = p\dot{q} - H$ is the Lagrangian of a dynamical system. Thus the Lagrangian is the Legendre transformation of the hamiltonian H (see, [21], p.210).

We consider the Hamilton system in a space $R^1_x \bigoplus R_{1p} \bigoplus R^1_t$. Let $x(x_0, t)$, $p(x_0, t)$ be solutions of the Hamilton system. Then equations

$$x = x(x_0, t), \quad p = p(x_0, t) \tag{20}$$

determines a manifold $\mathfrak{L}$ of dimension $n = 2$ in the $R^1_x \bigoplus R_{1p} \bigoplus R^1_t$ with boundary $\{t = 0\}$. We can prove that on this manifold the form

$$-H(x, p, t)dt + pdx = \Omega \tag{21}$$

is closed. The form is closed if $d\Omega = 0$. If the manifold $\mathfrak{L}$ is connected, then the form Ω is exact. Then, by definition, $\Omega = d\Lambda$, where Λ is a differential form.

Interpreting Hamiltonian system as a vector field (dynamic system) on a symplectic manifold $\mathfrak{M}$, we can consider the solutions of the Hamiltonian system as the integral trajectories of the vector field. The important question of solvability of equation $\Omega = d\Lambda$ (with respect Λ) is connected with the topological structure of the manifold $\mathfrak{M}$. It is known that there is a solution os problem

$$dS' = \Omega_{\mathfrak{L}'}, \quad S'_{t=0} = S_0(x). \tag{22}$$

Let U is an open neighbourhood in the space $R^1_x \bigoplus R^1_t$ of the set $\{t = 0\}$. We assume that U is the projection of the manifold $\mathfrak{L}$ such that equation $x = x(x_0, t)$ is solvable with respect x_0. It means that $x_0 = x_0(x, t)$. Then the function

$$S(x, t) = S'(x_0 = x_0(x, t), t) = (\pi_x^{-1})^* S'(x_0, t), \tag{23}$$

where $\pi_x : \mathfrak{L} \to R^1_x \bigoplus R^1_t$ is the projection, is a solution in a region U of the initial problem for Hamilton-Jacobi equations (see, [22], p.25).

A function S at surface L satisfies to the relation

$$d\hat{S} = -Hdt + pdx. \tag{24}$$

Then from (17),(18 it follows that

$$S(x(t), t) = S_0(x(t_0), t_0) + \int_{t_0}^{t} (-Hdt + p\dot{x}dt) \tag{25}$$

where $x_0(x, t)$ is determined by relation (16). From (25) it follows that

$$S(l, t) = S(x(0), 0) - Ht + p(l - x(0)) = h_1(t), \quad t \in [0, l/p) \tag{26}$$

where $x(0) = x$.

Now, we return to boundary conditions (**??**) and assume that $\hat{S}(S) := \Phi_1(S) : I \leftarrow I$, where I is some bounded open interval. Then $S(0, t) = \Phi_1(S(l, t)) = h_1(t)$, $t \in [0, l/p)$. It is easy to see that

$$dS = -Hdt + \frac{dx}{dt}dt. \tag{27}$$

Then

$$S(l, t + l/p) = S(0, t) - Hl/p + pl. \tag{28}$$

Then from boundary conditions

$$S(0, t) = \Phi_1[S(l, t)], \quad t > 0, \tag{29}$$

and from (28) it follows that

$$S(l, t + l/p) = \Phi_1[S(l, t)] + pl/2. \tag{30}$$

Equation (30) can be solved, step by step, if we know the initial function $h(t)$ on the interval $t \in [0, l/p)$. This function can be find by the formula

$$S(l, t) = S(x, 0) - Ht + px, \quad t \in [0, l/p). \tag{31}$$

We assume that Φ_1 is structural stable, and $\Phi_1 \in C^2$. Then a set $Per\,\Phi_1$ of periodic points of Φ_1 is $Per\,\Phi_1 = P^+ \bigcup P^-$, where a set P^+ is finite, and P^- is finite or countable. Separator D of the map Φ_1 is determined by formula

$$D = \bigcup_{n \geq 0} h^{-n}(\bar{P}^-)z \tag{32}$$

where $\bar{P}^-$ is closer of P^-. It is uncountable nowhere dense closed set on I of measure zero which is finite, countable or uncountable (see,[9],p.234).

Now, we consider the a $\Gamma = h^{-1}(D)$. Then the set Γ has the same properties as the set D if the transversal condition $\dot{h} \neq 0$, $t \in \Gamma$ is satisfied. Then each solution of difference equation (30) is $2^N l/p$ - asymptotic periodic piecewise constant function with finite, countable or uncountable points of discontinuities on a period, where N is least common multiple of periods of attractive circles of the map Φ_1. A number $pl/2$ is a parameter of bifurcations of limit solutions.

Further,

$$S(x, t) = S(0, t - x/p) + \int_{t-x/p}^{t} (-Hdt + pdx), \tag{33}$$

where $(0, t - x/p) = \Phi_1[S(0, t - x/p)]$.

Then

$$S(x, t) = \Phi_1[S(0, t - x/p)] - \frac{H}{p} + px. \tag{34}$$

If $\zeta = t - x/p \to +\infty$, then

$$S(x, t) = \Phi_1[P^+(\zeta)] - \frac{H}{p}x + px \tag{35}$$

where $P^+(\zeta) \in P^+$ for almost all point $\zeta \in [-l/p, \infty)$.

3 Boundary problem for transport equation

As above,

$$S(l,t+p/l)=S(0,t)-Hp/l+pl=\Phi_1[S(l,t)]-Hp/l+pl,\quad t\in[0,l/p). \tag{36}$$

Let us denote $\mu=-Hp/l+pl$. Then we obtain the difference equation

$$S(l,t+p/l)=\Phi_1[S(l,t)]+\mu \tag{37}$$

with the initial condition

$$S(l,t)=h_1(t),\quad t\in[0,l/p). \tag{38}$$

Then solutions of difference equation can be find step by step by iterations of initial function $h_1(t)$ with help of the map Φ_1.

This equation has asymptotic periodic piecewise constant solutions $P(t)$ with finite or infinite points of discontinuities on a period [9]. Since the solutions are constant about the characteristic $dx(t)/dt=p$ the same property has the function $u^2(\zeta)$ where $\zeta=t-x/p$. Then $\frac{\partial S}{\partial x}(x,t)=\frac{\partial S}{\partial\zeta}(\zeta)\to P(\zeta)$ for almost all points $\zeta\in R$. From the last equality it follows that the phase $S(\zeta)$ is asymptotic piecewise constant function with accuracy to some constant. Such type solutions known for Burgers equation with viscosity γ as $\gamma\to 0$ ([18] p.190). Such type solutions has been obtained for the boundary problem for the Ginzburg-Landau equation by computer simulation ([20] p.270).

