Tabular Islamic Calendar

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Abstract. We describe the arithmetic or computational Islamic calendar of medieval Muslim astronomers. We classify the different calendars of this type, also called tabular, finding the possible intercalation criteria. With the chronological Julian day, we obtain precise rules to convert this tabular calendar to the Julian or Gregorian calendar and vice versa. The Islamic tabular calendar is, on average, very close to astronomical reality; however, there is an error that is accumulative and that we determine precisely. This paper analyzes the Islamic era and its relationship with the pre-Islamic calendar that existed in Arabia before the arrival of Islam.

1. Introduction

Strictly speaking, the Islamic calendar is observational, that is, physical observation of the crescent Moon is needed to declare the beginning of the new month. However, there are visibility criteria, some exclusively astronomical and others astronomical and physical, to anticipate when the Moon will be visible after having been in conjunction with the Sun.

The lunar calendar is local, that is to say, that the observation depends on the place where the view is made, depending on the geographical coordinates, height, clarity of the sky, time of year, etc.

Besides, different techniques are applied in each territory to establish the time when the new month begins. Therefore it is impossible to ensure in advance when a new month will start in a particular place in the world.

There is a problem to date an event from the past. That is, to know the Julian or Gregorian date of an Islamic date. Although the Julian and Gregorian calendars comply with well-established rules, the same is not valid with the Islamic calendar because, for historical dates, we do not know the techniques that were used to determine the beginning of the month, nor is it known whether those rules were applied correctly.

In order to convert from Julian (or Gregorian) date to Islamic date and vice versa, Muslim astronomers of the Middle Ages devised a computational lunar calendar, which we call the Islamic tabular calendar, which although it does not precisely reproduce Islamic dates, at least there is an average match [1], [2], [3], [4], [5]. In historical periods the dates of the tabular calendar depart at most one day from the Islamic dates obtained by a correct observation of the first crescent.

The Islamic tabular calendar is of great importance in chronology; it also allows us with an excellent approximation to give us the Julian or Gregorian dates of a future Islamic date; therefore, it is appropriate to do this research. [6], [7], [8].

The tabular calendar that we expose next has years of 12 lunar months, that is to say, months that begin a little after the astronomical New Moon. The duration of the months is alternately 30 and 29 days. The first of the year has 30 days.With this structure, the calendar year would have 354 days, but since the astronomical lunar year is something more than those days, it is necessary to add from time to time one more day per year, which is added to the last month of the year. These extraordinary years, called abundant or embolismic, have 355 days.

Since the lunar year is shorter than the solar year, it happens that a solar year is normally found in two lunar years and on some occasion in three years. While the lunar year normally belongs to two solar years and is occasionally included exclusively in a single solar year.

The names of the months of the tabular calendar are the same as those used in the usual practice of Muslims and they are the same as the calendar that existed in Arabia before Islamization. These names are from the first onwards: Muharram, Safar, Rabi I, Rabi II, Jumada I, Jumada II, Rajab, Shaban, Ramadhan, Shawwal, Zul-Qida y Zul-Hijja.

The Islamic calendar use the week, a seven-day period that matches the Jewish and Christian weeks. The days of the week are numbered, except the Friday, which is called the day of the meeting; and the Saturday receives the Jewish name of the sabbath.

An important issue to keep in mind when investigating the Islamic calendar is that Muslim days begin with the sunset, differently than the civil day that begins at midnight. This means that the same Muslim day belongs to two different civil days and vice versa. For about six hours (from sunset to 0 hours), there is a match between civil and Muslim days, and for the remaining 18 hours, there is another different match. As a usual practice, we will identify the Muslim day with the civil day with which it coincides from the 0 hours.

2. Synodic month

We define the moments of the New Moon, first quarter, Full Moon, and last quarter as those in which the difference between the apparent ecliptic longitudes of the Moon and the Sun * are 0º, 90º, 180º and 270º respectively [9].

At the time of the New Moon, which is when the Sun and the Moon have the same ecliptic longitude, it is called a conjunction. Calling opposition when the difference between its longitudes is 180º, then it is the full Moon.

It is called elongation between the Sun and the Moon at the angle

$$\Delta = L' - L$$

where L' is the ecliptic longitude of the Moon, and L is the ecliptic longitude of the Sun, both corrected by nutation and aberration. So when Δ is 0º, 90º, 180º, and 270º, the Moon is in New Moon, crescent, full Moon, and waning. The nutation is counteracted when the longitudes difference is made; therefore, it does not intervene for the determination of the lunar phases.

The astronomical synodic month or lunation is the time between two consecutive conjunctions of the Sun and the Moon, that is, the period between two New Moons. We distinguish between true and mean lunation.

The true lunation is the time between two conjunctions of the Sun and the Moon. By the periodic terms of the movements of the Sun and the Moon, the true synodic month or astronomical lunation is variable with time.

The astronomical mean lunation or synodic month m_s is the time that has to elapse for the mean elongation between the Sun and the Moon increases by 360º. This mean synodic month has a secular variation with time but does not contain periodic terms. Its value is

$$m_s = 29^d.530\,588\,85 + 2^d.163 \cdot 10^{-7} T \tag{1}$$

T is the centuries of 36,525-days (called Julian centuries) that have elapsed since the beginning of the year 2000 of the common era. The synodic month (1) is expressed in units of terrestrial time (TT), a uniform time scale, which does not match the universal time (UT). The UT is the time scale used for civil purposes and therefore for use in calendars **.

The synodic month or lunation is a period that varies very slightly over time. As calculated

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* Ecliptic longitude is the angle measured in the ecliptic (the plane in which the Earth moves) from the point where the Sun is in the spring equinox (ascending node of Earth's orbit) and the center of the Sun or the Moon, measured in the sense of its movements as observed from Earth.
The apparent ecliptic longitude is corrected by nutation and aberration. Nutation is a small movement of the Earth's axis of rotation, which affects all celestial bodies equally. Aberration is caused by the time it takes for light to reach Earth. The geometric position is the one that the Sun or the Moon have when their light reaches the Earth, and the apparent position is the one they occupied when they emitted the light we observe. The distance from Earth to Sun is greater than between our planet and Moon, therefore, the correction for aberration is higher for the Sun than for the Moon.

** In reality, the civil time scale is Coordinated Universal Time, UTC, a combination between the uniform atomic time scale and the rotational time UT. However, since the UTC can only depart a maximum of 0.9 seconds from the universal time UT, we can consider both scales identical for calendar purposes.

from (1), each century the synodic month increases by 0.0187 seconds, so for periods that are not excessively long, we can consider the astronomical lunation to be constant.

3. Normal and embolismic years

We look for lunar calendars computational, that is, calendars constructed using defined mathematical rules. These calendars intended to be as close as possible to astronomy, so the average duration of the calendar month should be as close as possible to the astronomical synodic month (1).

The lunar calendars computational are grouped into cycles composed of *A* lunar years, which are formed by twelve months of the calendar. Another concept is the astronomical lunar year composed of twelve astronomical lunations and its mean duration for the year 2000 CE (comun era) is

$$a_a = 12 \cdot m_s = 354^d.367\ 0662 = 354^d\ 8^h\ 48^m\ 34^s.52 \qquad (2)$$

with a small increase for later years. As we want to make a regular calendar, that is to say, close to astronomy, the calendar year must have 354 or 355 days, and its duration on average to be as close as possible to (2). By (2), we see that the years of 354 days are more frequent than those of 355. Therefore the normal year is

$$a_n = 354^d.$$

We have normal years of 354 days, and occasionally years of 355 days. These extraordinary years are called generically embolismic or abundant and, in the case of the Islamic calendar, *Kabisa* years. The embolism *D* are the days to be added to form the abundant years, which for the lunar calendar is $D = 1$ *.

The number of abundant years *B*, of a cycle of *A* years, where abundant years are formed by adding *D* days is [10]

$$B = \text{cint}\left[\frac{(a_a - a_n)A}{D}\right] \quad \Rightarrow \quad B = \text{cint}\left[(354.367\ 0662 - 354)A\right], \qquad (3)$$

cint is the round function. Formula (3) makes the average calendar year as close as possible to the astronomical year. Formula (3) applies when the normal year is less than the astronomical year, but (3) can be extended to the opposite case, getting the same results.

The embolism is placed at the end of a month of 29 days, which would has 30 days, avoiding that there are months of 31 days, which is considerably higher than the average value of the astronomical lunation.

The following formulas calculate the number of months of 30 and 29 days

$$m_{30} = 6A + B; \quad m_{29} = 6A - B \qquad (4)$$

reversing the formulas (4)

$$A = \frac{m_{30} + m_{29}}{12}; \quad B = \frac{m_{30} - m_{29}}{2}.$$

The duration of the lunation of a cycle of *A* years with *B* embolisms of $D = 1$ is

$$m_m = \frac{354A + BD}{12A} \quad \Rightarrow \quad m_m = \frac{354A + B}{12A}.$$

Table 1 gives a list of possibles lunar cycles computational derived from formula (3).

4. The natural order of intercalation of embolisms

It is necessary to establish the rule of intercalation of embolisms to define a lunar calendar computational, that is, to say what years of the cycle are embolisms or years of 355 days. Let *k* be a whole number and Y_k the position of the embolismic year in the cycle; then if kA/B were a

* Theoretically, it is possible to take another value for the typical year of the calendar month, for example, a duration of 353 days. We can also take another duration of the embolismic year, that is, the embolism is several days. The text proposal has always been used because it is more regular.

Years of the cycle	Years abundants	Months of 30 days	Months of 29 days	Mean lunation of the calendar	$m_s - m_m$
5	2	32	28	$29^d\ 12^h\ 48^m\ 0^s.00$	$- 3^m\ 57^s.12$
8	3	51	45	$29^d\ 12^h\ 45^m\ 0^s.00$	$- 0^m\ 57^s.12$
11	4	70	62	$29^d\ 12^h\ 43^m\ 38^s.18$	$0^m\ 24^s.69$
19	7	121	107	$29^d\ 12^h\ 44^m\ 12^s.63$	$- 0^m\ 9^s.75$
30	11	191	169	$29^d\ 12^h\ 44^m\ 0^s.00$	$0^m\ 2^s.88$
79	29	503	445	$29^d\ 12^h\ 44^m\ 3^s.04$	$- 0^m\ 0^s.16$

Table 1.- *Examples of computational lunar calendars obtained from formula (3). The last column gives the difference between astronomical lunation and the average calendar lunation.*

whole number, the year would be embolismic

$$Y_k = k\frac{A}{B}.$$

If kA/B were not a whole number, then we would have two ways of choosing the position of the embolismic years: choose the whole number immediately below or choose the number immediately above, that is, we can choose two intercalation rules

$$Y_k = \mathrm{int}\left(k\frac{A}{B}\right); \quad Y_k = \mathrm{int}\left(k\frac{A}{B}+1\right)$$

if we choose the second possibility, the law that determines if a year is embolismic is

$$Y_k = \mathrm{int}\left(k\frac{A}{B}+1\right) - \mathrm{int}\left[\frac{B}{kA}\mathrm{int}\left(\frac{kA}{B}\right)\right] \tag{5}$$

the last sum is 1 in the case that kA/B is whole, otherwise, it is null.

The order of intercalation of embolisms derived from (5) is called the natural order, to distinguish it from other criteria that we will analyze next.

By applying formula (5) to the 30-year cycle, we obtain Table 2, where the embolismic years in the cycle are according to the natural order of intercalation.

