On the distance between the incenter and the circumcenter of a triangle

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Abstract: In this paper, using mainly the distance in the title, we present an alternative proof of Feuerbach’s theorem and some remarks.

K. W. Feuerbach proved on 1822 that the nine point circle of a triangle inscribed to the incircle and circumscribed to the excircles of a triangle. There are many proofs on Feuerbach’s theorem as Pythagoras’s theorem and Gauss’ law of quadratic reciprocity.

We shall confine our attention to acute triangles and incircles because our theorem can be proved in the same manner about other cases. This theorem is trivial for the equilateral triangle. We remove here this triangle. Notation \( C(U, V) \) means the circle with the center \( U \) and radius \( V \).

**Theorem** [Feuerbach’s Theorem]. The nine-point circle \( \mathcal{N} \) of a triangle tangents to each of the incircle and excircles of the triangle.

**Proof.** We set the circumcircle \( \mathcal{O} := C(O, R) \), incircle \( \mathcal{I} := C(I, r) \) and

![Figure (by Geogebra)](https://example.com/figure.png)
the orthocenter $H$ of $\triangle ABC$, respectively. Moreover, let $N$ and $A'$ be the midpoints of $OH$ and $BC$, respectively. The point $J$ is of $JN = NI$ on the line $NI$ and the other side from $I$. Points $K$ and $L$ are the midpoints of the segments $JO$ and $IH$, respectively.

Since $N$ is the midpoint of both lines $OH$ and $JI$, the quadrilateral $JOIH$ is a parallelogram. From this and two midpoints theorem, the segments $OI, KL$ and $JH$ are parallel and $N$ is on the line $KL$.

Let the line $\ell$ be perpendicular to the line $JI$ at $N$ and let $S$ (over $N$) and $T$ be the intersection points of $\ell$ and $\beta := \mathcal{C}(N, KN)$. Then $SN = NT = KN = OI/2$. The point $F$ is the intersection point of the nine point circle $N = \mathcal{C}(N, NA' = R/2)$ and the line $NI$ on $I$ side.

Let $\alpha$ be the circumcircle of $\triangle SJF$ and let $M$ be the intersection point of line $JF$ and the perpendicular bisector of $SJ$. Then the point $M$ is of $F$ side on the line $JF$ by the acute angle $\angle SJN$. Since $JF$ is the perpendicular bisector of $ST$ at the center $N$ of $\beta$, $\triangle SJT$ has the circumcircle $\alpha = \mathcal{C}(M, JM)$ by $JM = SM = TM$. Moreover $\triangle SJF$ is of right and three circles $N, \beta, \alpha$ are at position of line $JF$ symmetry.

$\alpha$ is inscribed at $F$ to $N$ by $MN = JM - JN$ from $JN = NI < NI + IF/2 = JF/2 = JM$. we obtain $IN = R/2 - r$ from the next

$$(R/2) \cdot NI = NF \cdot JN = SN^2 = KN^2 = (IO/2)^2 = (R/2)(R/2 - r).$$

Remarks.

R0. It is essential to make the right $\triangle SJF$ with $SN^2 = R/2(R/2 - r), NF = R/2$ and points $N, I$ on $JF$ as the Figure by the next reason. Since $\triangle FSN$ and $\triangle SJN$ is similar, $SN/FN = JN/SN$ and hence we have $NI = R/2 - r$ by the last equation in the proof of Theorem.
R1. Since midpoints $A', B', C'$ of three sides $BC, AC, AB$ and midpoints $A'', B'', C''$ of $AH, BH, CH$ are on the nine point circle $N$, we can see that $\triangle ABC, \triangle A'B'C'$ and $\triangle A''B''C''$ are positions of similarity with the centers; centroid and orthocenter $H$ with a ratio $2 : -1 : 1$, respectively.

R2. Since the pedal $\triangle A'B'C'$ of $\triangle ABC$ has the incercle $C(H, \rho)$ and the circumcircle $N = C(N, R/2)$, We have

$$OH^2 = 4 \cdot NH^2 = 4 \cdot R/2 \cdot (R/2 - 2\rho) = R^2 - 4\rho R \cdots (\sharp).$$

Thus using formula $OI^2, (\sharp), NI = R/2 - r$ and Pappus’ theorem to $\triangle OIH$, we can calculate the distance

$$IH^2 = 2r^2 - 2\rho R \cdots (b).$$

R3. From the equations $(\sharp), (b)$ in R2 and $NI = R/2 - r$, we can see the following inequalities. (1) $R > 2r$, (2) $r > 2\rho$, (3) $KN > NI$, (4) $JF > OI$, (5) $OI > JI$, (6) $OH > OI$, (7) $OH > IH$.

Proof.
(1): by $NI > 0$.
(2): $r^2 > R\rho > 2r\rho$ by $(b)$ and (1), and hence, $r > 2\rho$.
(3): $KN^2 = (R/2)(R/2 - r) > (R/2 - r)^2 = NI^2$.
(4): $JF^2 = (R - r)^2 = R(R - 2r) + r^2 = OI^2 + r^2 > OI^2$
(5): $OI = 2KN > 2NI = JI$ by (3).
(6): $OH^2 - OI^2 = 2R(r - 2\rho) > 0$ by $(\sharp)$ and (2).
(7): $OH^2 - IH^2 = R(R - 2\rho) - 2r^2 > 2r(R - 2\rho - r) > 2r(r - 2\rho) > 0$ by $(\sharp), (b), (1)$ and (2)
R4. If we give points $O, I, H$, then we set points $N, J, K, L$ the line $\ell$, $\beta = \mathcal{C}(N, KN)$, and points $S, T$ as the proof of Theorem and the Figure. The point $F$ is the intersection point of the line $JI$ and the perpendicular line to $SJ$ at $S$. We set $R := 2NF, r := IF, \mathcal{O} := \mathcal{C}(O, R)$ and $\mathcal{I} := \mathcal{C}(I, r)$. In case $r > 0$, we can draw a figure but by a size of a paper, we can not obtain a figure. Please correct the positions of $O, I, H$ such that $NI$ is not so small and large, where $N$ is the midpoint of $OH$. From equations $OI^2 = (2KN)^2 = (2SN)^2 = 4(NI \cdot NF) = R(R - 2r)$ using $NI + r = NI + IF = NF = R/2$ and [1, p.86, 155. Theorem ], we draw two tangents to the incircle $\mathcal{I}$ from any point on the circumcircle $\mathcal{O}$, we obtain triangles having the same circumcircle $\mathcal{O}$ and incircle $\mathcal{I}$.

Reference