An amplitude $\varphi(x,t)$ is a solution of the transport equation

$$\frac{\partial S}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial t}+\frac{1}{2}\varphi\triangle S=0. \tag{39}$$

On the lagrange surface L we have that $\partial S/\partial x=p$ and, hence, a function $\varphi(x,t)$ satisfies to the transport equation:

$$p\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial t}+\frac{1}{2}\varphi\frac{\partial^2 S}{\partial x^2}=0. \tag{40}$$

This equation can be written as

$$\frac{d\varphi}{dt}(x(t),t)=-\frac{1}{2}\varphi\frac{\partial^2 S}{\partial x^2} \tag{41}$$

as $dx(t)/dt=p$. Then from (41) it follows that

$$\varphi(l,t_0+l/p)=\varphi(0,t_0)-\frac{1}{2}\int_{t_0}^{t_0+l/p}\varphi(p(t-t_0,t)\frac{\partial^2 S}{\partial x^2}(p(t-t_0,t)dt. \tag{42}$$

From the boundary conditions

$$\varphi(0,t) = \Phi_2[\varphi(l,t)] \tag{43}$$

it follows that equation (42) can be written as

$$\varphi(l,t_0+l/p) = \Phi_2[\varphi(l,t)] - \frac{1}{2}\int_{t_0}^{t_0+l/p} \varphi(p(t-t_0,t)\frac{\partial^2 S}{\partial x^2}(p(t-t_0,t)dt. \tag{44}$$

Now we see that

$$\frac{\partial S}{\partial x}(x,t) = \Phi_1'[S(l,t-x/p)]\frac{\partial S}{\partial \zeta}(l,\zeta)(-p^{-1}) - \frac{H}{p} + p. \tag{45}$$

From (45) it follows that

$$\frac{\partial^2 S}{\partial x^2}(x,t) = \Phi_1''[S(l,\zeta)]\frac{\partial S}{\partial \zeta}(l,\zeta)(-p^{-1}) + \Phi_1'[S(l,\zeta)]\frac{\partial^2 S}{\partial \zeta^2}(l,\zeta)(-p^{-2}). \tag{46}$$

Let P_1^+ are attractive points of the map Φ_1. Then perturbations of phase at a neighbourhood of a fixed point $\Phi_{1,\mu}$ satisfies to the linear difference equation

$$S'(l,t+l/p) = \Phi_1'(S(l,t))S'(l,\zeta), \quad \zeta \in [-p/l,0), \tag{47}$$

$$S''(l,\zeta+l/p) = \Phi_1''(S(l,\zeta))(S'(l,\zeta))^2 + \Phi_1'(S(l,\zeta))S'(l,\zeta), \quad \zeta \in [-p/l,0), \tag{48}$$

At a fixed points we have that

$$S'(l,\zeta+l/p) = \Phi_1'(P_1^+)S'(l,\zeta), \quad \zeta \in [-p/l,0). \tag{49}$$

Solutions of this equation have the form $S'(l,\zeta) = e^{k_1\zeta}$ where $k_1 = \frac{p}{l}\ln|\Phi_{1,\mu}'(P_{1,\mu}^+)|$ where $|\Phi_{1,\mu}'(P_{1,\mu}^+)| < 1$ and, hence, $k_1 < 0$. Then from (50) we obtain that $|S'(\zeta)| \to 0$ as $\zeta \to \infty$, and from (50) it follows that $|S''(\zeta)| \to 0$ as $\zeta \to \infty$. As a result, we get that

$$S''(l,\zeta+l/p) = 2\Phi_{1,\mu}'(P_{1,\mu}^+)e^{2k_1\zeta}. \tag{50}$$

Note that from (50) it follows the relation

$$\frac{\partial^2 S}{\partial x^2}(x,t) = \Phi_1''(S(P_1^+)e^{2k_1(t-x/p)} + \Phi_1'(S(P_1^+)e^{k_1(t-x/p)}. \tag{51}$$

From (52) it follows that

$$\frac{\partial^2 S}{\partial x^2}(p(t-t_0,t) = \Phi_1''(S(P_1^+)e^{2k_1t_0} + \Phi_1'(S(P_1^+)e^{k_1(t_0)} \tag{52}$$

along each line $t - x/p = t_0$. Then the integro-difference equation can be rewritten as

$$\varphi(l,t_0+l/p) = \Phi_2[\varphi(l,t)] - \frac{1}{2}(\Phi_1''(S(P_1^+)e^{2k_1t_0} + \Phi_1'(S(P_1^+)e^{k_1(t_0)}) \int_{t_0}^{t_0+l/p} \varphi(p(t-t_0,t)dt. \tag{53}$$

Differentiating this equation on t_0, we obtain the equation

$$\varphi'(l,t_0+l/p) = \Phi_2'[\varphi(l,t)]\varphi'(l,t) - \frac{1}{2}(\Phi_1''(S(P_1^+)e^{2k_1t_0} + \Phi_1'(S(P_1^+)e^{k_1(t_0)})(\varphi(l,t_0) - \varphi(0,t_0)). \tag{54}$$

Since,

$$\varphi(l,t_0) = \Phi_2[\varphi(0,t_0+l/p)], \tag{55}$$

equation (54) can be written as

$$y'(t_0) = a(t_0)y'(t_0) \tag{56}$$

where

$$y(t_0) = \varphi(l,t_0+l/p) - \Phi_2[\varphi(0,t_0)], \tag{57}$$

$$a(t_0) = \frac{1}{2}(\Phi_1''(S(P_1^+)e^{2k_1t_0} + \Phi_1'(S(P_1^+)e^{k_1(t_0)}). \tag{58}$$

Solutions of equation (56) are

$$y(t) = y(0)\left(e^{-\frac{1}{k_1}}e^{\frac{1}{k_1}e^{k_1t}} + e^{-\frac{2}{k_1}}e^{\frac{2}{k_1}e^{2k_1t}}\right) \tag{59}$$

where $y(0) = \varphi(l,l/p) - \Phi_2[\varphi(0,0)]$. For each solution $\varphi(l,t)$ of equation (54) its restriction $\varphi(l,t))_{|t\in[0,l/p)}$ belong to one and only one of classes of initial functions

$$\Phi := \bigcup \Phi_\lambda \quad \text{where} \quad \Phi_\lambda := \{\varphi(l,t) \in \Phi : \varphi(l,l/p) - \Phi_2[\varphi(0,0)] = \lambda\} \tag{60}$$

where $\lambda = y(0)$. Any solution of equation (54) such that $\varphi(l,t))_{|t\in[0,l/p)} \in \Phi_\lambda$ is a solution of difference equation (56) with $y(0) = \lambda$ (see, [9], p.178).

Let us define $y_1(t) = e^{\frac{1}{k_1}e^{k_1t}}$, and

$$y_1(t) = e^{\frac{1}{k_1} e^{k_1 t}}, \quad y_2(t) = e^{\frac{1}{k_2} e^{k_2 t}}. \tag{61}$$

Then

$$\varphi(l, t_0 + l/p) = \Phi_2[P^+]\varphi(l,t) - \frac{1}{2}(\Phi_1''(S(P_1^+)y_2(t) + \Phi_1'(S(P_1^+)y_1(t)) \tag{62}$$

$$y_1(t + l/p) = e^{\frac{1}{k_1}} y_1(t) \tag{63}$$

$$y_2(t + l/p) = e^{\frac{1}{2k_1}} y_1(t) \tag{64}$$

These equations produce the $G : R^3 \to R^3$ which has a fixed point $z^* = (P^+, 0, 0)$. The Jacobi matrixes TG of the map is

$$\begin{pmatrix} \Phi_2'[P^+] - \kappa & \frac{1}{2}\Phi_1'(S(P_1^+) & \frac{1}{2}\Phi_1''(S(P_1^+) \\ 0 & e^{\frac{1}{2k_1}} - \kappa & 0 \\ 0 & 0 & e^{\frac{1}{2k_2}} - \kappa \end{pmatrix} \tag{65}$$

Then the eigenvalues of the equation $TG = \kappa G$ are $\kappa_1 = \Phi_2'[P^+]$, $\kappa_2 = e^{\frac{1}{k_1}}$, $\kappa_3 = e^{\frac{1}{2k_2}} e^{\frac{1}{2k_2}}$, where $|\Phi_2'[P^+]| < 1$ and $k_1, k_1 < 0$. For a repelling point, we have $|\Phi_2'[P^-]| > 1$.