5. Number of embolismic years elapsed

The rule (5) that we are using to find out the position of the embolismic years tells us that for a value of *k*, the position of the embolism that is associated fulfill the inequality

$$k\frac{A}{B}+1 > Y_k \geq k\frac{A}{B}. \tag{6}$$

Since Y_k is an integer, the above means that if kA/B is a whole number, then it gives us the position of embolism; otherwise, the embolism will be the following year.

Multiplying (6) by *B* and dividing by *A* we get

k	1	2	3	4	5	6	7	8	9	10	11
Y_k	3	6	9	11	14	17	20	22	25	28	30

Table 2.- *Natural order of intercalation of embolisms of a lunar calendar of 30 years. The coefficient k is the order number of the embolism, and* Y_k *is its position in the 30-year cycle.*

$$k+\frac{B}{A}>Y_k\frac{B}{A}\geq k \tag{7}$$

as Y_k must be an integer, then

$$\operatorname{int}\left(Y_k\frac{B}{A}\right)=k \tag{8}$$

the equality that will be maintained for the years after, until the next year embolismic is reached. If we call E the number of embolismic years elapsed in the cycle until year Y included, then from (8) we find

$$E=\operatorname{int}\left(Y\frac{B}{A}\right) \tag{9}$$

Y is a year of the cycle.

6. Method to know if a year is embolismic

Given a year Y of the cycle, we want to find a method to know if it is embolismic or not. With formulas (7) and (8) we find

$$\operatorname{int}\left(Y_k\frac{B}{A}\right)+\frac{B}{A}>Y_k\frac{B}{A} \quad\Rightarrow\quad Y_k\frac{B}{A}-\operatorname{int}\left(Y_k\frac{B}{A}\right)<\frac{B}{A} \tag{10}$$

by the definition of the mod function

$$\frac{(Y_kB)\bmod A}{A}=Y_k\frac{B}{A}-\operatorname{int}\left(Y_k\frac{B}{A}\right)$$

inserting this result to (10)

$$(YB)\bmod A<B \tag{11}$$

which is the condition for the year Y to be embolismic

7. Intercalation criteria

The natural order of intercalation can be shifted, and then other intercalation criteria are obtained. Indeed, we can begin to count the years of the cycle, not from year 1 of the natural order, but another different year, we will have a displacement of the order in which the embolismic years occur.

Let β be the number of years since first of the natural order until the beginning of the year that we will now use as the beginning of the displaced cycle, and α is number of embolisms that have elapsed between the two years. By (9)

$$\alpha=\operatorname{int}\left(\beta\frac{B}{A}\right). \tag{12}$$

To know the embolismic years in the displaced order, we make in (5) and (11) the substitutions

$$Y_k\to Y_k+\beta;\quad k\to k+\alpha$$

k	1	2	3	4	5	6	7	8	9	10	11
Y_k	3	6	9	11	14	17	20	22	25	28	30
Y_k	3	6	8	11	14	17	19	22	25	27	30
Y_k	3	5	8	11	14	16	19	22	24	27	30

Table 3.- *The table shows three intercalation criteria of the Islamic calendar. In the second row is what we have called normal intercalation (Table 2). The third row gives us the intercalar years when the beginning of the cycle years is 3 years ($\beta=3$) with respect to the normal cycle, and the last row is intercalation when the years begin to be counted 6 years later ($\beta=6$) in relation to the normal cycle.*

of (5) we get that the new intercalation rule of the abundant years is

$$Y_k = \operatorname{int}\left[(k+\alpha)\frac{A}{B}+1\right] - \operatorname{int}\left\{\frac{B}{(k+\alpha)A}\operatorname{int}\left[\frac{(k+\alpha)A}{B}\right]\right\} - \beta \tag{13}$$

while the rule to know if a year is embolismic is

$$\left[(Y+\beta)B\right] \bmod A < B. \tag{14}$$

8. 30-year calendar subcycles

In the 30-year cycle, 11 of them are abundant. As the calendar is regular, the distribution of the two types of years should be as homogeneous as possible. Since $30/11$ is between 2 and 3, then the abundant years should be every two or three years.

Let *u* be the number of times the embolismic day is inserted in the third year after another embolismic day and *v* the corresponding number for periods of two years

$$3u + 2v = A; \quad u + v = B \quad \Rightarrow \quad u = A - 2B; \quad v = 3B - A.$$

For the 30-year cycle calendar

$$u = 8; \quad v = 3,$$

that is, eight times the embolism will come after three years and three times after two years.

We look for the regularity of the calendar; therefore, these periods of 3 and 2 years must be homogeneously spaced; that is to say in the following order

$$2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 \tag{15}$$

alternatively, in an order shifted from the previous. The previous distribution means that the first embolism will come in year 2 (ie, two years after the previous embolism), the next in 5, then it will be on 8, 10, 13, 16, 19, 21, 24, 27, 30.

The 30-year cycle consists of two subcycles. One of them formed by 8 years, 3 of them embolismic. The other subcycle, which is repeated twice, is made up of 11 years, with 4 of them embolismic.

As we have said, the distribution of the intercalar years obeys the rule (15) or any other order displaced from the previous one, this is

$$\begin{array}{l}
2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - \rightarrow I \\
3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 \rightarrow II \\
3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3\ -2\ 3 \rightarrow III \\
3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2 \rightarrow IV \\
2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - \rightarrow V \\
3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3 \rightarrow VI \\
3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3 \rightarrow VII \\
3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2 \rightarrow VIII \\
2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - \rightarrow IX \\
3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3 \rightarrow X \\
3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2 \rightarrow XI
\end{array}$$

in total, 11 possibilities, which will lead to different calendars and that we have numbered with Roman numerals to define later the types of calendars that can be formed.

The last years of the cycle that can be embolismic are 28, 29, and 30. It cannot be the 27 or the previous ones, because in this case, the embolism would be the year 30 or earlier, and it would no longer be the last of the abundant years. Then, the first years that can be embolismic are 1 (for the combination of $28+3$ or $29+2$), year 2 (combining $29+3$ or $30+2$) and year 3 (result of $30+3$).

If the cycle begins with an embolismic year at number 1 of the 30-year cycle, we will have a type of calendar that we will identify with the letter *a*; if the first embolism were year 2, we would give it the letter *b* and finally if the beginning of the abundant years were 3, we would assign the letter *c*.

At first, we might think that we would have 33 different calendars, the result of multiplying the 11 kinds of calendars according to the order of intercalation by the three types according to the first abundant year. We will see that only 30 different calendars are possible.

9. Classification of lunar calendars

Formula (14) tells us the order in which the embolismic years follow according to parameter β. Next, we will examine all the possibilities.

- We assume $\beta = 0$, by (12) we calculate that $\alpha = 0$. The criterion of the intercalations that we find by application of (14) is

$$3, 6, 9, 11, 14, 17, 20, 22, 25, 28, 30 \;\rightarrow\; VIII - c$$

the numbers are the years that are embolismic in the 30-year cycle. The first number, that is, 3, indicates the years that have elapsed since the last embolism of the previous cycle. The order of the intercalation is 3 3 3 - 2 3 3 3 - 2 3 3 - 2, that is, it is of type *VIII*, and since the first intercalar year is 3, it is of class *c*, we have therefore the calendar *VIII-c*. Los demás tipos de calendarios son

- $\beta = 1,\ \alpha = 0$

$$2, 5, 8, 10, 13, 16, 19, 21, 24, 27, 29 \;\rightarrow\; VIII - b$$
$$3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2$$

- $\beta = 2,\ \alpha = 0$

$$1, 4, 7, 9, 12, 15, 18, 20, 23, 26, 28 \;\rightarrow\; VIII - a$$
$$3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2$$

- $\beta = 3,\ \alpha = 1$

$$3, 6, 8, 11, 14, 17, 19, 22, 25, 27, 30 \;\rightarrow\; VII - c$$
$$3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3$$

- $\beta = 4,\ \alpha = 1$

$$2, 5, 7, 10, 13, 16, 18, 21, 24, 26, 29 \;\rightarrow\; VII - b$$
$$3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3$$

- $\beta = 5,\ \alpha = 1$

$$1, 4, 6, 9, 12, 15, 17, 20, 23, 25, 28 \;\rightarrow\; VII - a$$
$$3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3$$

- $\beta = 6,\ \alpha = 2$

$$3, 5, 8, 11, 14, 16, 19, 22, 24, 27, 30 \;\rightarrow\; VI - c$$
$$3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3$$

- $\beta = 7,\ \alpha = 2$

$$2, 4, 7, 10, 13, 15, 18, 21, 23, 26, 29 \;\rightarrow\; VI - b$$
$$3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3$$

- $\beta = 8,\ \alpha = 2$

$$1, 3, 6, 9, 12, 14, 17, 20, 22, 25, 28 \;\rightarrow\; VI - a$$
$$3 - 2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3$$

- $\beta = 9,\ \alpha = 3$

$$2, 5, 8, 11, 13, 16, 19, 21, 24, 27, 30 \;\rightarrow\; V - b$$
$$2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 -$$

- $\beta = 10,\ \alpha = 3$

$$1, 4, 7, 10, 12, 15, 18, 20, 23, 26, 29 \;\rightarrow\; V - a$$
$$2\ 3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3$$

- $\beta = 11,\ \alpha = 4$

$$3, 6, 9, 11, 14, 17, 19, 22, 25, 28, 30 \;\rightarrow\; IV - c$$
$$3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2$$

- $\beta = 12,\ \alpha = 4$

$2, 5, 8, 10, 13, 16, 18, 21, 24, 27, 29 \ \rightarrow \ IV - b$
$3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2$

- $\beta = 13,\ \alpha = 4$

$1, 4, 7, 9, 12, 15, 17, 20, 23, 26, 28 \ \rightarrow \ IV - a$
$3\ 3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2$

- $\beta = 14,\ \alpha = 5$

$3, 6, 8, 11, 14, 16, 19, 22, 25, 27, 30 \ \rightarrow \ III - c$
$3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3$

- $\beta = 15,\ \alpha = 5$

$2, 5, 7, 10, 13, 15, 18, 21, 24, 26, 29 \ \rightarrow \ III - b$
$3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3$

- $\beta = 16,\ \alpha = 5$

$1, 4, 6, 9, 12, 14, 17, 20, 23, 25, 28 \ \rightarrow \ III - a$
$3\ 3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3$

- $\beta = 17,\ \alpha = 6$

$3, 5, 8, 11, 13, 16, 19, 22, 24, 27, 30 \ \rightarrow \ II - c$
$3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3$

- $\beta = 18,\ \alpha = 6$

$2, 4, 7, 10, 12, 15, 18, 21, 23, 26, 29 \ \rightarrow \ II - b$
$3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3$

- $\beta = 19,\ \alpha = 6$

$1, 3, 6, 9, 11, 14, 17, 20, 22, 25, 28 \ \rightarrow \ II - a$
$3 - 2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3$

- $\beta = 20,\ \alpha = 7$

$2, 5, 8, 10, 13, 16, 19, 21, 24, 27, 30 \ \rightarrow \ I - b$
$2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3$

- $\beta = 21,\ \alpha = 7$

$1, 4, 7, 9, 12, 15, 18, 20, 23, 26, 29 \ \rightarrow \ I - a$
$2\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3$

- $\beta = 22,\ \alpha = 8$

$3, 6, 8, 11, 14, 17, 19, 22, 25, 28, 30 \ \rightarrow \ XI - c$
$3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2$

- $\beta = 23,\ \alpha = 8$

$2, 5, 7, 10, 13, 16, 18, 21, 24, 27, 29 \ \rightarrow \ XI - b$
$3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2$

