Cantor Set and Fractals

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8 June 2019
Abstract

In his studies of abstract mathematics, Georg Cantor revolutionized the field of number theory, introducing innovative ideas in set theory and the idea of infinities. In a footnote on one of his papers, he introduced the Cantor Ternary Set, a now well-known set defined by the following mathematical construction:

\[ \mathcal{C} = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^n-1} \left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right) \]

Extrapolating the construction, the set can also be written as

\[ \mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^n-1} \left( \left[ \frac{3k+0}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[ \frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right) \]

0.1 Overview

The set can be generalized with different base ratios and different distributions during the “splitting” process of the set’s construction, with the generalizations often referred to as Cantor Sets. In this article, we present elementary proofs that the sets possesses the cardinality of the continuum, have a Lebesgue measure of zero, are nowhere dense, and that any number in [0, 2] can be produced by summing two members of the regular Cantor Sets.
1. Construction

The Cantor Ternary Set is formed by taking the interval $[0, 1]$ and progressively removing sections of the interval. Particularly, on the first step, the middle third $(\frac{1}{3}, \frac{2}{3})$ of the interval is removed. Then, on the next step, of each of the 2 remaining intervals, the middle thirds are removed. On the third step, of the remaining 4 sub-intervals, each of the middle thirds are removed, et cetera.

We can numerically determine an $n^{th}$ step generalization for the Cantor Set using its mathematical definition. On the first step, we remove the interval $(\frac{1}{3}, \frac{2}{3})$. On the second step, we remove the intervals $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$. Generalizing, in the $n^{th}$ step we remove

$$\left(\frac{1}{3^n+1}, \frac{2}{3^n}\right) \cup \left(\frac{4}{3^n+1}, \frac{5}{3^n}\right) \cup \cdots \cup \left(\frac{3^n-2}{3^n}, \frac{3^n-1}{3^n}\right)$$

What remains of the interval is the Cantor Ternary Set. The first 6 iterations of the construction of Cantor Ternary Set can be visualized as:

![Diagram of the Cantor Set]

We can define $C_n$ to be what remains of the interval $[0, 1]$ after $n$ iterations. The Cantor Ternary Set is the intersection of all of what remains:
\[ C = \bigcap_{n=d}^{\infty} C_n \]
2. Ternary Representations

**Theorem 2.0.1.** Let $c \in \mathbb{C}$. $c$ can be expressed as $0.x_1x_2x_3x_4... \forall n \in \mathbb{N}, \ x_n = 0, 2$

*Proof.* We will prove this using induction. Let $\mathcal{C}_n$ be the Cantor Set after the $n^{th}$ removal. We will show that the digits of each element in $\mathcal{C}_n$ are either 0 or 2.

Base Case: $n = 1$

The first division of the Cantor Set removes the interval $(\frac{1}{3}, \frac{2}{3})$, which can express this interval as follows (in base 3):

$$\left(\frac{1}{3}, \frac{2}{3}\right) = \{0.1x_1x_2x_3x_4..., x_n \in \{0, 2\}\}$$

Therefore, in order to perform the first division of the Cantor Set, we can simply express $[0, 1]$ in ternary and only include numbers of the form

$$[0, \frac{1}{3}] = 0.0x_1x_2x_3x_4...x_n \ \forall n \in \mathbb{N}, \ x \in \mathbb{Z}^*$$

$$[\frac{2}{3}, 1] = 0.2x_1x_2x_3x_4...x_n \ \forall n \in \mathbb{N}, \ x \in \mathbb{Z}^*$$

Note that $\frac{1}{3}$ can still be expressed as $0.0\overline{2}$. Hence we have proved that the 1st digit of each element in $\mathcal{C}_1$ is either a 0 or a 2.

Inductive Hypothesis: Given that the first $n$ digits in $\mathcal{C}_n$ are 0 or 2, the first $n + 1$ digits of $\mathcal{C}_{n+1}$ are 0 or 2.