Let $P = P^+ \bigcup P^-$, where P^+ is a finite set, and P^- is a finite or countable set. Then the map G contains contains a finite number of attractive fixed points $A^+ = (P^+, 0, 0)$ and finite or countable set of saddle point $A^\pm = (P^+, 0, 0)$. Let us assume that the map Φ_2 is is monotone increasing function for each $\varphi \in I$ which has two attractive fixed points a_1, a_3 and one repelling fixed point a_2. Then the behaviour of phase trajectories can be described as projection on a plane $(\varphi, y_1) \subset R^2$ or on a plane $(\varphi, y_2) \subset R^2$. The trajectories will be topologically equivalent. We see that each curve $\nu(t) = \nu^-(t) \bigcup \nu^+(t)$, $t \in [0, p)$ is such that the part $\nu^-(t)$ is attracted by a point a_1, and the part $\nu^+(t)$ is attracted by a point a_3. Of course, this curve depends on the initial values of the boundary problem. Lines $y_1 = 0$ and $y_2 = 0$ represents separatrixe in R^2 for trajectories of dynamical system. As a result, we get that amplitudes $\varphi(l,t)$ tends to a piecewise constant asymptotic $2l/p$ - periodic function $p(t)$, where $p(t \in a_1 \bigcup a_3$ (see, Fig.2).

Further

$$S(x,t) = S(0, t - x/p) + \int_{t_0}^{t_0 - x/p} (-Hdt + pdx(t)) = \Phi_1[S(l, t - x/p)] - Hx + px. \tag{66}$$

As a result,

$$S(x,t) = \Phi_1[p_1(t - x/p)] - Hx + px + o(t_0) \tag{67}$$

where $o(t_0) \to 0$ as $t_0 \to \infty$. Here $p_1(t - x/p)$ is $2^{N_1} l/p$ - periodic function on the variable $t - x/p$, where $N^1 \in Z^+$, as $t \to \infty$.

Further

$$\varphi(x,t_0)=\varphi(0,t_0-x/p)+\int_{t_0}^{t_0-x/p}\varphi(p(t-t_0+x/p,t)\frac{\partial^2 S}{\partial x^2}(p(t-t_0+x/p,t)dt. \tag{68}$$

But from (66) it follows that

$$S(x,t_0)\to\Phi_1[p_1(t_0-x/p)]-Hx+px. \tag{69}$$

Then

$$\varphi(x,t_0)\to\varphi(0,t_0-x/p)+(\Phi_1[p_1(t_0-x/p)]-Hx+px)\int_{t_0}^{t_0-x/p}\varphi(p(t-t_0+x/p,t)dt \tag{70}$$

as $t_0\to\infty$. Then

$$\varphi(x,t_0)'\to\varphi'(0,t_0-x/p)+(\Phi_1[p_1(t_0-x/p)]-Hx+px)'(\varphi(0,t_0-x/p)-\varphi(x,t_0). \tag{71}$$

Let

$$z(t_0)=\varphi(x,t_0)-\varphi(0,t_0-x/p). \tag{72}$$

Then equation (71) can be written as

$$z'(x,t_0)=-(\Phi_1'[p_1(t_0-x/p)]p_1'(t_0-x/p)z(t_0) \tag{73}$$

where x can be considered as a parameter. The function $p_1'(t_0-x/p)\to 0$ as $t_0\to\infty$ for almost all points $\zeta\in R^+$. From (71) it follows that

$$\frac{1}{2}[z^2(x,t_0)]'=-(\Phi_1'[p_1(t_0-x/p)]p_1'(t_0-x/p)z^2(x,t_0). \tag{74}$$

It is easy to prove that if the function $\Phi_1'[p_1(t_0-x/p)]$ is positive(negative) at a neighborhood of some fixed point, then the function $p_1'(t_0-x/p)$ is also positive (negative). Then from (74) it follows that $z^2(x,t_0)\to 0$ as $t_0\to\infty$ for almost each $0<x<l$. As a result,

$$\varphi(x,t_0)\to\varphi(0,t_0-x/p)=\Phi_2(\varphi(l,t_0-x/p)\to p_2(l,t_0-x/p) \tag{75}$$

where $p_2(\zeta)$ is asymptotic periodic 2^{N_2} - periodic function, $n_2\in Z^+$.

Thus the solutions of the origin initial boundary value problem are as

$$\psi(x,t)=e^{\frac{i}{h}p_1(l,t_0-x/p)}p_2(l,t_0-x/p) \tag{76}$$

where $p_1(\zeta)$ and $p_2(\zeta)$ have finite, countable or uncountable "points" of discontinuities on a periods. We call these solutions by solution of relaxation, pre-turbulent or turbulent type, correspondingly (see, Fig.1-3).

4 Properties of asymptotic solutions of difference equations

Thus, the problem can be reduced to difference equations

$$u(t) = f[u(t - l/p) \tag{77}$$

with continuous time. Indeed, we consider, as example, an unimodal map $f \in C^2(I \to I)$ and suppose that f has one unique critical point (a maximum). The map f is regular if there is a quadratic critical point, the map is hyperbolic, and its critical point is not periodic or pre-periodic. A point u_0 such that $f'(u_0) = 0$ is critical. If $f''(u_0) \neq 0$, we have the quadratic critical point. The set of regular maps coincide with the set of structurally stable unimodal maps $f \in C^2(I \to I)$, which are structurally stable, and open, and dense in all smooth topologies. Hence, the set of regular maps is open and dense in all smooth topologies. As a result, we can reduce the study of unimodal maps to the special case of unimodal maps which are quasiquadratic or topologically conjugate to quadratic maps (see, [9]).

Let us assume that a solution satisfies to the initial conditions

$$u_0(t) = (\varphi_1(-t),\, t \in [-l/p, 0),\, \varphi_2(t),\, t \in [0, l/p)). \tag{78}$$

If $f \in C^2(I \to I)$ is an unimodal and structurally stable map (in C^2 structurally stable maps form an form an open dense subset), then the set $Per\, f = P^+ \bigcup P^-$, where P^+ is finite, and P^- is finite or countable. If $f \in C^0(X \to X)$, where X is a topological or metrical space, the map $u \to f(u)$ produce on X the semigroup of continuous maps $f^n{}_{n=0}^{n=\infty}$ of the dynamical system. We consider dynamical systems which are given by the continuous maps $f \in C^0(I \to I)$, where I is a closed interval. It is known the classification of the dynamical systems with periodic points of the periods $2^i l/p$, $i = 0, 1, 2, \ldots$. Such systems are called systems of the type 2^∞. These systems form in the space $C^0(I \to I)$ nowhere dense set.A dynamical system is said to be simple if the periods of cycles are limited. If the map contains a periodic point, then the dynamical system is complex. Complex systems have cycles of any periods.

Next, let us consider a difference equation with initial data

$$u(t) = \varphi(t), \quad t \in [0, l/p], \tag{79}$$

and assume that

$$f \in C^0(I, I),\, \varphi \in C^0([0, 1], I), \tag{80}$$

and

$$\varphi(1) = f(\varphi(0). \tag{81}$$

Then a solution $u(t)$ of initial problem for difference equation, which is produced by the map f, is unique and can be represent in the form

$$u(t+nl/p) = f^n(\varphi(t)),\ t \in [0,l/p],\ n \in Z^+,\ Z^+ = \{0,1,...\}, \tag{82}$$

where f^n is $n-th$ iteration of f. We have that $u(t) \in C^0(R^+,I)$. So we got continuous and bounded solutions of the the difference equation.

However, the space $C^0([0,l/p],I)$ is not compact. Therefore, the trajectory of the dynamical system does not even have a partial limits. However, there are difference equations for which the path is still compact in $C^0([0,l/p],I)$ (see examples, [9]). Let's consider a compact space $C^\Delta([0,\Delta],I)$ upper semi-continuous functions $\psi : [0,\Delta] \to 2^I$, where 2^I is set of closed intervals on I with topology

$$dist\,[\psi_1,\psi_2] := \max\,[\sup_{z\in gr\,\psi_2} \varrho(z, gr\,\psi_1),\ \sup_{z\in gr\,\psi_1} \varrho(z, gr\,\psi_2)], \tag{83}$$

where $gr\,\psi_k$ is the graphic of a function ψ_k, $k=1,2$ as $t \in [0,\Delta]$, and $\varrho\,(z, graph\,\psi_k)$ is the distance in R^2 from a point z to the set $gr\,\psi_k$. The phase space C^Δ is compact and, hence, completely contains the limit set each trajectory of the dynamical system.

