THE ROCKERS FUNCTION

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Abstract. In this note we introduce and study the rockers function $\lambda(n)$ on the natural numbers. We establish an asymptotic for the rockers function on the integers and exploit some applications. In particular we show that

$$\lambda(n) \sim \frac{n^{n-\frac{1}{2n}-\frac{1}{2}}\sqrt{2\pi}}{e^{n+\Psi(n)-1}}$$

where

$$\Psi(n) := \int_{1}^{n-1} \frac{\sum_{1\leq j\leq t} \log(n-j)}{(t+1)^2} dt.$$

1. Introduction

The factorial function and the Gamma function by extension is an incredibly useful function. It shows up a lot in most theoretical scenarios and as well in many applications. It is defined in a natural way as

$$\Gamma(s+1) = s\Gamma(s)$$

for any $s \in \mathbb{R}$, where

$$\Gamma(s) := \int_{0}^{\infty} e^{-t}t^{s-1}dt$$

is valid in the entire complex plane except at $s = 0, -1, -2, \ldots$ with simple poles [1]. On the integers, this function is the well-known factorial function given by $\Gamma(s+1) := s! = s(s-1)\cdots 2.1$. Sterling's formula for the factorial function is endowed with important constants such as π and e given by (See [1])

$$n! \sim \frac{n^n\sqrt{2\pi n}}{e^n}$$

that makes it very rife and indispensable modeling tool. In this note, with the goal of illustrating a real-life situation, we introduce and study the rockers function closely suitable for our model. We first establish an asymptotic in the following result

Theorem 1.1. *For all $n \in \mathbb{N}$, the rockers function satisfies the following asymptotic relation*

$$\lambda(n) \sim \frac{n^{n-\frac{1}{2n}-\frac{1}{2}}\sqrt{2\pi}}{e^{n+\Psi(n)-1}}$$

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where

$$\Psi(n) := \int\limits_{1}^{n-1} \frac{\sum\limits_{1\leq j\leq t} \log(n-j)}{(t+1)^2} dt.$$

2. The rockers function

Definition 2.1. By the rockers function, we mean the function

$$\lambda : \mathbb{N} \longrightarrow \mathbb{R}^+$$

such that

$$\lambda(n) = 2^{\frac{n-2}{n-1}} 3^{\frac{n-3}{n-2}} \cdots (n-1)^{\frac{1}{2}} n.$$

We call $\mathcal{F}_k(\lambda(n)) = (n-k)$ for $k \geq 1$ the $(k+1)$ th factor of the iteration with index $\frac{k}{k+1}$. We denote the index of each factor $(n-k)$ by $\mathrm{Ind}(n-k) = \frac{k}{k+1}$.

The rockers function on the natural numbers is very much akin to the factorial function, except that each factor produced by an iteration is scaled down dramatically. This accounts in part for the relative slow growth rate when compared to the factorial function.

Theorem 2.2. *For all $n \in \mathbb{N}$ with $n \geq 1$, the rockers function satisfies the following asymptotic relation*

$$\lambda(n) \sim \frac{n^{n-\frac{1}{2n}-\frac{1}{2}}\sqrt{2\pi}}{e^{n+\Psi(n)-1}}$$

where

$$\Psi(n) := \int\limits_{1}^{n-1} \frac{\sum\limits_{1\leq j\leq t} \log(n-j)}{(t+1)^2} dt.$$

Proof. We observe that the rockers function can be written in the form

$$\lambda(n) = \frac{n!}{\prod\limits_{j=1}^{n-1} (n-j)^{\frac{1}{j+1}}}.$$

By taking the natural logarithm on both sides of the expression, we have

$$\log \lambda(n) = \log n! - \sum_{j=1}^{n-1} \frac{1}{j+1} \log(n-j).$$

By an application of partial summation to the second expression on the right, we have the following

$$\sum_{j=1}^{n-1} \frac{1}{j+1} \log(n-j) = \frac{1}{n} \sum_{j=1}^{n-1} \log(n-j) + \int\limits_{1}^{n-1} \frac{\sum\limits_{j=1}^{t} \log(n-j)}{(t+1)^2} dt.$$

It follows that

$$\log \lambda(n) = \log n! - \frac{1}{n}\sum_{j=1}^{n-1}\log(n-j) - \int_{1}^{n-1}\frac{\sum_{j=1}^{t}\log(n-j)}{(t+1)^2}dt.$$

The result follows from this relation. □

2.1. **Properties of the rockers function.**

In this section we examine some properties and subtle features of the rockers function.