Consider any interval $(\frac{a}{3^n}, \frac{a+1}{3^n})$ of $\mathcal{C}_n$ where $n \in \mathbb{N}$ and $0 \leq a \leq 3^n - 1$.

$$\frac{a}{3^n} = 0.a_1a_2a_3...a_n, n \in \mathbb{N}, a_n \in 0, 1, 2$$
The one-third mark of this interval can be expressed as
\[
\frac{3a + 1}{3^{n+1}} = 0.a_1a_2a_3...a_n0222..., n \in \mathbb{N}, a_n \in 0, 1, 2
\]

The two-thirds mark can be expressed as
\[
\frac{3a + 2}{3^{n+1}} = 0.a_1a_2a_3...a_n2, n \in \mathbb{N}, a_n \in 0, 1, 2
\]

In the \( n + 1 \) iteration, we remove the interval \((\frac{3a+1}{3^{n+1}}, \frac{3a+2}{3^{n+1}})\).

Therefore, we keep the intervals \([\frac{3a}{3^{n+1}}, \frac{3a+1}{3^{n+1}}]\) and \([\frac{3a+2}{3^{n+1}}, \frac{3a+3}{3^{n+1}}]\).

The first interval contains all elements in \((\frac{a}{3^n}, \frac{a+1}{3^n})\), that are in the following form

\[
c = 0.a_1a_2a_3...a_n0a_{n+1}a_{n+2}a_{n+3}..., n \in \mathbb{N}, a_n \in 0, 1, 2
\]

. Hence the first \( n \) digits of \( c \) are 0 or 2 and the \( n + 1 \)st digit is 0.

The second interval contains all elements in \((\frac{a}{3^n}, \frac{a+1}{3^n})\), that are in the following form

\[
c = 0.a_1a_2a_3...a_n2a_{n+1}a_{n+2}a_{n+3}..., n \in \mathbb{N}, a_n \in 0, 1, 2
\]

. Hence the first \( n \) digits of \( c \) are 0 or 2 and the \( n + 1 \)st digit is 2, and we have proven our inductive hypothesis.

Therefore the first \( n + 1 \) digits of \( C_{n+1} \) are 0 or 2. Taking the limiting set as \( n \) approaches \( \infty \), we get that all the digits of the elements of the Cantor Set are 0 or 2, proving our theorem.
3. Uncountability

Theorem 3.0.1. The Cantor Ternary set has a cardinality of the continuum (the cardinality of \([0, 1]\) and of \(\mathbb{R}\)).

Proof. By Theorem 2.0.1, the Cantor Ternary Set is the set of all real numbers in \([0, 1]\) that can be expressed as a ternary number not containing the digit “1”. Thus, there are two options for each digit, and there can be (countably) infinitely many digits (any real number in \([0, 1]\) can be written as a number with a ternary expansion beginning with “0”: “0.\(a_1a_2a_3a_4\ldots\)”).

\[ |C| = |\{0, 2\}^\mathbb{N}| = 2^{\mathbb{N}} \]

which is the same as \(|\{0, 1\}^\mathbb{N}|\), or the cardinality of all binary numbers. Notice this is also the cardinality of all numbers that can be written in binary as \(0.a_1a_2a_3a_4\ldots\), where each \(a_i \in \{0, 1\}\), which is the cardinality of \([0, 1]\). 

\(\square\)
4. Zero Measure

Lemma 4.0.1. At iteration $N$ of the Cantor Set, we have removed a total length of $\sum_{n=1}^{N} \frac{2^{n-1}}{3^n}$.

Proof. During the $n^{th}$ step, we remove $\frac{1}{3}$ of each of the $2^{n-1}$ intervals, whose length is $3^{n-1}$. Hence we remove a total length of $\frac{2^{n-1}}{3^n}$ during the $n^{th}$ step. Since these removed sections have not already been removed and don’t overlap, in total, we have removed $\sum_{n=1}^{N} \frac{2^{n-1}}{3^n}$ after $N$ iterations. \qed

Theorem 4.0.2. The Cantor Set has 0 length under the Lebesgue Measure.

Proof. We know that the Lebesgue Measure of $[0, 1]$ is $1$. By Lemma 4.0.1, we know that after the $N^{th}$ iteration, we have removed a total length of $\sum_{n=1}^{N} \frac{2^{n-1}}{3^n}$. As $N \to \infty$, this value converges to 1. For any $\epsilon > 0$, we can choose an $N$ large enough such that $\sum_{n=1}^{N} \frac{2^{n-1}}{3^n} > 1 - \epsilon$. Let $I$ be the union of the intervals that correspond to the aforementioned sum. Thus, in the universe of $[0, 1]$, $C = I^c$. We know that the $m(C) < \epsilon$. Because we can choose $\epsilon$ to be arbitrarily small, we determine that $m(C) = 0$. \qed
5. Loose Order (Nowhere Dense)

A nowhere dense set (referred to in the past as a set in loose order) is a set in which there are no contained intervals (continuous stretches of points).