In typical situations, there are asymptotically periodic piecewise constant solutions with finite or infinite points Γ_ρ, where values of limit functions equal I_ρ. Here the index may be finite, infinite countable, or infinite uncountable. Excluding the set Γ_ρ, values of limit function belong to a set $P^+(f)$, where P^+ is the set of attractive fixed points of the map f. If Γ_ρ is finite, we have the solutions of relaxation type. If Γ_ρ is infinite and countable, we have the solutions of preturbulent type. If Γ_ρ is infinite and uncountable, we have the solutions of turbulent type. Thus, we have shown that the linear system of Schrodinger equations with nonlinear boundary conditions allow oscillating solutions with non-decreasing amplitude.

Here, the map $\Phi \in C^2(I,I)$ is structural stable and hyperbolic. Then the set of periodic points $Per\,\Phi := \Phi_1 \circ \Phi_2$, where $\circ$ is the superposition of corresponding functions, splits on the union of attractive periodic points P^+ and repelling periodic points P^-, which organize circles in the following ordering [9]:

$$3,5,7,9,11,...,2\times 3, 2\times 5,...,2^2\times 3, 2^2\times 5,...,2^3,2^2,2,1. \tag{84}$$

If map has the circle of the period k, then the map has the circle of the period k', which follows after k as shown by the sequences (84). Each trajectory asymptotically tends to a circle if (and only if) the set of points of circles is closed. There are conditions when this set of the circles is not closed. For example, when the dynamical system contains a circle of period $\neq 2^i$, $i = 0,1,2,...$.

Two subsets A and B of a topological space X is called a functional separable if there is defined on the whole space X limited continuous real-valued function which takes in all the points of the set A one value (set, a) at all points of the set B- the value $b \neq a$. There is always can be assumed that $a = 0$, $b = 1$ and $0 < f(x) < 1$ for all $x \in X$. The Urysohn lemma claims that in normal space any two disjoint closed sets are functional separable. The solutions of the difference equations contains a family of such functions $f(x)$ on periodically repeated sets of type $A \bigcup B$. The problem of determining all the general spaces with compact Hausdorff extension was solved by Tichonoff (see, [10].

The set of attractive circles P^+ is finite for the structural stable maps, but the set of repelling circles is finite or countable. To characterize the set of "discontinuities" of limit solutions of difference equation, which are, as $t \to +\infty$, piecewise constant functions, satisfying to the Urysohn lemma, we must introduced the set of points

$$D = \bigcup_{n \geq 0} h^{-n} \bar{P}^-, \tag{85}$$

which called by the separator of the map. he set D is nowhere dense on I closed set of measure zero: empty, countable or uncountable; and $\bar{P}^-$ is the closure of the set P^-.

The separator D is uncountable if (and only if) the map has circles of periods $\neq 2^i$, $i = 0, 1, 2, \ldots$. If the map has circles of periods $\neq 1, 2$ and not has circles of periods $\neq 2^i$, $i = 0, 1, 2, \ldots$, the set D is uncountable [9].

Now, we can construct the set

$$\Gamma = h^{-1}(D). \tag{86}$$

Suppouse that $h(t)$ satisfies to the transversal condition

$$h(t) \neq 0, \quad t \in \Gamma. \tag{87}$$

Then the set Γ is closed and nowhere dense on $[0, 1]$, and the Lebesgue measure $mes\, \Gamma = 0$. From these properties it follows that the transversal condition is satisfies for an open dense set of functions $h \in C^2([0, 1], I)$.

Let

$$\Pi_\alpha^\beta = [0, 1] \times [\alpha, \beta], \quad -\infty \leq \alpha < \beta \leq +\infty \tag{88}$$

(in particular, $\Pi_0^\infty = \Pi$). For any α, β, we consider the space $C^\Delta(\Pi_\alpha^\beta, I)$ upper semicontinuous functions $\psi : \Pi_\alpha^\beta \mapsto 2^I$ with topology, which is given by the Hausdorff metric

$$\Delta_{\Pi_\alpha^\beta}(\psi_1, \psi_2) = \max\{ \sup_{z \in gr\psi_1} \rho(z, gr\psi_2); \ \sup_{z \in gr\psi_2} \rho(z, gr\psi_1)\} \tag{89}$$

where $gr\psi$ is the graphic of function $\psi(x, t)$ at $(x, t) \in \Pi_\alpha^\beta$; $\rho(z, gr\psi)$ is the metric in R^2 from a point z to the set $gr\psi$. Similarly we can determine the set $C^\Delta(\Pi_\alpha^\beta, I \times I)$. The set C^Δ contains as subset the space C^0, but C^0 is not compact, while C^Δ is compact and contains functions, which belong to the attractor of the considered problem.

Now, we consider the locally maximal ω - limit set (or the basic set). F is a locally maximal set if there is a neighbourhood which not contain larger ω - limit set $F' \supset F$. If a map $f \in C^0(I, I)$ contains a circle of the period $\neq 2^i$, $i = 0, 1, 2, \ldots$, then the map has maximal ω - limit set $F_{\max}$, containing circle. $F_{\max}$ is Cantor set or it consists of multiple intervals. In the first case there are solutions of pre-turbulent type. In the second case there are random solutions of the difference equation. Hence, the same type of solution can be found for the original quantum problem in the zero approximation.

Thus with accuracy $o(h^2)$ the quantum boundary problem is reduced in WKB - approximation to the difference equations. In particular, for such equations, in book [9] has been proved the asymptotic stability of solutions in the Skorokhod metric and the Hausdorff metric for classical problem. Hence the same result holds for quantum problem. Indeed, a solution of the problem $v = (v_1, v_2)$ call stable to perturbations of initial and boundary conditions if arbitrarily small C^2 - perturbation of the functions Φ and $h_i, i = 1, 2$ lead to arbitrarily small variations of the functions v in the Hausdorff metric (Skorokhod metric).

Recall that the Skorokhod metric is defined as [9]:

$$s(v, \tilde{v}) = \sup_{\alpha \in \Lambda} \{||v \circ \alpha - \tilde{v}||_{C^0(\Pi, I \times I)} + ||\alpha - Id||_{C^0(\Pi, \Pi)}\} \tag{90}$$

where Λ is set of homeomorphisms $\Pi \mapsto \Pi$, $Id \in \Lambda$ is identical homeomorphism.

Let us say that a function $\psi \in C^{\Delta}(\Pi, I)$ tends, as $t \to \infty$, to the function $\tilde{\psi} \in C^{\Delta}(\Pi.I)$ if

$$\Delta_{\Pi_T^\infty}(\psi, \tilde{\psi}) \to 0, \quad t \to \infty. \tag{91}$$

Similarly, the convergence is determined for functions from $C^{\Delta}(\Pi, I \times I)$. Denote by m - the least common multiple of the periods of attractive circles P^+ and consider the $\Phi^{\Delta} \in C^{\Delta}(I, I)$. Define the map $\Phi^{\Delta} : I \mapsto 2^I$ as [9]

$$\Phi^{\Delta} = \lim_{n \to \infty} \Phi^{n!}, \tag{92}$$

where the symbol lim is defined as the limit in C^{Δ}, and consider the sequences of maps $\Phi^n \circ \Phi^{\Delta} \in C^{\Delta}(2^I, 2^I)$, $n = 0, 1, \ldots$. Then from [9] it follows that the map Φ^{Δ} exists, commutes with Φ and satisfies to the relation $\Phi^{\Delta} \circ \Phi^{\Delta} = \Phi^{\Delta}$. Then $\{\Phi^{\Delta} \circ \Phi^{\Delta}\}$ is semigroup of maps.