- $\beta = 24,\ \alpha = 8$

$1, 4, 6, 9, 12, 15, 17, 20, 23, 26, 28 \ \rightarrow \ XI - a$
$3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2$

- $\beta = 25,\ \alpha = 9$

$3, 5, 8, 11, 14, 16, 19, 22, 25, 27, 30 \ \rightarrow \ X - c$
$3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3$

- $\beta = 26,\ \alpha = 9$

$$2, 4, 7, 10, 13, 15, 18, 21, 24, 26, 29 \ \rightarrow \ X - b$$
$$3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3$$

- $\beta = 27,\ \alpha = 9$

$$1, 3, 6, 9, 12, 14, 17, 20, 23, 25, 28 \ \rightarrow \ X - a$$
$$3 - 2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3$$

- $\beta = 28,\ \alpha = 10$

$$2, 5, 8, 11, 13.16, 19, 22, 24, 27, 30 \ \rightarrow \ IX - b$$
$$2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3$$

- $\beta = 29,\ \alpha = 10$

$$1, 4, 7, 10, 12, 15, 18, 21, 23, 26, 29 \ \ IX - a$$
$$2\ 3\ 3\ 3 - 2\ 3\ 3\ 3 - 2\ 3\ 3$$

Therefore, we find 30 different calendars, as far as their order of intercalation is concerned. The most used criterion is *VII-b* that we will call al-Khwarizmi or al-Battani style (9th century). The style of al-Burini (11th century) is *VIII-b*. Type *V-b* is associated with al-Hasib (10th century), *III-b* with Ulugh Beg (14th century). Ibn Futuh «Sevillian» (13th century) is associated with another calendar, not regular, which has embolismic years of the following order 2, 5, 8, 10, 13, 16, 18, 21, 24, 26 y 29 [11].

10. Pre-Islamic calendar

In the year 10 of the Hijrah Muhammad implanted the lunar calendar that the Muslims have used since then. Before this time, the Arabs followed a calendar we call pre-Islamic. According to numerous ancient authors, this calendar was lunisolar, that is to say, that it had years formed by twelve lunar months and occasionally, every two or three years, they intercalated a month or embolism, so that the months remained adjusted to the seasons, especially the last month of the year in which the pilgrimage to Mecca was made, which was in spring, making it coincide with the Jewish Passover and Christian Easter [14], [15].

The names of the months of the pre-Islamic calendar were the same as the current, but no ancient author gives a name to the embolismic month, which leads us to assume that this month did not have a proper name.

Following Abu Ma'shar, an astronomer of the ninth century who is the oldest author dealing with this matter, the pre-Islamic Arabs had a peculiar system of intercalation of embolismic months, which has made some think that the calendar was only lunar *.

As many old references cite, the intercalation operation was called *nasi*, a word that means to postpone and which consisted of transferring the sacred character from one month to the following month.

The last two months of the year and the first of the following year were considered sacred. So when they wanted to make intercalation, the first month of the year was included in the previous year, so that this last year was thirteen months while the second month became the first of the following year and was also given the character of sacred.

For example, suppose the first month of the year was Muharram. If the year became abundant, the month of Muharram of the following year was included in that year, and then the month of

* Al-Tabari Persian historian of the ninth century thinks that in pre-Islamic times there was intercalation: «They would go on a pilgramage in Zul-Hijja for two years, Muharram for the following two years, and Safar for the next two years. Thus they went on a pilgramage by shifting the months of pilgramage to subsequent month every two years» [11]. Al-Biruni expresses himself confusedly. He says that the pre-Islamic calendar was lunisolar and that the intercalation was done by displacement of one month. But while in a paragraph he states that the months changed their name, he later says that at the time of the Hijrah, the first month was Shaban, which indicates that there was no change of name months when the intercalation was made [13].

Order of the months at the beginning of the cycle	Order of the months in the first embolismic year	Order of the months the year following the first embolism	Order of the months in the second embolismic year.	Order of the months the year following the second embolism
Muharram	Muharram	**Safar**	Safar	**Rabi I**
Safar	Safar	Rabi I	Rabi I	Rabi II
Rabi I	Rabi I	Rabi II	Rabi II	Jumada I
Rabi II	Rabi II	Jumada I	Jumada I	Jumada II
Jumada I	Jumada I	Jumada II	Jumada II	Rajab
Jumada II	Jumada II	Rajab	Rajab	Shaban
Rajab	Rajab	Shaban	Shaban	Ramadhan
Shaban	Shaban	Ramadhan	Ramadhan	Shawwal
Ramadhan	Ramadhan	Shawwal	Shawwal	Zul-Qida
Shawwal	Shawwal	Zul-Qida	Zul-Qida	Zul-Hijja
Zul-Qida	Zul-Qida	Zul-Hijja	Zul-Hijja	Muharram
Zul-Hijja	Zul-Hijja	**Muharram**	Muharram	Safar
	Muharram		**Safar**	

Table 4.- *Order of the months in the first two embolismic years of the intercalation cycle.*

Safar of the following year became the first month of that year, while the last of that same year was again Muharram. For the following embolismic year, the process was repeated, then Safar became the last month of the embolismic year, and the following year began with Rabi I and ended with Safar, and so on (see Table 4).

It seems that the names of the months did not change, so that their position in the year varied, until complete cycle intercalations. However, according to some authors, when intercalating, names of months were changed, so that the year began with the month of Muharram. But if this had been the case, the month of Muharram would have to be repeated when the embolism was placed, that is, the embolismic month would have to be called Muharram and would be located in the last position of the year, and immediately after, the new year also it would begin with a month called Muharram, repeating this name twice consecutively. But it does not appear in any of the ancient authors that this repetition of the name of the month would have occurred.

With the intercalation method described above of not changing the names, the succession of the months always followed the same order, as if it were a purely lunar calendar. But if we refer to the position of the month in the year and not its name, we see that with the method described, there are years of thirteen months and other years of twelve (see Table 4). The seasons were stable in the year, although there was no coincidence between the names of the months and the seasons. Therefore, the calendar was lunisolar if we identify the months by their position in the year and not by their names. In this way, the last month of the year (whatever its name) was in the spring, the appropriate date to make the pilgrimage.

The name of the sacred months was changing, but not its position in the year, since the first month of the year, and the last two were still sacred, although the name of those sacred months was changing.

Active support for the idea that we expose (that the months did not change their name when the intercalation was done) is presented by Mahmoud Effendi [25]. This author identifies three events whose dates are known in the Julian calendar and the pre-Islamic. These dates are the death of Ibrahim, son of the Prophet, on the 29th of Shawwal that coincided with January 27, 632; the entrance of Muhammad in Medina that supposes the 8 of Rabi I, that makes coincide with the September 20, 622 and the birth of the Prophet the 9 of Rabi I that Effendi identifies with the April 20, 571.

Comparing the days between the dates mentioned above, we verify that there is a coincidence if the calendar had been lunar and not intercalar. And this is what should happen if, in the pre-Islamic calendar, the name of the months was not changed. As we have said, if we limit ourselves to the name of the months, as Effendi does in its calculation, the calendar follows the pattern of the lunar calendar, as in fact it is verified. The aforementioned coincidence would not occur if the name of the months changed due to the intercalation.

For purposes of converting dates to the Julian (or Gregorian) calendar, we assume that the pre-Islamic calendar was exclusively lunar, and we use the techniques that we will see later, as long as the conversion is made for months named and not numbered since the beginning of the year. Although in this type of transformation, the techniques we will expose do not always give us the right year.

To make the conversion of the Julian (or Gregorian) calendar to a pre-Islamic date and vice versa it is necessary to know the intercalation criteria of the abundant years, on which we can only speculate.

It seems that this intercalation system was in use in Arabia from about two hundred years before the Hijrah. But the criteria followed for intercalations is unknown. While for some it was done every two years, for others it was every three years; there are authors who say that they followed the rule of intercalation of the Jews, others that intercalated 9 months every 24 years or that there were 11 embolisms in a 30-year cycle *.

It seems that the intercalation cycle began with the year in which Muharram was the first month, and the cycle concluded when again this month occupied the first position of the year. This circumstance seems to have occurred the year Muhammad established the Islamic calendar, as follows from his own words: «Time had circulated as on the day when Allah created the heavens and the earth», indicating that after the intercalation cycle, Muharram had taken the first place in the year.

As it is said, in the year 10 of the Hijrah Muhammad forbade the *nasi* ** or intercalation of the months, prohibiting that the months that were profane became sacred and establishing that the sacred months will permanently occupy the eleventh, twelfth positions, and the first position, respectively, according to the Qur'an: «The nasi is an addition of unbelief. Those who have disbelieved are led astray, thereby. The make it profane one year and make it sacred another year, in order to adjust the number which Allah has made sacred. Then make profane what Allah has made sacred». (Kor 9, 37).

11. Islamic era

With the name of Hijrah, we understand the migration of the Prophet from Mecca to Medina and the formation of the first Muslim community, the historical event that is used to establish the Islamic era [16], [17].

The Islamic era or simply Hijrah was implanted by Caliph Umar ibn al-Jattab in the year 7 or 8 after the death of the Prophet. It is, therefore, a proleptic era; that is, it was established after the historical event that gave rise to it. As we have said, the same year of his death or 10 years of the Islamic era, Muhammad established the lunar calendar in force today. This calendar can be extended to the past, until the same year in which the Hijrah took place or year 1; it would be a proleptic calendar, which was not the one that existed during the first ten years of Islamism, which, as we have said, was the pre-Islamic lunisolar calendar.

The Islamic calendar (that is, the lunar) always begins with the first day of Muharram, so it

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* According to Coussin de Perceval, the intercalation was always every three years, which means that less intercalary months were put. This researcher assumes that in two hundred years that the lunisolar calendar lasted there was a shift in the date of the last month of the year, the one dedicated to the pilgrimage, going from November at the beginning of the intercalar system in 412 CE, to April at the time of the Prophet. This hypothesis does not seem satisfactory, because the function of the *nasi* would not be understood, which according to numerous authors proclaimed a year in advance whether or not there would be an embolism. If was known that the embolism would come every three years unchanged, no one was needed to announce it in advance, nor would the *nasi* have been a prominent character, as confirmed by ancient references [14].

** In addition to intercalation, word *nasi* was used when, for some reason, the sacred character was transferred from one month to another. For example, if instead of Muharram the month of Safar was sacred, without it meaning no intercalation.

was for all the years before the implementation of this calendar, that is, for the period in which it was a proleptic calendar. For reasons that we will see later, the first day of Muharram of year 1 of the Hijrah was the Julian date of Friday, July 16, 622, an identity that allows us to compare Islamic and Julian (or Gregorian) dates.

It should be noted, from the above, that both in the Islamic lunar calendar and the pre-Islamic lunisolar, the Julian date of Muharram day 1 of year 1 is July 16, 622. The difference is that in the proleptic Islamic calendar That month was the first of the year, but it did not have that place in the lunisolar calendar of that first year, as we will now see.

There are doubts about the exact date on which Muhammad left Mecca to go to Medina. However, most of the ancient authors agree that the arrival of the Prophet to Medina (then called Yathrib) was 12th of Rabi I; therefore, the departure from Mecca was a few days earlier, on a date that is disputed by the different historians. The departure from Mecca and the arrival to Medina occurred in the same lunar year, and this is the data that interests us.