**Lemma 5.0.1.** In base $b > 1$, the “decimal” representation $0.ddddd...$ of a real number is unique, where $d$ is a digit not equal to $b - 1$.

Update: all non-terminating real numbers in a given base $b$ that do not end with a repeating digit equal to $b - 1$ have a unique expanded representation.

**Proof.** Let $n = 0.(ddd...)_b$. For the sake of contradiction, we will assume that it has another base-$b$ representation, $n = a = 0.a_1a_2a_3...$. Let $a_x$ be the first digit $a$ that is not equal to $d$. We have two cases $a_x > d$ and $a_x < d$, for which we will prove that there is no alternate representation separately.

Case 1: $a_x > d$

Let’s create a new number

$$m = 0.d_1d_2d_3...d_{x-1}a_x \leq 0.d_1d_2d_3...d_{x-1}a_xa_{x+1}... = a,$$

where each $d_i = d$. Now let’s create a new number

$$j = 0.d_1d_2d_3...d_{x-1}f, \text{ where } f = d + 1.$$

We know $j \leq m$ since $a_x > d$. We can rewrite $j$ as $0.d_1d_2d_3...d_zwww...$, where $w = b - 1$. Now we subtract $n$ from $j$, and we get $0.0_10_20_3...0_xkkkkk...$, where $k = w - d > 0$. This difference is positive. Hence, we get $n < j < m < a = n$, which is a contradiction. Hence, the number $n = 0.(ddd...)_b$ has a unique base $b$ representation.
Case 2: $a_x < d$

Let’s create a new number

$$m = 0.d_1d_2d_3 \ldots d_{x-1}a_x \leq 0.d_1d_2d_3 \ldots d_{x-1}a_xa_{x+1} \ldots = a,$$

where each $d_i = d$. Now let’s create a new number

$$j = 0.d_1d_2d_3 \ldots d_{x-1}f,$$

where $f = d + 1$.

We know $j \geq m$ since $a_x < d$. We can rewrite $j$ as $0.d_1d_2d_3 \ldots d_xwww\ldots$, where $w = b-1$. Now we subtract $n$ from $j$, and we get $0.0_10_20_3\ldots 0_3k\ldots$, where $k = w - d < 0$. This difference is positive. Hence, we get $n > j > m > a = n$, which is a contradiction. Hence, the number $n = 0.(ddd\ldots)_b$ has a unique base $b$ representation.

Therefore we have proved from both cases that the number $n = 0.(ddd\ldots)_b$ has a unique base $b$ representation.

\[\square\]

**Corollary 5.0.1.1.** In base $b > 1$, “decimal” (expanded) representations are unique for all numbers “ending” in a repeating digit, as long as the repeating digit is not $b-1$ or 0.

**Proof.** Take a number $x$ in base $b$ that begins to repeat with digit $d$ after $n$ digits succeeding the base point.

$$x = d_1d_2d_3d_4d_5\ldots d_k.d_{k+1}d_{k+2}d_{k+3}\ldots d_{k+n}ddd\ldots$$

$$x \cdot b^n = d_1d_2d_3d_4d_5\ldots d_{k+n}.ddd\ldots$$

$$= d_1d_2d_3d_4d_5\ldots d_{k+n} + 0.dds\ldots$$

Since the $0.dds\ldots$ has a fixed representation and the RHS is an integer (either $n.000\ldots$ or $(n-1)(b-1)(b-1)(b-1)\ldots$) plus that quantity, the only way that the LHS can have multiple representations is if both $0.dds\ldots$ and $0.ee\ldots$, $e = d + (b-1)$ are valid numbers. In any base, the only digits that differ by $(b-1)$ are “0” and “$b-1$”, which are exactly $b-1$ and 0.

\[\square\]

**Theorem 5.0.2.** $C$ is nowhere dense.
Proof. Assume that $\mathcal{C}$ is not nowhere dense. By our assumption, $\mathcal{C}$ must contain some open interval as a subset. Suppose $\mathcal{C}$ does contain an interval $(a, b)$, with $a \neq b$. Because $(a, b)$ is a continuous interval contained within $\mathcal{C}$, there is some real number $x$, member of the Cantor Ternary Set, s.t. $a < x < b$. Write $x$ in base 3.