Thus, solutions $y(t)$ tends, as $t \to \infty$, to $4m$ - periodic distribution in Hausdorff metric so that

$$p(t) = \Phi^{4m-1} \circ \Phi^{\Delta} \circ h(t - 2(2m - 1)),\ t \in [4m - 3, 4m - 1),\ m = 1, 2, \ldots \tag{93}$$

with points of discontinuities $\Gamma_{R^+} = \bigcup_{n=1}^{\infty} \{t : t - 2n \in \Gamma\}$. At these points derivatives of smooth solutions of the difference equation tends to infinity as $t \to \infty$. Then, as follows from [9], solution of Sharkovsky initial boundary value problem (v_1^0, v_2^0) is stable in the space C^2 in Hausdorff and Skorokhod metrics and tends, as $t \to \infty$ to a $4m$ - periodic on t distribution p^0.

4.1 Remark

We again consider solutions of difference equation of the form

$$u(t + 1) = f[u(t)],\ t > 0, \tag{94}$$

where $f \in C^2(I, I)$ is a given function, and we assume, for simplicity, that the delay argument is 1. Then from (97) it follows that

$$u'(t + 1) = f'[u(t)]u'(t). \tag{95}$$

If $|f'[u]| < 1$ for some $u \in I$, then $u'(t) \to 0$ as $t \to \infty$. Indeed, let $\beta_1, \ldots, \beta_m \in I$. Let $f(\beta_i) = \beta_{i+1}$, $i = 1, \ldots, m - 1$, $f(\beta_m) = \beta_1$. We will continue a function $f(u)$ in a continuous manner. Then the map $u \to f(u)$ has periodic trajectory $\beta_1, ..., \beta_m, \beta_1, ...$ of period m. At a neighbourhood of each point β_i we have

$$|f(u) - \beta_{i+1}| \approx\approx |f'(\beta_i||u - \beta_i|. \tag{96}$$

As a result

$$|f^m(u) - \beta_i| \approx \prod_{j=1}^{m} |f'(\beta_j||u - \beta_i|. \tag{97}$$

If

$$\prod_{j=1}^{m} |f'(\beta_j| < 1 \tag{98}$$

and a point $u_0 \in O_\delta(\beta_i$, then a trajectory $f^n(u_0) \to (\beta_1, ..., \beta_m)$, where $O_\delta(a)$ is a neighbourhood of a point a, and $\delta > 0$ is small enough. Since in previews sections we use inequality (98), we assume that, for difference equations, the initial data $u_0(t) \in O_\delta(\beta_i$, where $t \in [-l/p, 0)$. The same is true for the initial data of the initial boundary value problem. Then convergence of solutions to piecewise constant periodic distributions follows from the structural stability of the map f or the corresponding map in the nonlinear boundary conditions.

Further,

$$u''(t+1) = f''[u(t)][u'(t)]^2 + f'[u(t)]u''(t). \tag{99}$$

Hence,

$$u''(t+1) = f''[u(t)][u'(t)]^2 + f'[u(t)]u''(t). \tag{100}$$

Hence,

$$|u''(t+1)| \leq \nu|u''(t)| + O(1/t), \tag{101}$$

where $0 < \nu < 1$. Hence, $|u''(t)| \to 0$ as $t \to \infty$.

For solutions of difference equations, points of discontinuities of a limit solution are produced by repelling fixed points $u^- \in A^-$ of a map f, where $|f'(u^-)| > 1$ and by preimages $U = f^{-n}(u^-)$, $n = 1, 2, 3, ...$ of the repelling points. If f is a monotone and this map has one repelling point, then the set U is empty and we obtain solutions of relaxation type with one point of discontinuities on a period. If set U is countable, we have solutions of pre-turbulent type. If set U is uncountable, we have solutions of turbulent type.

Suppose that X is a locally compact topological space. We say that Λ is a mixing set, if for each set $W \subset \Lambda$, which is open in Λ, and for each finite open set $\Sigma = \{\sigma_j, 1 \leq j \leq n\}$ of the set Λ there are m and s, depending on W and Σ, such that $f^i(\bigcup_{q=0}^{s-q}\bigcap n\sigma_j \neq \emptyset$ as $i \geq m$ and $1 \leq j \leq n$.

From the definition of mixing set it follows that there are points $a \in \Lambda$ such that trajectories $f^i(u)_{i=0}^{\infty}$ dense on Λ. Since the trajectories on Λ are unstable, the set Λ is stable and attracts all trajectories. If there is a neighbourhood $U \subset \Lambda$ such that $\bigcap_{i>0} f^i(U) = \Lambda$, then Λ is a mixing attractor. Indeed, let us introduce

$$\mathfrak{B}(\mathfrak{A}) = \{u \in E | \mathfrak{A}(u) = \mathfrak{A}(u)\} \tag{102}$$

where $\mathfrak{A}(u)$ is an attractor of a trajectory.

If a mixing set Λ has a neighbourhood U such that all points on U Λ go out from U, then Λ is mixing repeller. Mixing repeller may include mixing attractor. If a map $f \in C^r(R \to R)$, $r \geq 1$, then the mixing attractor is interval or several intervals, cyclically transformed into each other under iterations of f. The mixing repeller is a set which is homeomorphic to the Cantor set or interval or several intervals. For example, the map $f : u \to \lambda u(1-u)$ has a repeller. If $\lambda = 4$, then repeller is interval $[0,1]$. If $\lambda^3 - 2\lambda^2 - 4\lambda - 8 = 0$, then $\lambda = 3.678$ and the map f has attractor which is interval $\left[\frac{\lambda^2}{4}\left(1 - \frac{\lambda}{4}\right), \frac{\lambda}{4}\right]$. There are mixing repellers or mixing attractors if and only if there are circles of periods $\neq 2^i$, $i = 0, 1, 2, \ldots$.

Whether attracted to the attractions almost all points in space by the Lebesgue measure? If repeller contains an interval, then the answer is negative. If repeller is the Cantor, set then the answer is positive. Indeed, if $f \in C^2$ and on a repeller Λ hyperbolic conditions are satisfied, then measure Λ is zero. It means that there is $N \geq 1$ and $o > 1$ such that if $|df^N(u)/du| \geq o$ for $u \in \Lambda$, then measure $\Lambda = 0$. These statement follows from [?, ?]. Almost all points $x \in \Lambda$ form on Λ a dense set of type G_σ that is these points are represented as the intersection of a countable number of open sets. For example, the derivative Cantor staircase is defined and equal to zero at all points except for the Cantor set.

Below this observation will be used to prove existence of the Lebesgue integral for solutions of turbulent type for integro-difference equations for which an origin initial value boundary problem (IVBP) may be reduced. This allows us to prove the convergence of these equations to the piecewise constant asymptotic periodic functions $\mathfrak{P}_{1,\mathfrak{h}}$, $\mathfrak{P}_{2,\mathfrak{h}}$ (for $h < h_0$ as $t \to \infty$) with finite, infinite countable or infinite uncountable points of discontinuities $\mathfrak{d}_\mathfrak{h}$ on a period where $\Gamma_h \to \Gamma_0$ as $h \to 0$.