Instead of starting the Islamic lunar calendar on the same day of the Hijrah, it was decided that this year would begin with the month of Muharram, that is, a time before the Hijrah so that the sequence of the months of the pre-Islamic calendar it was not altered. As we have said, the implementation of the Islamic lunar calendar did not produce discontinuity with the pre-Islamic calendar, but was a continuation, with the new feature of eliminating embolismic months and therefore being Muharram permanently the first month of the year, which was also the first month in the year of the Hijrah in the lunisolar calendar.

We cannot know the duration of the lunar months since its beginning is determined by the observation of the first crescent moon, which is greatly affected by local circumstances. However, they must always have 29 or 30 days. From 12th of Rabi I, date of the arrival of Muhammad to Medina, to the 1 of Muharram of the same year 69, 70 or 71 days passed, if the months of Muharram and Safar had 30 or 29 days. That is to say, that 2th of Rabi I that we are considering should have been September 25, 24, or 23 of the 622 Julian year.

If as historians suppose, Rabi 12 was Friday, that is, on the same day of the week as Muharram 1, then between both dates, there must have been an integer number of weeks, that is, 70 days, no 69 or 71, and therefore Muhammad's arrival in Medina must have been Friday, September 24, 622 of the Julian calendar.

Another problem that arises is the date on which year 1 of the Hijrah began according to the lunisolar calendar. This problem depends on the number of embolisms that occurred in the first ten years of the Hijrah, that is, during the period in which the lunisolar calendar was in force from the Hijrah until it was replaced by the lunar calendar. According to those who have dealt with this matter, there must have been three interleaves or perhaps four.

If there were, as most historians think, three intercalations, the month of Muharram of the first year of the Hijra was the fourth month in that year, that is to say, the year should have started with the month of Shawwal; so that after 10 years, 3 of them intercalary, Muharram was placed as the first month, as happened when Muhammad established the lunar calendar.

We know the duration of the months that were embolismic in the first ten years of the Hijrah. We know that three consecutive 30-day months are possible, and even in very exceptional conditions, there can be up to 4 consecutive months of that duration. While in the most optimal conditions, there can only be three consecutive months of 29 days.

Then the three embolismic months that took place during the first ten years together would have a duration of 90, 89, 88, or 87 days, depending if the months had 30 or 29 days *. These days are those between the beginning of the lunisolar year and the 1st of Muharram, which, as we know, coincided with July 16, 622. Therefore, the Julian dates on which the lunisolar calendar could begin in the year of the Hijrah are 17, 18, 19, or 20 April of 622. Now we do not know the weekly day of the date, so we can not have security in the correspondence, as in the case analyzed of the arrival of Muhammad to Medina. If we stick to the duration of the months as given in the tabular calendar, then there would be 88 days (two months of 29 days and one of 30) between the two dates

* There is a tradition that there are no more than two months of 30 days followed. If this rule was applied at that time, the duration of 90 days should be ruled out.

considered and therefore the beginning of the year, or day 1 of Shawwal of year 1, it would be the Julian date April 19, 622.

A problem arises with the Islamic era. According to the Muslim calculation, the days begin at sunset, while in the civil calculation, the day starts at midnight. Then the day of the Hijrah began being still Thursday, July 15, 622.

Then there are two different eras: the one that starts on July 15 and the one that begins on July 16. We will identify the first era with the letter T (Thursday or astronomical style) and the second era with F (Friday or civil style). The calendar difference, according to the chosen era, differs permanently in one day. The most used, and the one we consider next, is the Friday or civil style, which according to the previous notation, we will call *VII-b-F.*

If we take into account the two types of eras, we find that there are $2 \times 30 = 60$ different calendars with the 30-year cycle.

12. Short cycle and long cycle of the 30-year calendar

In the calendar we are considering, every 30 years is repeated the intercalation of the abundant years in the same positions. This cycle is called short.

Each short cycle has 10,631 days, a period that is not divisible by 7, that is, at the end of a cycle, the weekly day of the first day does not coincide with that corresponding to the previous period. 10,631 days consists of 1,518 weeks and 5 days. Therefore, the weekly day of the first day of each cycle moves 5 days with relation to the previous year. That is, if one cycle begins with Monday, the next cycle will start with Saturday and the next cycle with Thursday. 7 cycles of 30 years must pass before there is an agreement with the weekly days of the first day of the year.

Therefore, after 210 years ($= 7 \times 30$), not only do the intercalar years repeat with the same pattern, but the dates will be repeated on the same weekly day. The 210 years is called the long cycle.

If we take into account the seven possible types of years according to the weekly day of the first day of the cycle, we will have a total of $2 \times 30 \times 7 = 420$ different calendars with the 30-year cycle.

13. Regular monthly and signature of the year

The regular monthly * is a fixed number associated with each month, which allows us to determine the weekly day of a date. For its definition, we assume that the duration of the months is alternately 30 and 29, having the first month 30 days and the last 29, except in the abundant years that it has 30 [18].

The first month of the calendar has, arbitrarily, the number 7, if from here we count the days in weekly periods numbered from 1 to 7, it is easy to verify that the first day of the second month will have the number 2, the third month number 3 and so on. We call these numbers regular monthly, which we identify with the letter r, which follows the sequence

$$7-2-3-5-6-1-2-4-5-7-1-3 \tag{16}$$

are permanently associated with each of the twelve months of the year.

It is now necessary to associate the numbers with the weekly days. This association is arbitrary, but taking Sunday as the first day of the week, we identify it with day 1. It is called the signature of the year to the weekly day of its first day. So if it starts on Sunday, its signature will be number 1 and 2 if it begins on Monday.

In each 210-year long cycle, the weekly days of all dates are repeated, in particular, the weekly day of the first day of the year; therefore, we have a perpetual cycle. Since the typical year has 354 days, it exceeds a whole number of weeks by 4 days; this means that the weekly day of a year that is after a normal year is four days after the first day of the previous year. If the year is 355 days, then its duration exceeds 5 to an integer number of weeks; therefore, the year following an embolismic year has a weekly day 5 days after the first day of the previous year. With these rules it is easy to build a table that gives us the weekly days of the first day of each year in a period of 210

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* We copy the terminology used in the computus or technique to determine the date of the Christian Easter.

years (Table 5), taking into account that the first day of the cycle has the same weekly day as the first day of the 1 H (Hijrah) year, that is Friday or 6.

To determine the weekly day of the first day of the month, the first we have to do is determine the position *n* of year in the 210-year cycle

$$n = (Y_I - 1)\bmod 210 + 1 \tag{17}$$

Y_I is the year of the Hijrah. Note that we subtract one unit from Y_I and subsequently add 1 to get the first year of the cycle to be 1 and the second cycle 211 and so on.

Knowing the order *n* of the 210-year cycle, we go to Table 5 and find the signature of the year *s*. To find the day of the week *d* of the beginning of any month we use the regular lunar *r* and apply the formula

$$d = \left(r + s - 1\right)\bmod 7 + 1 \tag{18}$$

We have adapted the formula so that the first day of the week is Sunday, which we identify with the number 1.

If we want to determine the weekly day d' we apply the formula

$$d' = \left(d + D_I - 1\right)\bmod 7 \tag{19}$$

D_I is the monthly day. In formula (19), we subtract 1 to get that when $D_I = 1$, $d' = d = 1$.

* *Example 1*.- Determine the weekly day of the 12th of Rajab of the year 1220 H.

- Taking $Y_I = 1220$, we find by (17) that year has the order 170 in the 210-year cycle.
- From Table 5, we find that the signature of that year is $s = 2$, that is, that year begins with a Monday.
- Since the month of Rajab is the seventh, by (16), we find that the regular of that month is $r = 2$.
- The weekly day of the first day of Rajab is calculated by (18), obtaining $d = 4$, that is to say, that the day 1 of Rajab of the year 1220 H is Wednesday.
- Finally, we apply (19) and find $d' = 1$, that is, the day of Rajab 12 of the year 1220 H is Sunday.

* *Example 2*.- Determine the weekly day of Muharram 15 of the year 751 H.

- By (16), we find $n = 121$, which is the order of the year 750 H in the 210-year cycle.
- From Table 5, we find that $s = 1$, that is, the year begins with Sunday.
- As the month of Muharram is the first, $r = 7$, according to (15).
- The weekly day of Muharram's first day is $d = 1$ by (18).
- By (19), we find that $d' = 1$, so Muharram 15 of 751 H is a Sunday.

In section 19, we will present a method to determine the weekly day of a date, calculated using its chronological Julian day.

14. Auxiliary calendar associated with the Islamic calendar

The Julian period is a chronological dating technique introduced by Scaliger in 1582 and subsequently applied to the computation of the days by Herschel in 1851 [19], [20], [21].

The Julian period is the product of the following three cycles: solar (28 years), Meton (19 years), and indiction (15 years). The product of the three periods is 7,890 years, which is the duration of the Julian period. The beginning of the three cycles coincided in the year -4712 or 4713 before the common era (BCE), which is the origin of the first Julian period. The second cycle will begin in the year 3178 of the common era (CE).

The count of astronomical Julian days begins on January 1 of the year 4713 BCE or year -4712 of the Julian calendar at 12 hours of terrestrial time (or another time scale). Although the chronological Julian day begins just like the day civil, that is to say at midnight; that is, at 0 o'clock on January 1 of the year 4713 BCE. The Julian days count starts with the number 0; therefore, on January 1, 4713, BCE was the Julian day 0.

The Julian chronological day is always an integer value, but not the astronomical Julian day that can have decimals corresponding to the fraction of a day. In what follows, we will understand that we use the chronological Julian day expressed in terrestrial time (TT), which we will represent with the *JD* notation.

The problem we intend to solve is given date in the tabular Islamic calendar Y_I, M_I, D_I determine the corresponding chronological Julian day *JD*. The calculations shall apply the al-Battani

Years of 210-year cycle	Weekly day first day of the year	Years of 210-year cycle	Weekly day first day of the year	Years of 210-year cycle	Weekly day first day of the year	Years of 210-year cycle	Weekly day first day of the year
1	6	***54***	1	107	2	***160***	7
2	3	55	6	***108***	6	161	5
3	1	***56***	3	109	4	162	2
4	5	57	1	110	1	***163***	6
5	2	58	5	***111***	5	164	4
6	7	***59***	2	112	3	165	1
7	4	60	7	113	7	***166***	5
8	2	61	4	***114***	4	167	3
9	6	***62***	1	115	2	***168***	7
10	3	63	6	***116***	7	169	5
11	1	64	3	117	5	170	2
12	5	***65***	7	118	2	***171***	6
13	2	66	5	***119***	6	172	4
14	7	***67***	2	120	4	173	1
15	4	68	7	121	1	***174***	5
16	1	69	4	***122***	5	175	3
17	6	***70***	1	123	3	***176***	7
18	3	71	6	124	7	177	5
19	1	72	3	***125***	4	178	2
20	5	***73***	7	126	2	***179***	6
21	2	74	5	***127***	6	180	4
22	7	75	2	128	4	181	1
23	4	***76***	6	129	1	***182***	5
24	1	77	4	***130***	5	183	3
25	6	***78***	1	131	3	184	7
26	3	79	6	132	4	***185***	4
27	1	80	3	***133***	1	186	2
28	5	***81***	7	134	6	***187***	6
29	2	82	5	135	3	188	4
30	7	83	2	***136***	7	189	1
31	4	***84***	6	137	5	***190***	5
32	1	85	4	***138***	2	191	3
33	6	***86***	1	139	7	192	7
34	3	87	6	140	4	***193***	4
35	7	88	3	***141***	1	194	2
36	5	***89***	7	142	6	195	6
37	2	90	5	143	3	***196***	3
38	7	91	2	***144***	7	197	1
39	4	***92***	6	145	5	***198***	5
40	1	93	4	***146***	2	199	3
41	6	94	1	147	7	200	7
42	3	***95***	5	148	4	***201***	4
43	7	96	3	***149***	1	202	2
44	5	***97***	7	150	6	203	6
45	2	98	5	151	3	***204***	3
46	6	99	2	***152***	7	205	1
47	4	***100***	6	153	5	***206***	5
48	1	101	4	154	2	207	3
49	6	102	1	***155***	6	208	7
50	3	***103***	5	156	4	***209***	4
51	7	104	3	***157***	1	210	2
52	5	105	7	158	6	-	-
53	2	***106***	4	159	3	-	-

Table 5.- *Islamic tabular perpetual calendar. The abundant years are in italics. The numbers associated with each year is your signature or weekly day of the first day. Sunday is associated with 1* [22].

