Three-Dimensional Flow Impinging Obliquely on a Rigid Cylinder

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An exact solution of the Navier-Stokes equation is given which represents steady three-dimensional flow of a viscous fluid impinging on a rigid cylinder obliquely. Numerical discussions of the relevant functions as well as the structure of the flow field are made. A comparison with an existing theory is also given.

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Due to the inherent nonlinearity of the Navier-Stokes equation, only three true exact three-dimensional solutions are known. Namely:
- the conical jet of Slezkin [4], generalized to the case of swirling flow by Holstein [5] and Yih [6];
- Himenz flow [1], generalized to the case of an oblique flow by Stuart [7] and Dowgialo [8].

This note presents a new exact solution to the Navier-Stokes equation, which belongs to the same class as the three listed above. This is the case of a spatial flow obliquely running onto a rigid cylinder.

To construct a solution of this class, the corresponding ideal fluid flow is used as a basis, which is at the same time a solution of the Navier-Stokes equation, which is nonlinear, and a simpler, linear equation of the vortex-free flow

\[ \Omega = \nabla \times \vec{u} = \vec{0}, \]

in which the velocity field, represented as a vector product of the gradients of its integral surfaces \( \psi_i^0, i = 1, 2 \)

\[ \vec{u} = \nabla \psi_1^0 \times \nabla \psi_2^0 \]

With a special view of surfaces \( \psi_i^0 \)

\[ \psi_1^0 = f_0^0(x_1) + f_1^0(x_1)f_2^0(x_2), \psi_2^0 = x_3 - \int_{f_1^0(x_1)}^{f_3^0(x_1)} dx_1 \]

the variables in equation (1) are separated, which makes it possible to reduce it to a system of ordinary differential equations.

Further, in order to extend the ideal solution (3) to the case of a viscous flow, the form of the function \( f_2^0(x_2) \) is preserved, and the remaining functions are searched again, assuming their asymptotic desire for their "ideal" analogues:

\[ \psi_1 = f_0(x_1) + f_1(x_1)f_2^0(x_2), \psi_2 = x_3 - \int_{f_1(x_1)}^{f_3(x_1)} dx_1 \]

We choose the Cartesian coordinates \((x, y)\) in the plane of the cylinder section and the coordinate \(z\) in the direction of its axis. A non-viscous version of the current stream given in terms \( \psi_i^0, i = 1, 2 \) of the coordinates of the source function, \([x, y, z] \rightarrow [l, o, z]\), where

\[ l = \frac{\ln(x^2 + y^2)}{2}, o = \arctan(y, x) \]

after substituting (3) into the vortex-free flow equation (1), the differential equations for
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$f_0^0(l), f_1^0(l), f_2^0(\alpha), f_3^0(l)$ take the form

\begin{equation}
\begin{align*}
f_0^0'' &= 0, f_1^0'' = 0, f_3^0' = 2f_3^0, \frac{d^2 f_2^0}{do^2} = 0
\end{align*}
\end{equation}

As a result, for the flow of an ideal fluid, we obtain

\begin{equation}
\begin{align*}
\psi_1^0 &= a\ell + (l + b)\alpha, \psi_2^0 = z - c\int \frac{e^{2\ell}}{l} dl
\end{align*}
\end{equation}

where, $a, b$ and $c$ are scale constants. The ideal flow functions are shown in Fig. 1 & 2. The velocity field of the ideal flow in this case has the form

\begin{equation}
\begin{align*}
\vec{u}^0 &= \left[ (a + \alpha) \sin(\alpha) + (l + b) \cos(\alpha) \right] e^{-t}, \left[ (b + l) \sin(\alpha) - (a + \alpha) \cos(\alpha) \right] e^{-t}, c
\end{align*}
\end{equation}

If fluid viscosity is taken into account, a boundary layer appears along the wall. We assume a generalization of (3’) in the form of (4), assuming $f_2^0(\alpha) = \alpha$:

\begin{equation}
\begin{align*}
\text{Re} \cdot \psi_1 &= f_0^0(l) + f_1^0(l) \cdot \alpha, \psi_2 = z - \int \frac{f_1^0(l)}{f_1(l)} dl
\end{align*}
\end{equation}

Where $\text{Re} = \frac{u_s d}{v}$ is the Reynolds number.

Then the stationary Navier-Stokes equation

\begin{equation}
\begin{align*}
\vec{V} \wedge (\text{Re} \cdot \vec{\Omega} \wedge \vec{u} + \vec{V} \wedge \vec{\Omega}) = \vec{0},
\end{align*}
\end{equation}

gives ordinary differential equations for $f_0^0(l), f_1^0(l), f_2^0(l)$:

\begin{equation}
\begin{align*}
f_1''' + (f_1 + 4)f_1'' + (f_1' + 2f_1 + 4)f_1' = 0, \\
f_0'' - (f_1 + 1)f_0' - f_0 + (f_1 + f_1 + 1)f_0 = 0, \\
f_2'' - (f_1 + 2)f_2' - (f_1' - 2f_1)f_2' = 0,
\end{align*}
\end{equation}
where the dashes denote differentiation by \( l \). Suitable boundary conditions follow from the expression for the flow rate

\[
\text{Re} \cdot \ddot{u} = \left[ (of_0' + e' f_0) \sin(\theta) + f_1 \cos(\theta) \right] e^{-l}, \left( f_1 \sin(\theta) - (of_0' + e' f_0) \cos(\theta) \right) e^{-l}, f_3 \right],
\]

and have the form:

\[
f_0(0) = 0, f_1(0) = 0, f_1'(0) = 0, f_3(0) = 0
\]

\[
f_0(\infty) = 0, f_1(\infty) \to 0, f_1'(\infty) = 1, f_3(\infty) = \nu_3
\]

The solutions of equations (6) - (8) and the components of the velocity field are presented in Figs. 3 & 4.

### References