Solutions on Fig.1 and Fig.2 are similar to solutions which has been obtained in ([9], p.209, Fig.78), as the solutions of the differential-difference equation

$$hu'(t) = -u(t) + f(u(t-1)). \tag{103}$$

5 The structural stability and the stability of perturbations of invariant solutions with respect of initial data

Since $W := \Phi_2'$ is structural stable, a set P^+ of points of stable circles has the following property of the local attraction: there is $\delta_1 > 0$ such that for each small enough $\delta > 0$

$$W(O_\delta(P^+)) \subset O_{(1-\delta_1)\delta}(P^+) \tag{104}$$

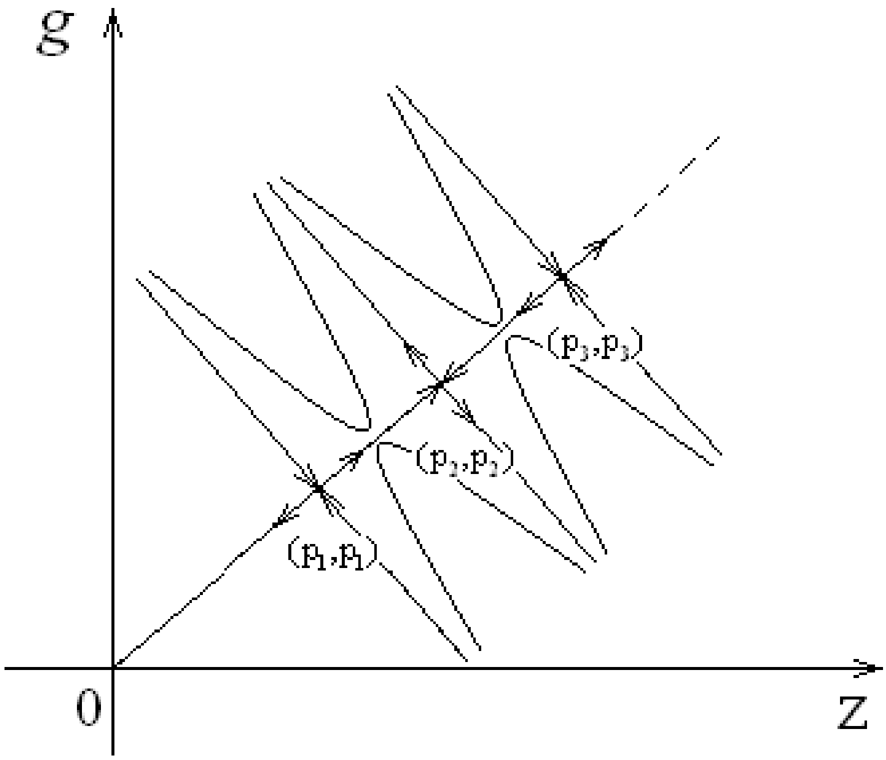

Figure 1: The trajectories of hyperbolic dynamical systems with attractive and saddle points in a plane.

(see, [9],p.242)).

On the other hand, the so called separator $D := \bigcup_{n\geq 0} S^{-n}P^{-}$ of the map S has the similar property of the local repulsion: there is $\mu_1 > 0$ such that for each small enough $\mu > 0$

$$W(I\backslash O_{\mu}(D)) \subset I\backslash O_{(1+\mu_1)\mu}(D). \tag{105}$$

Besides, the map W has the following property: for all $\delta, \mu > 0$ there is $N = N(\delta, \mu) < +\infty$ such that

$$W^{N}(I\backslash O_{\mu}(D)) \subset I\backslash O_{\delta}(P^{+}). \tag{106}$$

The last inclusion means that all points, which do not belong to the μ - neighbourhood of the set D, fall into the δ - neighborhood of the set P^{+} by N iterations. This properties allows to prove asymptotic stability of solutions of the initial boundary value problem with respect to perturbations of the boundary conditions and the initial data in the metric C^2 almost all points. For chaotic solutions the stability can be proved in Schorohod metric [10].

6 Applications to incoherent optical solitons

Below we consider the example of applications the formulated results in the linear optic median with the Kerr type nonlinear boundary conditions. The similar problem has been

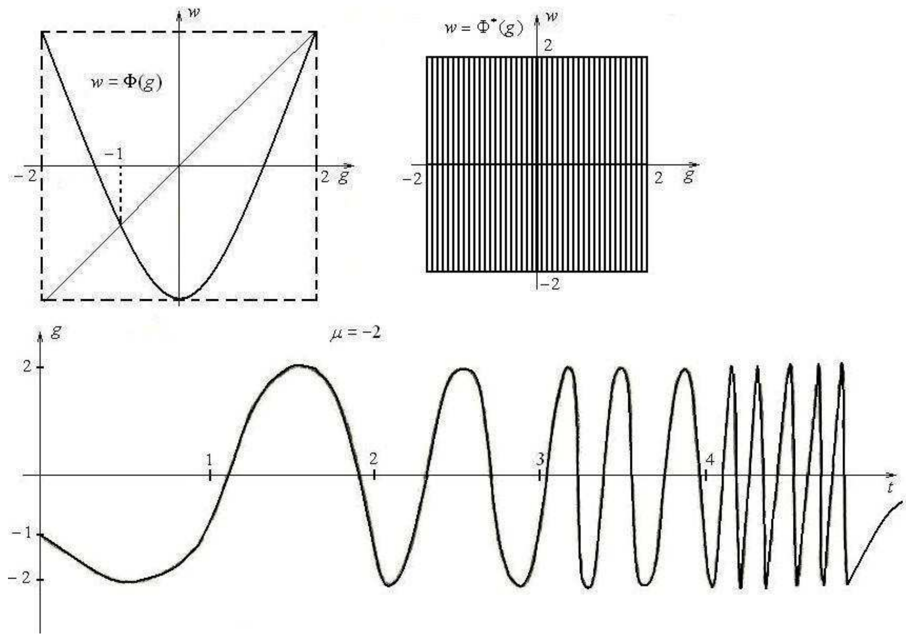

Figure 2: Limit distributions of turbulent type with uncountable points of discontinuities on a period.

considered in [14], where was analyzed "the dynamics of modulation instability and periodic waves in the coupled nonlinear Schrodinger equations describing light propagation in birefringent dispersive Kerr media" [14]. But the difference is in the formulation of the problem because we consider opposite problem: a linear medium with nonlinear boundary conditions. The asymptotic spatial temporal surface-induced oscillations may be interpreted as "white and black" solutions on the language of optic. The phenomenon of the emergence of optical (space) solitons is determined by a dynamic balance between the two competing factors: 1) by the tendentious of the optical beam to expand its own support by the diffraction; 2) by the tendentious of the optical beam to minimise its own support by the self-focusing. The experiments [17] shows the possibility of existence of solitons which are spatially incoherent and quasi-monochromatic; 3) the solitons are incoherent simultaneously on the space and time variables. These experiments subsequently initiated many theoretical works on the noncoherent solitons [16, 17]. However, these papers limited by the case 3) and, hence, the corresponding theory could not model, for example, incoherent white light that is to study spatiotemporal coherence properties of solitons and the further evolution of the spectral density.

In this section, we consider the spatiotemporal (in the two variables) light. We suppose that: 4)the spatial profile of the light belongs to the interval of frequencies $[\omega, \omega + d\omega]$; 5) the spatial correlation length (across the soliton) is greater at low frequencies and smaller at high frequencies.

We begin our exploration with the following equation:

$$i\left(\frac{\partial f^{\omega}}{\partial z} + \theta\frac{\partial f^{\omega}}{\partial x}\right) + \frac{1}{2k_{\omega}}\frac{\partial^2 f^{\omega}}{\partial x^2} + \frac{k_{\omega}}{n_0}\delta n(I) f^{\omega}(x, z, \theta) = 0 \tag{107}$$

where f^{ω} is the coherent density on the given frequency of the optical beam; $k_{\omega} = n_0\omega/c$, where n_0 is the refractive index, ω is the frequency, c is a velocity of the light; the parameter

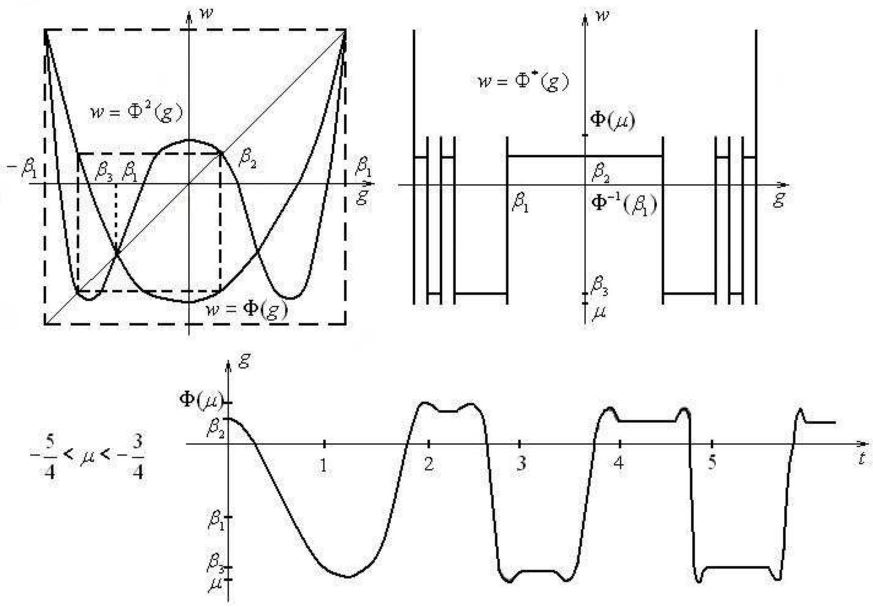

Figure 3: Limit distributions of relaxation type with finite points of discontinuities on a period.