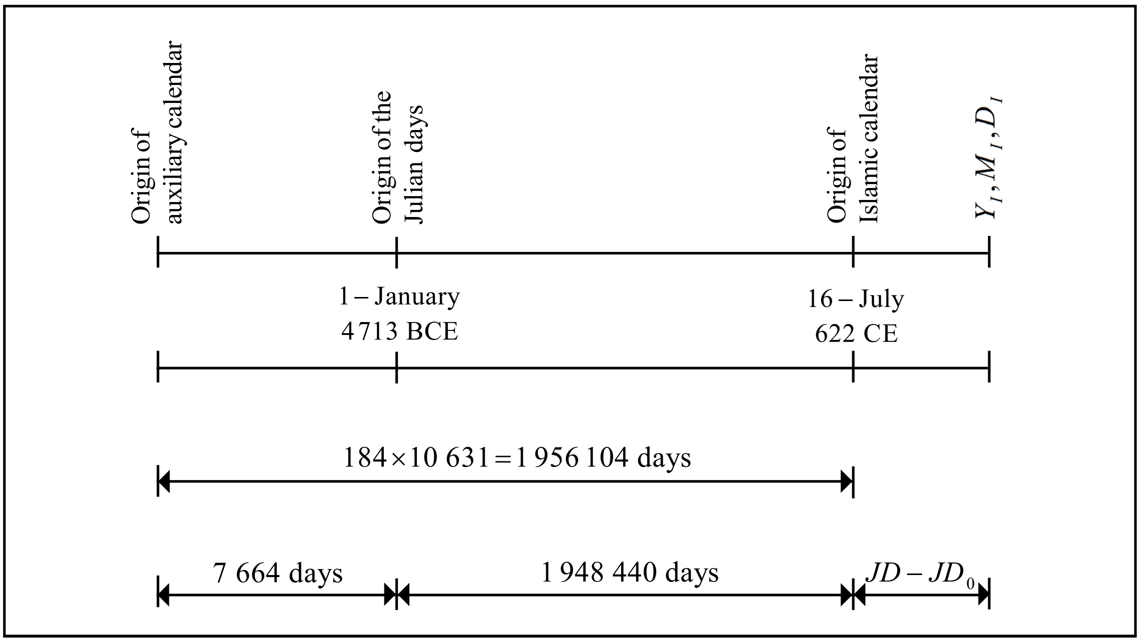

Illustration 1.- *Representation of the origins of the Islamic auxiliary calendar and the Julian days account.*

style or type of calendar *VII-b-F* described above. As a starting point, we have that the Julian day of the origin of the Islamic calendar, July 16, 622, is

$$JD_c = 1\,948\,440.$$

The Islamic auxiliary calendar is the one whose first cycle begins immediately before the origin of the Julian days. The number of cycles of 30 lunar years elapsed since the origin of the Julian days ($JD = 0$) until the beginning of the calendar (JD_c) is

$$\frac{1\,948\,440 - 0}{10631} = 183.279\,0895$$

where 10 631 are the days of a 30-year lunar cycle

$$30 \times 354 + 11 = 10\,631,$$

therefore the origin of the auxiliary calendar took place 184 cycles before July 16, 622. That is

$$184 \times 10\,631 - 1\,948\,440 = 7\,664$$

what are the days before the origin of the Julian days, as shown in illustration 1.

The date in the Islamic auxiliary calendar is Y'_I, M'_I, D'_I. Between the origin of the auxiliary calendar and the Hijrah, there are $184 \times 30 = 5{,}520$ lunar years. But as the years of the Hijrah begin with the number 1, and the years of the auxiliary calendar start it in year 0, then

$$Y'_I = Y_I + 5\,519$$

Y_I is the year of the Islamic calendar date, and Y'_I is the corresponding year in the Islamic auxiliary calendar.

As the days in the auxiliary calendar also begin with the number 0

$$D'_I = D_I - 1.$$

Also, the months of the auxiliary calendar begin with month 0

$$M'_I = (M_I + 11) \bmod 12$$

an expression that decreases M_I in a unit. In short, the date in the auxiliary calendar of a date in the Islamic calendar is

$$Y'_I = Y_I + 5\,519; \quad M'_I = (M_I + 11) \bmod 12; \quad D'_I = D_I - 1. \tag{20}$$

15. Julian day of Islamic date

We want to find a relationship between the days elapsed since the origin of the auxiliary calendar until the beginning of the year Y'_I. Since the relation is closely linear, we will check if

Years x	Days elapsed y	$y-Rx$	Years x	Days elapsed y	$y-Rx$
0	0	0	**15**	5 315	-15/30
1	354	-11/30	16	5 670	4/30
2	709	8/30	**17**	6 024	-7/30
3	1 063	-3/30	18	6 379	12/30
4	1 417	-14/30	19	6 733	1/30
5	1 772	5/30	**20**	7 087	-10/30
6	2 126	-6/30	21	7 442	9/30
7	2 481	13/30	22	7 796	-2/30
8	2 835	2/30	**23**	8 150	-13/30
9	3 189	-9/30	24	8 505	6/30
10	3 544	10/30	**25**	8 859	-5/30
11	3 898	-1/30	26	9 214	14/30
12	4 252	-12/30	27	9 568	3/30
13	4 607	7/30	**28**	9 922	-8/30
14	4 961	-4/30	29	10 277	11/30

Table 5.- *y are the days elapsed from the beginning of the 30-year cycle until the beginning of the year x. We take R = 10631/30. The first year of the cycle is 0. The embolismic years are in bold type.*

Zeller's theorem applies [23], which tells us that, for the correspondence $x \to y$, if there is a fractional number *R* such that

$$\max\left(y-Rx\right)-\min\left(y-Rx\right)<1$$

then

$$y=\operatorname{int}\left(Rx+S\right)$$

with

$$S=\max\left(y-Rx\right).$$

We refer to a 30-year cycle of the auxiliary calendar, and we build Table 5 taking $R=10\ 631/30$, and the days elapsed *y* until the beginning of year *x*, numbered since the beginning of the cycle. Other values of R can be found, but the one we take is the coefficient if the relationship were linear, and therefore it is the most immediate to test whether Zeller's theorem is fulfilled. From the data obtained we obtain

$$\max\left(y-Rx\right)-\min\left(y-Rx\right)=\frac{14}{30}-\left(-\frac{15}{30}\right)<1$$

therefore the requirement demanded by Zeller's theorem is satisfied

$$y=\operatorname{int}\left(\frac{10\ 631x+14}{30}\right). \tag{21}$$

Months x	Days elapsed y	$y-Rx$	Months x	Days elapsed y	$y-Rx$
0	0	0	6	177	0
1	30	1/2	7	207	1/2
2	59	0	8	236	0
3	89	1/2	9	266	1/2
4	118	0	10	295	0
5	148	1/2	11	325	1/2

Table 7.- *y are the days elapsed from the beginning of the year until the beginning of the month x. We take* $R=59/2$. *The first month of the year is 0.*

We want to show that the relation (21) is fulfilled for the years elapsed since the origin of the auxiliary calendar and not only for those passed in a cycle. Let's assume a year Y_I' that belongs to the cycle $N+1$ counted from the origin of the auxiliary calendar, then

$$Y_I' = Y_I'' + 30N$$

Y_I'' is the year in the $N+1$ cycle. The days elapsed from the start of the auxiliary calendar to the beginning of the year Y_I' is

$$J_1 = \operatorname{int}\left(\frac{10\ 631Y_I''+14}{30}\right) + 10\ 631N = \operatorname{int}\left[\frac{10\ 631\left(Y_I''+30N\right)+14}{30}\right] = \operatorname{int}\left(\frac{10\ 631Y_I'+14}{30}\right) \quad (22)$$

(22) demonstrates that Zeller's theorem applies to any year of the auxiliary calendar.

Now we determine the days elapsed from the beginning of the year to the month of the date. Also, in this case, Zeller's theorem applies. Taking $R = 59/2$, we make Table 7 where we find

$$\max\left(y-Rx\right) - \min\left(y-Rx\right) = \frac{1}{2} - 0 < 1$$

which is the validity condition of Zeller's theorem. If we call J_2 the days elapsed in the year until the beginning of the month M'_I, then

$$J_2 = \operatorname{int}\left(\frac{59M'_I+1}{2}\right). \quad (23)$$

$J_1 + J_2 + D'_I$ are the days since the start of the auxiliary calendar to date. By Illustration 1 we conclude that the Julian day of the date is

$$JD = \operatorname{int}\left(\frac{10\ 631Y_I'+14}{30}\right) + \operatorname{int}\left(\frac{59M'_I+1}{2}\right) + D'_I - 7\ 664 \quad (24)$$

7,664 are the days between the beginning of the auxiliary calendar and the beginning of the Julian days. With formulas (20) and (24), we determine the Julian day of a date of the 30-year Islamic tabular calendar.

16. Islamic date of Julian day

Now we solve the inverse problem, that is, known the Julian day to determine the date in the Islamic calendar.

Another option of Zeller's theorem is

$$y = \operatorname{int}\left(Rx+S\right) \quad \Leftrightarrow \quad y = \operatorname{int}\left(\frac{px+q}{r}\right).$$

The inverse formula is

$$x = \operatorname{int}\left(\frac{ry+s}{p}\right)$$

the following equation is fulfilled [23]

$$q + s = r - 1,$$

so the inverse equation of (21) is

$$x = \operatorname{int}\left(\frac{30y+15}{10\ 631}\right) \quad \Rightarrow \quad Y_I' = \operatorname{int}\left(\frac{30J_1+15}{10\ 631}\right). \quad (25)$$

J_1 are the days that have passed since the beginning of the auxiliary calendar until the beginning of the year. J' represents the days from the beginning of the auxiliary calendar to the date we are trying to determine, therefore

$$J' = JD + 7\ 664. \quad (26)$$

Q are the days elapsed in the year of the date, therefore

$$Q = J' - J_1$$

By (22) and (25)

$$Q = J' - J_1 = JD + 7\ 664 - \operatorname{int}\left\{\left[10\ 631\operatorname{int}\left(\frac{30J'+15}{10\ 631}\right)+14\right]\Big/30\right\} \quad (27)$$

It is indifferent to put in (27) J' or J_1 since both days belong to the same year.

The inverse relationship of (23) is

$$M'_I = \mathrm{int}\left(\frac{2Q}{59}\right) \tag{28}$$

Q are the days elapsed from the beginning of the year.