$$x = 0.d_1d_2d_3\ldots, d_i \in \{0, 2\}$$

Because $x \in \mathcal{C}$, each $d_i \in \{0, 2\}$ (by Theorem 2.0.1). Let $\delta$ be the lesser of $(x - a)$ and $(b - x)$. Since both $(x - a)$ and $(b - x)$ are positive, by the Archimedean Property, we can choose a $k$ such that the decreasing towards 0 function $3^{-k} < \delta$. Now we will construct a new number $y = 0.d_1d_2\ldots d_k1111\ldots$ As the it has the same first $k$ digits after the ternary point as $x$, it is impossible for this new number $y$ to differ from $x$ by more than (or equal to) $3^{-k}$; $|x - y| < 3^{-k} < \delta$. This means that $y$ must be in $(a, b)$. Notice that $y$, when written in ternary, “ends” in the infinitely repeating digit “1”. By Lemma 5.0.1, Corollary 5.0.1.1, there is no way to rewrite $y$ as a ternary number without the digit “1” (or alternately whatsoever). We know, by Theorem 2.0.1, that all members of $\mathcal{C}$ can be written in ternary without the digit “1”. This means that $y$ cannot be in the Cantor Ternary Set $\mathcal{C}$. $(a, b)$ was chosen to be $\subset \mathcal{C}$, so $y \subset (a, b)$ and $y \not\subset \mathcal{C}$ is a contradiction (by transitivity), so the Cantor Ternary Set cannot contain an interval $(a, b)$, with $a \neq b$. Thus $\mathcal{C}$ is nowhere dense. \qed
6. $\mathcal{C} + \mathcal{C} = [0, 2]$

The Minkowski Sum $A + B$ of two sets $A$ and $B$ is := \{a \in A + b \in B\}. Despite having 0 length and being nowhere dense, the Cantor Set summed with itself yields the entire interval $[0, 2]$.

**Theorem 6.0.1.** $\mathcal{C} + \mathcal{C} = [0, 2]$

**Proof.** By definition, $\mathcal{C} + \mathcal{C} = \{a + b \mid (a, b) \in \mathcal{C}^2\}$. $\mathcal{C} + \mathcal{C}$ being equal to $[0, 2]$ is equivalent to $\forall d \in [0, 2] \exists (a, b) \in \mathcal{C}^2$ s.t. $a + b = d$.

The proof is now reduced to showing that every real in $[0, 2]$ can be formed by summing two elements of the Cantor Set. This is the same as showing that every real in $[0, 1]$ can be produced by summing two elements of the Cantor Set and then dividing by 2 i.e. every real in $[0, 1]$ is the arithmetic mean of two elements of the Cantor Set.

By Lemma 2.0.1, the Cantor Set contains all numbers in $[0, 1]$, which when written in ternary do not contain the digit “1” (ternary numbers in $[0, 1]$ that can be expressed with only the digits “0” and “2”). Meanwhile, the entire interval $[0, 1]$ is the set of all ternary numbers in $[0, 1]$ that can be expressed with the digits “0”, “1”, and “2”.

Choose any number $d$ in $[0, 1]$, and write it in ternary as $0.d_1d_2d_3\ldots$, where $d_n$ are ternary digits, either “0”, “1”, or “2”. We can create two numbers $a = 0.a_1a_2a_3\ldots$ and $b = 0.b_1b_2b_3\ldots$ both members of the Cantor Set such that when summed and divided by two, equal $d$: For each $d_n$,

1) If $d_n$ is 0, we choose $a_n$ and $b_n$ to both be 0.
2) If $d_n$ is 1, we choose $a_n$ to be 0 and $b_n$ to be 2.
3) If $d_n$ is 2, we choose $a_n$ and $b_n$ to both be 2.
Following these rules, the average of $a$ and $b$ will be equal to $d$. We now have an algorithm to take a real in $[0, 1]$ and generate two numbers, elements of the Cantor Set, which when summed and divided by two equal that real. This means that there is a way to sum two numbers in the Cantor Set and arrive at any real in $[0, 2]$. ☐
7. Fat Cantor Set