θ determines an angle between vector of propagation of a line (in a plane (z, x)) and the axes Oz. We assume that $\theta = 0$.

Spatial and temporal coherence properties of the beam can be explored in terms of the spectral density so that

$$B_\omega(x_1, x_2, z) = \int_{-\infty}^{+\infty} d\theta \exp\left[ik_\omega(x_1 - x_2)\right] f^\omega(x_1, z, \theta) f^\omega(x_2, z, \theta). \tag{108}$$

Note that equation (107) is equivalent to the corresponding equation for the spectral density (108).

We assume that the optical medium is dispersed. Assuming that $\partial\delta n(I)/\partial t \equiv 0$, the dispersion can be included in the consideration with the help of the dependence $n_0 = n_0(\omega)$. Instead of equation (107) we consider the Schrodinger equation with optical with the optical source. For simplicity we restrict ourselves to one-dimensional rod. The equation has the form

$$ih\left(\frac{\partial f}{\partial t} + \theta\frac{\partial f}{\partial x}\right) + \frac{h^2}{2k}\frac{\partial^2 f}{\partial x^2} + \frac{k}{n_0}\delta n(I) f = 0, \tag{109}$$

where index ω was omitted.

6.1 Solutions of problem

The state of the system is described by the mutual spectral density $B_\omega(x_1, x_2, t)$, where x is the spectral coordinate, t denotes propagation axis coordinate. The mutual spectral density $J_\omega(x, t) = B_\omega(x_1, x_2, t)$. The total intensity is $J(x, t) = \int d\omega B_\omega(x_1, x_2, t)$. In general, the function $B_\omega(x_1, x_2, t)$ is expressed in terms of modes $\psi_{\omega,m}(x, t)$ and their modal weights $d_{\omega,m}$ so that

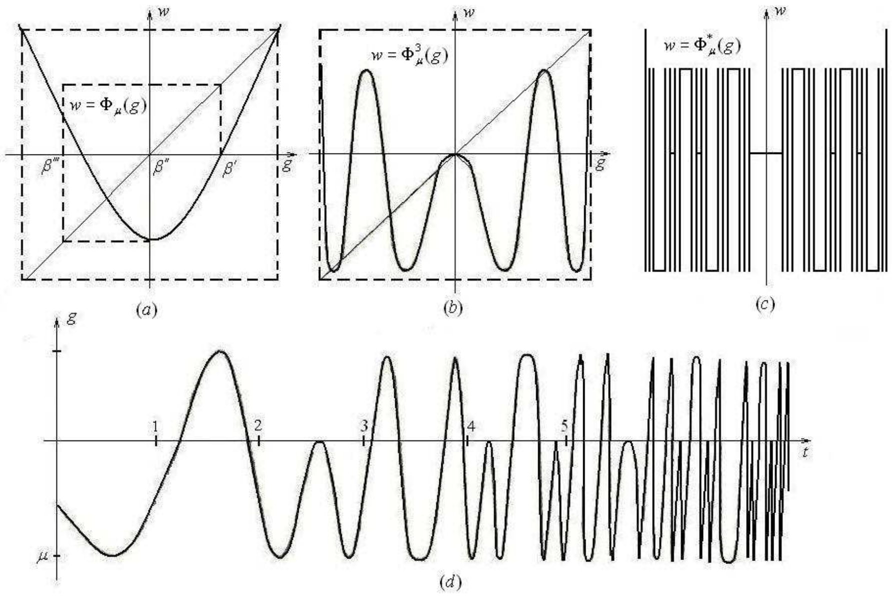

Figure 4: Limit distributions of pre-turbulent type uncountable type with finite points of discontinuities on a period.

$$B_\omega(x_1, x_2, t) = \sum_m d_{\omega,m}\psi_{\omega,m}(x_2, t)^*\psi_{\omega,m}(x_1, t). \tag{110}$$

We consider the evolution only one mode $\psi_{\omega,m}(x, t)$ for fixed values ω and m. As a result, the correlation function $B_\omega(x_1, x_2, t)$ along the t axis can be described by the equation [17]:

$$ih\frac{\partial\varphi}{\partial t} + +\frac{h^2}{2k}h^2\frac{\partial^2\varphi}{\partial x^2} - \frac{k_\omega}{n_0}V\psi = 0. \tag{111}$$

Here, the potential $V(x, t) = \delta n(J_\omega(x, t))$, where $J_\omega(x, t) = \sum_m d_{\omega,m}|\psi_{\omega,m}(x_2, t)|^2$ is the intensity structure at frequency ω. Particulary, for self0focusing media we have $\partial\delta n(J, \omega)/\partial J > 0(see, [17])$.

Let us consider the boundary conditions

$$\psi\psi^*_{|x=0} = F_1(\psi\psi^*)_{|x=l}, \tag{112}$$

$$\psi|\psi|^{-1}_{|x=0} = F_2[\psi|\psi|^{-1}]_{|x=l} \tag{113}$$

where F_1, F_2 are given functions. Here, $|\psi|^2$ is the density of probability in quantum mechanics, and $\psi\psi^*$ is the mouser in classical mechanics. Relation (112) can be written also as

$$\varphi^2_{|x=0} = F_1(\varphi^2)_{|x=l} \tag{114}$$

where $\varphi^2 = |\psi|$. Relation (115) can be written also as

$$e^{iS}_{|x=0} = F_2[e^{iS}]_{|x=l} \tag{115}$$

where $S \to S/h$.

Now, we assume that the map $F_2(t) := e^{iS(0,t)}$ has angle function for each $t \in A \subset R$. This function is determined with accuracy $2\pi k$, $k \in Z$. It is known that continuous map $f : I \to S^1$, where S^1 is a circle, has angle function S such that $S(0) = \alpha$ is given argument of number $f(0)$ (see, [7],p.60). Thus if we assume that $F_2(0) = \alpha$, then $S(l,t)$ is the angle function. Then from the known topological theorem (see, [7],p.60) it follows that

$$F_2(e^{iS}) = e^{iF_2(S)}. \tag{116}$$

From (116), (115) it follows the two-points boundary conditions:

$$S_{|x=0} = F_2(S)_{|x=l}. \tag{117}$$

The boundary conditions have a simple physical meaning. We can use the device which has been considered for the phase observation of the Mandelbrot set [1]. There is the electronic device which models the of the complex map $z \to z^2 + c$, where $z \in C$ and a parameter $c \in C$. The device contains the multiplier N which produce the nonlinear connection between input and output impulse, and the resistor R, which is connected to the amplifier $\Im$, allows to change the some parameter λ. In works [1, 2] also obtained piecewise constant periodic on time impulses, which are produced by the transformation between input and output impulses. Here, the multiplier N transforms, for example, the phase S into the phase $2S$, and amplifier $\Im$ transforms the amplitude φ into the φ^2. As a result, the parameter $c \in C$ is described by the the Mandelbrot set. Of course, these method can be prolonged on the map of interval into itself or the map from R^2 into R^2.