The days elapsed in the month of the date is

$$D'_I = Q - \mathrm{int}\left(\frac{59M'_I + 1}{2}\right) = Q - \mathrm{int}\left\{\left[59\,\mathrm{int}\left(\frac{2Q}{59}\right)+1\right]\Big/2\right\}. \tag{29}$$

With the equations (26), (25), (27), (28) and (29) the date of the auxiliary calendar is determined, it is only necessary to find the inverse equations to (20)

$$\begin{aligned} Y_I &= Y'_I - 5519 \\ M_I &= M'_I \bmod 12 + 1 \\ D_I &= D'_I + 1. \end{aligned} \tag{30}$$

and we already know the date of the Islamic calendar known the Julian day.

17. Julian day of dates in the Julian calendar and the Gregorian

The method we use to convert the Islamic calendar to the Julian or Gregorian calendar and do the inverse operation, is to determine the Julian day in the Julian or Gregorian calendar. With this data, find the date of the Islamic calendar according to techniques developed in the previous section.

The method to find the Julian day of the Julian and Gregorian calendars is similar to that described in the previous sections, so we will not develop the techniques but exclusively put the final results [24].

The date in the Julian auxiliary calendar is

$$\begin{aligned} D'_J &= D_J - 1 \\ M'_J &= \left(M_J + 9\right) \bmod 12 \\ Y'_J &= Y_J + 4716 - \mathrm{int}\left(\frac{14 - M_J}{12}\right) \end{aligned} \tag{31}$$

Y_J, M_J, D_J is the date on the Julian calendar. The associated Julian day is

$$JD = \mathrm{int}\left(\frac{1461 \cdot Y'_J}{4}\right) + \mathrm{int}\left(\frac{153 \cdot M'_J + 2}{5}\right) + D'_J - 1\,401. \tag{32}$$

To do the inverse operation, that is to find the date known the Julian day, we apply the following formulas

$$\begin{aligned} Y'_J &= \mathrm{int}\left[\frac{4 \cdot \left(JD + 1\,401\right) + 3}{1\,461}\right] \\ Q &= \mathrm{int}\left\{\frac{\left[4 \cdot \left(JD + 1\,401\right) + 3\right] \bmod 1\,461}{4}\right\} \\ M'_J &= \mathrm{int}\left(\frac{5 \cdot Q + 2}{153}\right) \\ D'_J &= \mathrm{int}\left[\frac{\left(5Q + 2\right) \bmod 153}{5}\right] \end{aligned} \tag{33}$$

with the previous formulas, we determine the date in the Julian calendar

$$\begin{aligned} D_J &= D'_J + 1 \\ M_J &= \left(M'_J + 2\right) \bmod 12 + 1 \\ Y_J &= Y'_J - 4716 + \mathrm{int}\left(\frac{14 - M_J}{12}\right). \end{aligned} \tag{34}$$

To find the Julian day of a Gregorian calendar day we use the formula

$$JD = \mathrm{int}\left(\frac{1461 \cdot Y'_J}{4}\right) + \mathrm{int}\left(\frac{153 \cdot M'_J + 2}{5}\right) + D'_J - \mathrm{int}\left[\frac{3}{4}\mathrm{int}\left(\frac{Y'_J + 184}{100}\right)\right] - 1\,363 \qquad (35)$$

Y'_J, M'_J, D'_J is the date of the auxiliary calendar, calculated by (31).

To find the date in the Gregorian calendar known Julian day, we calculate the coefficient s

$$\varepsilon = \mathrm{int}\left\{\frac{3}{4}\left[1 + \mathrm{int}\left(\frac{4 \cdot JD - 9\,222\,033}{146\,097}\right)\right]\right\} + 10 \qquad (36)$$

and we calculate the Julian day that would have the same date but expressed in the Julian calendar

$$JD_j = JD + \varepsilon, (37)$$

and with this new value of the Julian day, we apply the formulas (33) and (34) and find the Gregorian date.

18. Examples

* *Example 3*.- Determine the Julian day of the 13th of Safar of the year 720 of the Hijrah.

- By (20) we find the date in the Islamic auxiliary calendar

$$Y'_I = 6\,239; \quad M'_I = 1; \quad D'_I = 12$$

- We apply (24) and calculate the Julian day

$$JD = 2\,203\,372.$$

* *Example 4*.- Find the Islamic date of Julian day 2,450,320.

- With (26) and (27) we calculate the auxiliary quantities

$$J_1 = 2\,457\,984; \quad Q = 97.$$

- With (25), (28) and (29) we calculate the date of the auxiliary calendar

$$Y'_I = 6\,936; \quad M'_I = 3; \quad D'_I = 8.$$

- With (30) we calculate the date of the Islamic calendar

$$Y_I = 1\,417; \quad M_I = 4; \quad D_I = 9$$

9th day of Rabi II of the year 1417 of the Hijrah.

* *Example 5*.- Find the date of the Julian calendar of the Islamic date 15 of Shawwal 840 H.

- By (20) and (24) we determine the Julian day of the date

$$JD = 2\,246\,034.$$

- By (33) we found

$$Y'_J = 6\,153; \quad Q = 52; \quad M'_J = 1; \quad D'_J = 21.$$

- By (34) we determine the corresponding Julian calendar date

$$Y_J = 1\,437; \quad M_J = 1; \quad D_J = 22$$

January 3 of the year 1447 of the common era.

* *Example 6*.- Find the Islamic date of March 13 of the year 950 CE.

- Since the date corresponds to the Julian calendar, formulas (31) and (32) apply, so the corresponding Julian day is

$$JD = 2\,068\,117.$$

- With (26) and (27) we calculate the auxiliary quantities

$$J' = 2\,075\,781; \quad J_1 = 2\,075\,526; \quad Q = 255.$$

- With (25), (28) and (29) we calculate the date of the auxiliary calendar

$$Y'_I = 5\,857; \quad M'_I = 8; \quad D'_I = 19.$$

- With (30) the date of the Islamic calendar is calculated

$$Y_I = 338; \quad M_I = 9; \quad D_I = 20$$

that is, on the 20th Ramadhan 338 H.

* *Example 7*.- Find the date of the Gregorian calendar of the Islamic date 8 Ramadhan 1505 H.

- By (20) and (24) we determine the Julian day of the Islamic date

$$JD = 2\,481\,650.$$

- By (35) and (36) we find

$$\varepsilon = 13; \quad JD_J = 2\ 481\ 663.$$

- Applying formulas (33) and (34), we find that the Gregorian date is June 4, 2082 CE.

* *Example 8.*- Find the Islamic date of the Gregorian date October 23, 2043.

- Since the date corresponds to the Gregorian calendar, formulas (31) and (32) apply, so the corresponding Julian day is

$$JD = 2\ 467\ 546.$$

- From (26) and (27) we calculate the auxiliary quantities

$$J_1 = 2\ 475\ 210; \quad Q = 313.$$

- From (25), (28) and (29) we calculate the date of the auxiliary calendar

$$Y'_I = 6\ 984; \quad M'_I = 10; \quad D'_I = 18.$$

- With (30) the date of the Islamic calendar is calculated

$$Y_I = 1\ 465; \quad M_I = 11; \quad D_I = 19$$

that is, 19th Zul-Qida 1465 H.

Another problem is to determine the Julian or Gregorian date of the Islamic date of an observational calendar. As the observation of the crescent, with which the lunar month begins, it cannot be predicted for reasons as evident as the presence of clouds, erroneous view, change in the transparency of the sky, etc., we cannot have an exact conversion algorithm.

So, what we do is assume that the Islamic date corresponds to the tabular calendar and then apply the above algorithms to determine the Julian or Gregorian date. Still, this procedure will not give us for sure the equivalence between both dates.

However, if, in addition to the Islamic date deduced from an observational calendar, we know its weekly day, it is possible to find the equivalence with the Julian or Gregorian calendar, given the circumstance that the week is the same in both calendars.

* *Example 9.*- 1 Rabi-I 1235 H was a Sunday, find out the Gregorian date that corresponds to it.

- By (20) we find the date in the Islamic auxiliary calendar, assuming that the Islamic date corresponds to the tabular calendar

$$Y'_I = 6\ 754; \quad M'_I = 2; \quad D'_I = 0$$

- We apply (24) and calculate the Julian day of the date of the Islamic tabular calendar

$$JD = 2\ 385\ 787.$$

- By (36) and (37) we find

$$\varepsilon = 12; \quad JD_J = 2\ 385\ 799.$$

- Applying formulas (32) we find

$$Y'_G = 6535; \quad T = 292; \quad M'_G = 9; \quad D'_G = 17.$$

- By (34) we find that the Gregorian date of 1 Rabbi 1235 H of the tabular calendar is

$$D_G = 18; \quad M_G = 12; \quad Y_G = 1819.$$

- By (38) we find that the Julian day of the Islamic date of the tabular calendar has Saturday as a weekly day. Still, the day that had that date was Sunday, this means that the Gregorian date is not what we have found before, but a day after (to pass from Saturday to Sunday). Therefore the Gregorian date that corresponds to the Islamic date obtained by the observational calendar is Sunday, December 19, 1819.

19. Calculation of the weekly day of Islamic date

Since the Julian days count and the succession of the weeks have not been interrupted, it is possible to obtain an algorithm to determine the weekly day of date if its Julian day is known. Julian day 0 was a Monday, then the weekly day *W* of a Julian day *JD* is

$$W = (JD + 1) \bmod 7 + 1 \tag{38}$$

if the result of (38) is 1 the day is Sunday, if it is 2 the day is Monday and so on, until it reaches 7 which corresponds to Saturday.

Then to determine the weekly day of an Islamic date, we first calculate the corresponding chronological Julian day (24) and then calculate the weekly day by (38).

20. Islamic calendar and astronomy

The average duration of the calendar month in a 30-year cycle is

$$m_c = \frac{354A + B}{30 \cdot 12} = 29.530\,555\,56 = 29^d\,12^h\,44^m\,0^s$$

somewhat less than the average astronomical duration, which for the year 2000 is

$$m_s = 29^d.53058885 = 29^d\,12^h\,44^m\,2^s.88.$$

This small difference of 2.88 seconds in each month, accumulates over time. Besides, the average astronomical lunation increases with time, as shown (1). The result is that the Islamic tabular calendar is gradually moving away from astronomy, or more concretely said that the New Moon moves concerning the date of the first month of the computational calendar.

JD_0 is the Julian day of the beginning of the year 2000, JD is the Julian day of the date and JD_c the Julian day of the beginning of the Islamic calendar (July 16, 622 CE).

T is the Julian centuries since the beginning of the year 2000 to date

$$T = \frac{JD - JD_0}{36\,525}.$$

P is the cycles of 30 lunar years that have elapsed since the beginning of the Islamic calendar

$$P = \frac{JD - JD_c}{10\,631}.$$

The synodic month is for (1)

$$m_s = p + qT \tag{39}$$

T is the Julian centuries of 36,525 days of 24 hours of terrestrial time TT. We express (39) according to the lunar cycles elapsed

$$\begin{aligned} T &= \frac{JD - JD_0}{36\,525} = \frac{JD - JD_0 + JD_c - JD_c}{36\,525} = \frac{JD - JD_c}{36\,525} + \frac{JD_c - JD_0}{36\,525} = \\ &= \frac{JD - JD_c}{10\,631}\frac{10\,631}{36\,525} + \frac{JD_c - JD_0}{36\,525} = \nu P + \mu \end{aligned} \tag{40}$$

ν and μ are numerical constants

$$\nu = \frac{10\,631}{36\,525}; \quad \mu = \frac{JD_c - JD_0}{36\,525}$$

μ is the Julian centuries since the beginning of the Islamic calendar until the beginning of the year 2000.

Inserting (40) in (39)

$$m_s = p + qT = p + q(\nu P + \mu) = (p + q\mu) + q\nu P$$

and the astronomical synodic month is a function of the 30-year lunar cycles that have elapsed since the beginning of the calendar.