At the n-th iteration of the Fat Cantor set, one removes an interval of Lebesgue measure $a^n$ from each of the $2^{n-1}$ intervals. In total, we remove a length equal to the following sum. $\sum_{n=1}^{\infty} a^n 2^{n-1}$, which sums to $\frac{a}{1-2a}$. Therefore the Lebesgue measure of the items that remain is $1 - \frac{a}{1-2a} = \frac{1-3a}{1-2a}$. This quantity is positive as long as $a < \frac{1}{3}$. Additionally, this set is nowhere dense because it contains no intervals. The proof is the same as for a regular Cantor Set. Hence, we have shown that there exists a set with a positive Lebesgue measure that also has loose order properties.
8. Fractal Geometry of 
*n*-dimensional Analogs of 
Cantor Set

Hausdorff dimension is a measure of roughness or chaos within a fractal, a self similar entity. It is measured by the formula \( N = r^D \). \( N \) is the scale factor by which the measure of an object (length for 1D, area for 2D, and volume for 3D) would increase if its linear size was increased by a factor of \( r \) and its dimension is \( D \). We can rewrite this formula to solve for \( D \), \( D = \log_r N \).

If we carefully observe the Cantor Set, we can see that it is composed of 2 smaller sets that are \( \frac{1}{3} \) of the size of the original set. Hence, its \( r \) value corresponds to 3, and its \( N \) value corresponds to 2. As such, its Hausdorff dimension is \( \log_3 2 \approx 0.63 \).

On the left, we present the 2-dimensional analog of the Cantor Set. It sides are the Cantor Set, and it is formed in the same way as its 1-D counterpart. If we carefully observe the Cantor Dust, we can see that it is composed of 4 similar, smaller parts that are \( \frac{1}{3} \) of the original figure. Therefore its dimensionality is \( \log_3 4 \approx 1.26 \).

We can analogously extend this to the \( n \)-th dimension. At each of the vertices of the \( n \)-th dimensional Cantor Dust, there exists another smaller Cantor Dust that is self-similar to the Cantor Dust. Hence, there are \( 2^n \) copies of the original solid, whose side lengths are \( \frac{1}{3} \) of the original solid. Therefore its fractal dimension is \( \frac{\log 2^n}{\log 3} \approx 0.63 \times n \).
9. Applications

A dynamic system describes the time dependence of a point in geometric space. While many different situations can be modeled with dynamic systems, population growth is an often-used example. Consider the plot below.

In this figure, the x-axis represents a population over time and the y-axis represents the "growth potential." In the real world, as the population grows, its potential for greater reproduction increases as well. Thus, the potential for growth increases with the population, and vice versa.

However, because every environment has a carrying capacity, a species’ population growth is ultimately limited. In this case, the box represents the area for sustainable population growth. When the population increases too much, the growth potential grows to be unsustainable, eventually causing the population to fall again.

Dynamic systems can often be modeled by iterating functions over many compositions. In this case, taking the composition of this population graph
many times simulates the regular rise-and-fall of most population systems over time. As it turns out, the Cantor Set can be used to model these basic dynamic systems and offer insight to their unpredictable behavior.

Consider Figure 9.1a above, a revised version of the original figure. The dark shaded lines represent a starting population that immediately corresponds with unsustainable growth potential. Note than removing this interval corresponds with the first "cut" of the Cantor Set.
Understanding this population to be a dynamic system that can be modeled by a series of compositions, we see that the dotted lines in Figure 9.1b which enclose specific "growth potentials" as the output of one function create an unsustainable input (shown in Figure 9.1b).

As seen in Figure 9.1c, the two side intervals enclosed by darkened lines return an output within the dotted lines, which return an unsustainable growth potential after one composition. Thus, any starting population in these intervals will eventually lead to an unsustainable growth potential. Removing these intervals corresponds with the second "cut" of the Cantor Set.

Using a similar pattern of thought, any starting population within the four new enclosed intervals in Figure 9.1d will also eventually lead to an unsustainable growth potential. Removing these intervals corresponds with the third "cut" of the Cantor Set. As we continuing "cutting" out intervals that, over many compositions, will lead to an unsustainable growth potential, we are left with $C$, the Cantor Set.

The presence of the Cantor Set in dynamic systems illustrates their volatility. Because $C$ has length of 0, there does not exist a single interval of values that provide a sustainable starting point for population growth. Instead, the only possible successful starting populations are in the Cantor Dust. If the starting population begins anywhere outside this set of zero length, it will eventually lead to unsustainable growth. As such, the Cantor Set model provides a fascinating look into one of the most basic tenets of dynamic systems: their tendency toward chaos.
10. Bibliography

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