Proposition 2.1. *The following relation hold for the rockers function*

$$\prod_{k=1}^{n-2}\mathrm{Ind}(\mathcal{F}_k(\lambda(n))) = \frac{1}{n-1}$$

for all $n \in \mathbb{N}$ *with* $n \geq 3$.

Proof. The relation is easily established by noting that

$$\prod_{k=1}^{n-2}\mathrm{Ind}(\mathcal{F}_k(\lambda(n))) = \prod_{j=1}^{n-2}\left(\frac{j}{j+1}\right)$$
$$= \frac{1}{n-1}$$

by a simple induction argument. □

It follows from this relation we can somehow relate the arguments of the rockers function to their indices in the following way

$$\frac{1}{\prod_{k=1}^{n-2}\mathrm{Ind}(\mathcal{F}_k(\lambda(n)))} + 1 = n.$$

3. **Distribution of the rockers function on the integers**

In this section we examine the distribution of the rockers function $\lambda(n)$ for all $n \in \mathbb{N}$. We examine the distribution for the first twelve values of the integers, in the following tables.

Table 1

n	1	2	3	4	5	6	7	8	9
$\lambda(n)$	1	2	4.243	10.998	34.983	134.176	608.491	3205.596	19322.113

Table 2

n	10	11	12
$\lambda(n)$	131557.4713	1000838.66	8428867.597

It can be Inferred from the above table that the growth rate of the rockers function is not as dramatic and superfluous compared to the factorial function. Though it does exhibit some dramatic growth, we can in most cases get control of the growth rate and can use it to model several real life phenomena which we will lay down in the sequel.

4. **Inequalities involving the rockers function**

Proposition 4.1. *The rockers function $\lambda(n)$ satisfies the following inequality*

$$\frac{n^{n-\log n}\sqrt{n}\sqrt{2\pi}}{e^n} \leq \lambda(n) \leq \frac{n^n\sqrt{n}\sqrt{2\pi}}{2^{\log n}e^n}$$

for sufficiently large values of n.

Proof. We observe that the rockers function can be written in the form

$$\lambda(n) = \frac{n!}{\prod\limits_{j=1}^{n-1}(n-j)^{\frac{1}{j+1}}}.$$

The desired inequality follows from the inequality

$$2^{\log n} \leq \prod_{j=1}^{n-1}(n-j)^{\frac{1}{j+1}} \leq (n)^{\log n}$$

and applying Sterling's formula. □

5. **Applications**

A criminal from a crime scene, subject to a final arrest, has some amount of momentary maneuvers before his final apprehension. At the crime scene, he has

(i) n for $n \geq 3$ possible routes of momentary escape.

(ii) For each of the n routes of escape he resolves to take, he has

$$\lfloor (n-1)^{\frac{1}{2}} \rfloor$$

possible routes to take to outwit an arrest.

(iii) Again for each of the possible $\lfloor (n-1)^{\frac{1}{2}} \rfloor$ routes he has at his disposal, he has

$$\lfloor (n-2)^{\frac{2}{3}} \rfloor$$

routes of momentary escape.

(iv) In general, on the $(k+1)$ th route from the crime scene, he has

$$\lfloor (n-k)^{\frac{k}{k+1}} \rfloor$$

possible routes of momentary escape. If we denote the total number of possible routes of escapes from the crime scene with n possible routes of escapes before final arrest by $\mathcal{A}(n)$, then

$$\begin{aligned}
\mathcal{A}(n) &= \lfloor 2^{\frac{n-2}{n-1}} \rfloor \lfloor 3^{\frac{n-3}{n-2}} \rfloor \cdots \lfloor (n-1)^{\frac{1}{2}} \rfloor n \\
&\sim 2^{\frac{n-2}{n-1}} 3^{\frac{n-3}{n-2}} \cdots (n-1)^{\frac{1}{2}} n \\
&\sim \frac{n^{n-\frac{1}{2n}-\frac{1}{2}}\sqrt{2\pi}}{e^{n+\Psi(n)-1}}
\end{aligned}$$

where

$$\Psi(n) := \int_{1}^{n-1} \frac{\sum_{1\leq j\leq t} \log(n-j)}{(t+1)^2} dt.$$

[1].

References

1. Sebah, Pascal and Gourdon, Xavier, *Introduction to the gamma function*, American Journal of Scientific Research, 2002.

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