The number of months in the 30-year cycle is $12 \times 30 = 360$, and the astronomical duration of those months is $360 m_s$. As in a 30-year cycle, there are 10,631 days then the error in a cycle is

$$E = 10\,631 - 360 m_s = \left[10\,631 - 360(p + q\mu)\right] - 360 q\nu P \tag{41}$$

error E is the difference between calendar and astronomy in the period P of 30 years, period counted from the beginning of the Islamic era. If with Δ_c we represent the time difference between the beginning of the calendar month and the astronomical New Moon at the beginning of the cycle P, at the end of this cycle the difference will be $\Delta = \Delta_c + E$. As astronomical lunation varies very slowly over time, we have assumed that during a period it remains constant.

The chronological Julian day of the first day of the year 2000 is $JD_0 = 2{,}451{,}545$ counted from 0 hours on January 1, 2000. On the other hand $JD_c = 1\,948\,440$, then

$$\nu = \frac{10\,631}{36\,525} = 0.291\,061; \quad \mu = \frac{JD_c - JD_0}{36\,525} = -13.774\,264$$

and for (1) we have

$$p = 29^d{,}530\,58885; \quad q = 2{,}163 \cdot 10^{-7}.$$

Elapsed Lunar Cycles N	Accumulated error $E(N)$	Difference with astronomy $\Delta = \Delta_c + E(N)$
0	0	$2^d\ 1^h\ 21^m$
10	$-2^h\ 39^m$	$1^d\ 22^h\ 42^m$
20	$-5^h\ 21^m$	$1^d\ 20^h\ 0^m$
30	$-8^h\ 7^m$	$1^d\ 17^h\ 14^m$
40	$-10^h\ 55^m$	$1^d\ 14^h\ 26^m$
50	$-13^h\ 47^m$	$1^d\ 11^h\ 34^m$
86	$-\ 1^d\ 0^h\ 34^m$	$1^d\ 0^h\ 47^m$

Table 8.- *Accumulated errors in the Islamic tabular calendar according to the cycles of 30 years elapsed. When 86 cycles or 2,580 lunar years have passed, the accumulated error will have exceeded one day. In the column on the right is the difference between calendar and astronomy, that is, the difference between the beginning of the month of the calendar and the beginning of the astronomical New Moon.*

With the previous numerical data (41) it is

$$E = -0^d.010\,913 - 2^d.266\,434 \cdot 10^{-5} P.$$

To determine the error in a cycle, we consider that the average astronomical lunation remains constant, but we cannot make this assumption when considering a wide interval of time so that the error accumulated after N cycles of 30 years is

$$E(N) = \sum_1^N (10\,631 - 360 m_s)$$

using (41)

$$E(N) = N\left[10\,631 - 360(p + q\mu)\right] - \sum_1^N 360 q\nu P = N\left[10\,631 - 360(p + q\mu)\right] - 180 q\nu P N(N+1)$$

giving numerical values

$$E(N) = -0^d.010\,913N - 1^d.133\,217 \cdot 10^{-5} N(N+1) \qquad (42)$$

which tells us how much the difference between the calendar and astronomy increases in N cycles counted since the beginning of the Islamic era.

Formula (42) always gives a negative result, which means that the astronomical lunation is higher than the calendar lunation. Table 8 shows the values of the formula (42) for various periods. In the same table, it is indicated that after 86 cycles from the beginning of the Hijrah, the accumulated error of the calendar with astronomy will reach one day.

By direct calculation it is determined that the difference between the beginning of the day of the tabular calendar and the time of the New Moon at the beginning of the Hijrah is $\Delta_c = 2^d 1^h 21^m$ approximately *, understanding that the calendar day begins the same as the civil day, that is to say at 0 hours, according to the first meridian or meridian of Greenwich. The conjunction of the Moon, or New Moon, happens sometime before the beginning of the calendar month, and this is what happens in the tabular calendar. The value of Δ_c we have given refers to the first meridian, which

* We obtained this value averaging the difference between the first day of the month and the date of the New Moon during the first ten years of the Hijrah.

Lunar cycles elapsed N	Difference with the astronomy (calculated) $\Delta = \Delta_c + E(N)$	Difference with astronomy (theoretical) $\Delta = \Delta_c + E(N)$
0	0	0
20	$1^d\ 20^h\ 47^m$	$1^d\ 20^h\ 0^m$
40	$1^d\ 14^h\ 8^m$	$1^d\ 14^h\ 25^m$
60	$1^d\ 11^h\ 14^m$	$1^d\ 8^h\ 38^m$

Table 9.- *Difference between calendar and astronomy calculated by two procedures. In the second column are the results of comparing the dates of the beginning of the months with the date of the astronomical New Moon. In the third column, the value obtained with the formula (42). There is a small difference because of the periodic variations of astronomical lunations.*

means that if we consider a point East of Greenwich as a reference, Δ_c will take a smaller value. For example, if we assume that the comparison is made for Mecca (which is at a geographical longitude of 39º 50' East) then the value of Δ_c will be $1^d\,22^h\,42^m$ (Illustration 2).

The day begins at midnight, but as we have said, the beginning of the Islamic day is with the sunset, at approximately 18 hours local time. If we consider this beginning, the value of Δ_c for Mecca will be $1^d\,16^h\,42^m$ (Illustration 2), which is the time between the New Moon and the first day of the calendar in the year of the Hijrah.

As time passes, and as a result of the accumulated error, the time between the New Moon and the beginning of the lunar month decreases, as seen in Table 8.

Table 9 shows the error accumulated after several 30-year cycles, calculates comparing the date of the calendar with the time of the astronomical New Moon. The calculation has been made averaging the differences in the first year of the cycle, so the results are very approximate but agree with those obtained by the theoretical formula (42).

Table 8 shows that in the year 3124 CE ($= 622 + 2502$), the accumulated error will have exceeded one day, but there will still be a difference of one day between the New Moon and the beginning of the month, which would make it possible use the tabular calendar.

21. Terrestrial time and universal time

We have said that the time scale in which the synodic month (1) is expressed is the terrestrial time TT, a uniform time scale whose unit is the second atomic. But as we have also noted, calendars

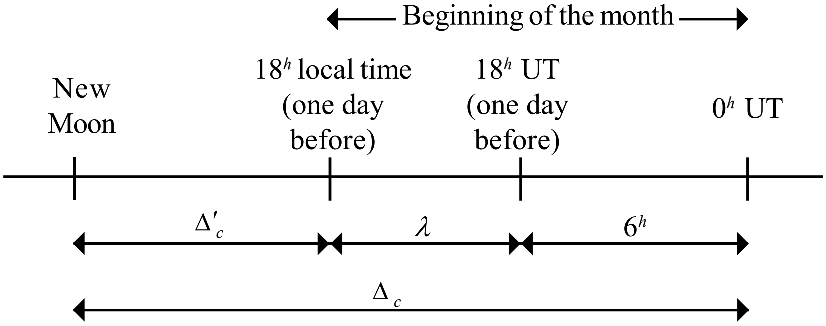

Illustration 2.- Δ_c *is the time between the New Moon and the first day of the calendar, assuming that it starts* at 0^h *UT.* Δ'_c *is the difference between the New Moon and the first day of the month, considering that it begins at 18 hours local time from a place of geographical longitude* λ *(which in the illustration we assume East of Greenwich). The picture shows that* $\Delta'_c = \Delta_c - 6 - \lambda$ *(with* λ *expressed in units of time).*

follow the civil time scale, called coordinated universal time UTC.

The universal time UT is a rotational time; that is, it is defined by the apparent rotation of the Sun around the Earth. It is not a uniform scale because of the irregularities of the Earth's rotation. However, it is the time that gives us the position of the Sun in the sky and the time that human life follows.

The civil scale or coordinated universal time is a mixture of terrestrial time and universal time. It uses the second atomic, however when it separates excessively from universal time due to the decrease in the rotation of the Earth, a second is inserted at the end or half of the year to get the UTC to be closest to UT.

The measured time difference on the TT and UT scales is

$$\Delta T = TT - UT\,,$$

from ancient astronomical observations, Stepheson obtains the following empirical formula

$$\Delta T = \alpha + \beta T + \gamma T^2 = 80^s.44 + 111.6T + 31T^2 \tag{43}$$

valid for historical times. In (43), ΔT is expressed in seconds, and T are Julian centuries since the beginning of the year 2000 [25].

Since the difference between the UT and the UTC is at most 0.9 seconds, we can consider for practical purposes that the scale that calendars follow is universal time. We must, therefore, modify the calculations made in the previous section and study the relationship between calendar and astronomy when universal time is used and not terrestrial time.

The first problem to solve is to express the synodic month (1) on the UT scale. t_1 and t_2 are the moments in TT when two consecutive New Moons occur, then the synodic month on the TT scale is $m_s = t_2 - t_1$. In UT the moments of the New Moons are $t_1 - \Delta T$ and $t_2 - \Delta T$, the difference between these two moments will be the synodic month in the UT scale

$$\begin{aligned} m'_s &= t_2 - \Delta T(t_2) - t_1 + \Delta T(t_1) = (t_2 - t_1) - \left[\Delta T(t_2) - \Delta T(t_1)\right] = m_s - \left[\Delta T(t_2) - \Delta T(t_1)\right] = \\ &= m_s - \frac{\Delta T(t_2) - \Delta T(t_1)}{t_2 - t_1}(t_2 - t_1) \approx m_s - \Delta T'(t_1) m_s = m_s\left[1 - \Delta T'(t_1)\right] \end{aligned} \tag{44}$$

ΔT varies very slowly, and its monthly variation is negligible, for this reason, we have chosen as the time to calculate ΔT the moment t_1; the same result is obtained if it is calculated for time t_2. (44) is a formula that applies to any period. By (39), (43) and (44)

$$m'_s = (p + qT)(1 - \beta - 2\gamma T) = p(1 - \beta) + \left[q(1 - \beta) - 2\gamma p\right]T - 2\gamma q T^2$$

therefore

$$m'_s = p' + q'T + r'T^2 \quad \Rightarrow \quad p' = p(1 - \beta); \quad q' = q(1 - \beta) - 2\gamma p; \quad r' = -2\gamma q \tag{45}$$

T is the time when the synodic month is calculated, expressed in Julian centuries since the year 2000.

Now it is necessary to express the numerical values in appropriate units. The derivative of ΔT in days per day is

$$\Delta T' = \frac{1}{86\,400}\frac{1}{36\,525}\frac{d\Delta T}{dT} = 3.5364 \cdot 10^{-8} + 1.9647 \cdot 10^{-8} T$$

then

$$\beta = 3.5364 \cdot 10^{-8}; \quad 2\gamma = 1.9647 \cdot 10^{-8}$$

they are dimensionless quantities. By (1) and (45) we find that the synodic month measured on the UT scale is

$$m'_s = 29^d.530\,5878 - 3.639 \cdot 10^{-7} T \tag{46}$$

we do not consider the second-order term in T because it is minimal. (46) shows us that if the measurement of the lunation is made on the UT time scale, then it decreases overtime at approximate 0.031 seconds per century.