The method, which is developed in [2], reduce the problem to the coupled systems. As a result, we obtain a system of two coupled difference equations. But this method, in our case, can be applied only to the the Schrödinger equation with boundary conditions (6) which in this paper is not considered

As a result, boundary conditions (115) can be written as

$$e^{iS}_{|x=0} = F_1[e^{iS}]_{|x=l} \tag{118}$$

$$ih\frac{\partial \varphi}{\partial t} + +\frac{h^2}{2k}h^2\frac{\partial^2 \varphi}{\partial x^2} - \frac{k}{n_0}\delta n(J)f = 0, \tag{119}$$

Let be $\bar{x} = \sqrt{k}x$. Then equation can be written as

$$-ih\frac{\partial \varphi}{\partial t} + +\frac{h^2}{2}\frac{\partial^2 \varphi}{\partial \zeta^2} - \frac{k}{n_0}\delta n(d\varphi^2)\varphi = 0 \tag{120}$$

where for one mode $J = d\varphi^2$.

The boundary conditions can be written as

$$\frac{\partial \varphi}{\partial t} = G_1[\delta n_1(d_1\varphi^2)]\varphi \quad \text{at} \quad x = 0, \tag{121}$$

$$\frac{\partial \varphi}{\partial t} = G_2[\delta n_2(d_2\varphi^2)]\varphi \quad \text{at} \quad x = \sqrt{k}l, \tag{122}$$

where G_1, G_2 are given functions. The boundary conditions describes injection of photons input (or output) the optical medium. We assume that system of ordinary differential equations is integrable. It means that there is an integral $Q[\varphi(0, \sqrt{k}t), \varphi(\sqrt{k}(l,t)] = \gamma$, where $\gamma \in R$. We assume that this functional equation is globally solvable such that for each z_1, $z_2 \in I$ from the relation $W(z_1, z_2) = 0$ it follows that $z_1 = \Upsilon_\gamma(z_2)$. It means that the boundary conditions can be written as the family of two-points boundary conditions:

$$\varphi(0,t) = \Upsilon_\gamma[\varphi(0, \sqrt{k}t)], \tag{123}$$

where $\nu = \varphi(0,0) - \Upsilon_\gamma[\varphi(0,0)$. Thus a problem is reduced to the problem which has been considered in previous sections.

As a result, the problem can be reduced to the difference equations

$$S(\sqrt{k}l,t) = F_1[S(l,t-\sqrt{k}l)/p], \tag{124}$$

$$\varphi(\sqrt{k}l,t) = \Upsilon_\gamma[\sqrt{k}\varphi(l,t-\sqrt{k}l)/p)], \tag{125}$$

where F_1, $\Upsilon : I \to I$. The limit solution of the first equation is $p_1(\sqrt{k}l,t)$ is $2^{N_1}\sqrt{k}l)/p$ - periodic piecewise constant distribution, and the limit solution of the second equation is $p_2(\sqrt{k}l,t)$ is $2^{N_2}\sqrt{k}l)/p$ - periodic piecewise constant distribution, where N_1, $N_2 \in Z$. Then, as indicated above, $p_1(\sqrt{k}x,t) = p_1(l,t-\sqrt{k}l)/p)$ and $p_2(\sqrt{k}x,t) = p_1(l,t-\sqrt{k}l)/p)$. The limit functions depends also of parameters δn_1, δn_1 and d_1, d_2. There are regions of these parameters, where we obtain limit solutions of relaxation, pre-turbulent and turbulent type.

7 Deterministic chaos in quantum boundary problem

Deterministic chaos is the existence of continuous solutions of difference equation which describes the asymptotic behavior of these equations with help of stochastic processes. More precisely, attractor of solutions of difference equations can be composed from random functions. A special role is played separator of a map f that is the set $D(f)$ such that $y \in I$ and trajectories $f^n(y), n = 0, 1, ...$ are Lyapunov's unstable. Points of the set $D(f)$ divide basins of attraction of attractive cycles of the map f. For points $y_* \in D(f)$ arbitrarily small error $y - y_*$ in the definition of exact value leads to arbitrarily large error between values $f^n(y)$ and $f^n(y^*)$. Therefore it is impossible to determine what values decides at large t. So that

$$\sup_{t \in D_n(f,\varphi)} |x^1_\varphi - x^2_\varphi| \geq L \quad \text{as} \quad n > N_* \tag{126}$$

where φ is an initial function for difference equation and $D_n(f, \varphi) = \{t \in (n, n+1) : \varphi(t-n) \in D_{sen}$. Here $D_{sen}(f) \subset D(f)$ is a set where trajectories are divided.

Indeed, for any point $y_* \in D(f)$ exists a number $d > 0$ such that for any $\varepsilon > 0$ there is a point $\tilde{y} \in y_* - \varepsilon, y_* + \varepsilon) \bigcap I$ and a number m such that

$$|f^m(y) - f^m(\tilde{y})| > d. \tag{127}$$

This situation has principal character if $D(f)$ contains a set $D_{sen}(f)$ of additional meager such that

$$L := \inf y \in D_{sen}(f) d(y) > 0. \tag{128}$$

Then

$$|f^n(y) - f^n(\tilde{y})|_{\tilde{y}:|y-\tilde{y}|<\varepsilon;\, n>N} \geq L \quad \text{as} \quad \varepsilon > 0,\ N > 0. \tag{129}$$

As a result, arbitrarily small errors in the quantities of the order L leads to the fact that the solution is on the horizon of predictability. Such solutions are similar to random process. Consequently, approximate solutions $x_{\varphi_1}(t)$ and $x_{\varphi_2}(t)$ can be considered as analog of realizations of random process. The set of initial functions, producing such solutions, is open subset in C^0 metric.

7.1 Example

Let us consider the map $\tilde{h}$

$$x(t+1) = 1 - 2|x(t) - 1/2| \tag{130}$$

that is the "tent"- map. This map is stretching so that

$$|h(y') - h(y'')| = 2|y' - y''| \tag{131}$$

as $y', y'' \in [0, 1/2] \bigcup [1/2, 1]$. As a result, $D(\tilde{h}) = [0, 1]$ and each interval $J \subset [0, 1]]$ of the length ε, after $n > \lg 1/\varepsilon / \lg 2$ iterations, contains $[0, 1]$ so that $\tilde{h}^n(J) = [0, 1]$. Then for the initial functions $\varphi(t)$

$$\sup |x_\varphi(t+n) - x_\varphi(\tilde{t}+n)| = 1 \tag{132}$$

as $n > \lg 1/\varepsilon / \lg 2$. It means that with an accuracy $\varepsilon = 10^{-20}$ of computer simulation we can not say what is a value of a solution if $t > t_0$, where $t_0 = 20/\lg 2 \approx 70$. However, we can determine the probability of whether the solution falls to a specified interval $[a, b]$.

Let $h = \chi/l$, where χ is the length of wave of particle. Here $\chi = \hbar/\sqrt{2mE}$, where m is the mass, E is

surface energy of interactions between the two particles. Let $L = 1$ and $h = \varepsilon$. Then as $t > \lg 1/h/\lg 2$, trajectories of particle are "non-observable" and we enter in the regions of "quantum mechanics".

8 Conclusion

Thus the initial boundary value problem for the Schrödinger equation with two-points nonlinear boundary conditions has been considered. The problem describes the formation of surface-induced wave structure in confined resonators with feedback which have the form of asymptotic periodic piecewise constant impulses with finite or infinite points of discontinuities on a period. It is shown that the problem can be reduced to the integro-difference equation which can be reduced to the family of integrable differential-difference equation. The solution of these equations are solutions of relaxation, pre-turbulent and turbulent type.

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