The error accumulated in a period of N cycles is again the formula (42) but with the coefficients p' and q' corresponding to the measurement in UT of the synodic month

$$\begin{aligned} E'(N) &= N\left[10\,631 - 360(p' + q'\mu)\right] - 180 q'\nu N(N+1) = \\ &= -0^d.013\,412 N - 1^d.906\,51 \cdot 10^{-5} N(N+1). \end{aligned} \tag{47}$$

Years era common	Frequency in %			δ
	0	+1	-1	
700	63.33	3.33	33.33	-30%
1200	66.66	16.66	16.66	0%
1700	66.66	23.33	10.00	+13%
2100	66.66	30.00	3.33	+26.66%

Table 10.-*Frequency, as a percentage, of days between the observation of the lunar crescent and the date of the tabular calendar. The calculations are made for Mecca and correspond to the average of the first days of the 30-year cycle years that begin in the year of the left column. It is verified that the most frequent is that there is a coincidence between the date of the calendar and the observation of the crescent. It is also confirmed that with time, it is more frequent for the Moon to be observed before the calendar date. In the last column is the coefficient δ, which is the average of the difference between the day of observation of the crescent and the first day of the calendar, expressed as a percentage.*

The numerical results found with (47) are different from those found with (42). In the first case, the error is expressed in 24-hour days of UT time, and in the second case it is 24-hour days of TT, that is, the numerical difference found is by the different time scales in which they are measured.

The accumulated error will exceed the 24-hour UT in the cycle 69 when 2,070 lunar years have elapsed, while, as seen in Table 8, there are 86 cycles that must pass so that the accumulation is of 1 day expressed in terrestrial time.

22. Determination of the first day of the year of the Hijrah

We try to compare the tabular calendar with the observation of the lunar crescent, which, as we know, is some time after the conjunction of the Moon with the Sun. To make this calculation, we will use the visibility criterion of Yallop [26], [27], [28]; which allows us to know theoretically the moment when the crescent is observable in a specific place. To fix the calculations, we will assume that the observation is made from Mecca.

We want to compare the date of the beginning of the calendar month with the day of the first visibility. Let us realize that the result is always an integer. It would be 0 if there is a match between the calendar and the observation, the value would be negative if the view occurs after the first day of the calendar and the difference is a positive number if the observation occurs before the first day of the calendar. That is, the coefficient δ we are looking for is the average, expressed as a percentage, of the difference of the day of the observation of the crescent and the day of the beginning of the month according to the calendar.

As a result of the error that the calendar accumulates, the value of δ depends on the time, so that the calendar dates are increasing as compared to the observation day. Table 10 shows that at the beginning of the Hijrah, δ is negative, indicating that the observation of the crescent occurs, with some frequency, after the calendar date. For the year 1200 CE, the parameter δ becomes positive; that is, occasionally, the observation day is before the calendar date.

Table 10 shows that, at least in historical times, the occasions when there is a coincidence between calendar date and the observation of the crescent are more frequent, although we must bear in mind that these results are sensitive to the longitude and latitude of the observation site.

So to find the date of the first day of the Islamic calendar, we will assume (and this is a hypothesis) that the calendar that follows the observation of the crescent (that is, the calendar that has been followed), fits on average to the tabular calendar. Therefore, we assume that in most cases, there is a coincidence between the day of observation and the first day of the tabular calendar, at least in historical times.

With the previous hypothesis, we can determine the Julian date of the first day of the Islamic calendar. We take a set of days whose Julian (or Gregorian) and Islamic dates are known. On some occasions, the day of the observation will have been before the first day of the month of the calendar, and on other times it will have been after, but more often, there will have coincidence, as the hypothesis we have made assumes.

Suppose we know the Islamic and Julian (or Gregorian) dates of the same day. We assume that the Islamic date, which has been determined by the observation, coincides with the tabular calendar date. Then we can calculate the days *J* elapsed since the beginning of the Hijrah or the first day of the first month of the year 1 H.

Once the Julian (or Gregorian) date is known, we find its Julian day *JD* by (32) or (35). Therefore we will have

$$JD_c = JD - J + 1 \tag{48}$$

we add a unit because the first day of the year 1 H is day 1. (48) allows us to determine the Julian day JD_c of the first day of the Islamic calendar and with it its corresponding Julian or Gregorian date.

To calculate the *J* value, we adapt the formulas (22) and (23), but now those formulas count the time since the beginning of the Hijrah, whose first day is 1 and not 0 as it happens when the account is made from the beginning of the auxiliary calendar Therefore we have

$$J = \operatorname{int}\left[\frac{10631(Y_I - 1) + 14}{30}\right] + \operatorname{int}\left[\frac{59(M_I - 1) + 1}{2}\right] + D_I \tag{49}$$

Y_I, M_I, and D_I is the date in the Islamic calendar.

For example, suppose the date of 1 of Muharram 133 H, which corresponds to the Julian date September 8, 750 CE (as we can deduce from the visibility criteria of Yallop). By application of (31) and (32), we find that the Julian day of the Julian calendar date is

$$JD = 1\ 995\ 246$$

and by (49)

$$J = 46\ 807$$

then the Julian day of the first day of the Islamic calendar is

$$JD_c = JD - J + 1 = 1\ 948\ 440$$

by (33) and (34) we find the corresponding date of the Julian calendar, July 16, 622.

This operation is performed for several date pairs, determining in each case the origin date of the calendar. The value that is adopted is the most frequent, that is, the one that makes the tabular calendar fit as best as possible to the calendar based on the observation, which, as we have already verified (Table 10) is July 16, 622 CE.

The date found is that corresponding to the first day of the tabular calendar, but we do not know if it was the first day in the observational calendar, that maybe it was a day before or a day later.

23. Islamic day and Islamic lunation

The Islamic day is an uninterrupted count of days in the tabular calendar that begins on the first day of the year of the Hijrah, which takes the value 0. Then the Islamic day J_I is

$$J_I = JD - JD_c$$

JD is the Julian day of the Islamic date and JD_c the Julian day of July 16, 622, or the first day of the Islamic tabular calendar. Another option of Islamic day is to count until the day when the crescent is observed; we call this definition astronomical Islamic day. In other definitions, the first day of the first year is taken as 1, instead of 0 as we have done in our definition.

There are several numbering systems for the lunations. Brown's lunation number, which is the most widely used system, defines lunation 1 as the first New Moon of 1923, the year Brown's lunar theory was introduced. This first lunation took place on January 17, 1923.

Jean Meuss's lunation number defines lunation 0 as the first New Moon of the year 2000, which was on January 6 of that year.

Goldstein's lunation number begins with the first New Moon of the year 1001 BCE, which was on January 11, 1001 BCE. The Hebrew lunation number begins with New Moon 1 that began on December 7, 3761 BCE.

The Islamic lunar number is counted from the beginning of the Islamic era, that is, from July 16, 622 CE. It should be noted that Islamic lunations begin with the month of the tabular calendar and, therefore, may differ some days from the astronomical lunation.

To find the number of Islamic lunation we calculate the months from the first Islamic year, that is, if L is the Islamic lunation

$$L = 12\left(Y_I - 1\right) + M_I \tag{50}$$

Y_I and M_I is the Islamic date.

* *Example 10*.- For the Islamic date 15 Ramadhan 1420 H find the Islamic day and the Islamic lunation on which that date is found.

- By (20) and (24) the Julian day of the Islamic date is calculated

$$JD = 2\ 451\ 536.$$

- With this data, we calculate the Islamic day

$$J_I = JD - JD_c = 2\ 451\ 536 - 1\ 948\ 440 = 503\ 096$$

are the days elapsed from 0 hours of day 1 Muharram 1 H to 0 hours of day 15 Ramadhan 1420 H.

- From (50), we determine the number of lunation

$$L = 17\ 037.$$

24. Displacement of the seasons

A solar calendar is characterized by keeping the dates of the seasons stable. We cannot get the seasons to start every year on the same day, either because of the variable duration of the year or, mainly, because of the intercalation every four years of the leap day. But at least the seasons start around the same calendar days. The lunar calendar is not related to the apparent movement of the Sun; therefore, the dates of the seasons are not fixed and can be on any Islamic date.

The time between the beginning of a season and the next season of the same type is called a seasonal year. Four types of seasonal years must be distinguished, depending on the season considered. For example, the year of spring had an average duration of $365^d\ 5^h\ 49^m\ 1^s.11$ in 2000. The tropical year is the average of the four seasonal years, its average value varies with time, being its value $365^d\ 5^h\ 48^m\ 45^s.19$ in the year 2000 [32].

The average duration of the year of the 30-year Islamic tabular calendar is $354^d\ 8^h\ 48^m$, that is, a difference with the tropical year of $10^d\ 21^h\ 0^m\ 45^s.19$, very close to 11 days.

The duration of the spring year is, as we have said, $365^d.242\ 374$, so between the day on which spring begins and the day on which the next spring begins, there must be 365 days or on some occasions 366 days, since this interval is an integer and spring can begin at any time of the day. Normally, every four years there are three in which the seasons are 365 days apart and one in which they are 366 days, although it is a rule that is not always fulfilled, either by the application of

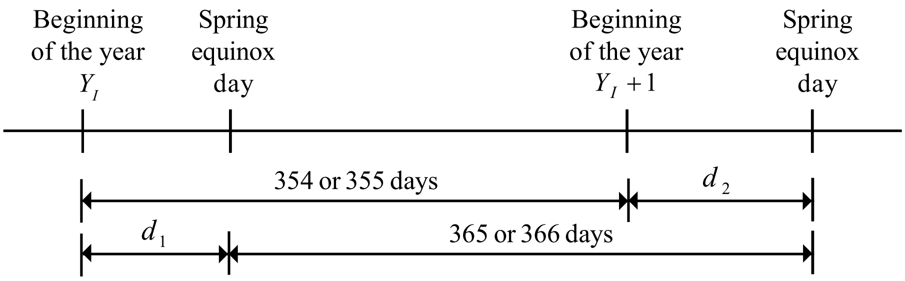

Illustration 3.- *Determination of the displacement of the seasons in the Islamic calendar. The days elapsed between the beginning of two consecutive Islamic years can be 354 or 355 days. The days between the start of two successive seasons can be 365 days (most frequently) or 366 days. d_1 are the days from the beginning of the year Y_I and the beginning of spring (or another season) that corresponds to that year, and d_2 are the days from the beginning of the year $Y_I + 1$ to the next start of spring. The displacement of spring is $d_2 - d_1$, which may be 10, 11, or 12 days.*

the Gregorian correction of intercalation or for the variable duration of the seasonal years [33].

Since the duration of the Islamic year is 354 or 355 days, the result is that the days between a spring and the next in the Islamic calendar can be 10, 11 or 12 days (and the same result for the remaining seasons).

In Illustration 3, d_1 are the days from the beginning of the year Y_I to the beginning of spring and d_2 the days from the beginning of the year Y_I+1 and the beginning of spring of that year. For Illustration 3 we have

$$d_1+(365 \text{ or } 366)=(354 \text{ or } 355)+d_2 \quad \Rightarrow \quad \Delta d=d_2-d_1=(365 \text{ or } 366)-(354 \text{ or } 355)$$

then Δd or displacement of days of the seasons in the Islamic calendar can be 10, 11, or 12 days; 11 being the most frequent interval, since the years of 365 days and 354 days are the most common.

25. Eight year calendar

From the formula (3), we find that if the lunar cycle is composed of 8 years, the number of embolismic years is 3. Of the equations (4) it is found that the months of 30 and 29 days are

$$m_{30}=51; \quad m_{29}=45$$

96 months in total, which means that the calendar lunation is

$$m_m=\frac{354A+B}{12A}=29^d.531\,25=29^d\,12^h\;45^m\,0^s$$

$57^s.12$ higher than the average astronomical duration for the year 2000.

The average duration of the calendar year based on the 8-year cycle is

$$\frac{51\cdot 30+45\cdot 29}{8}=354^d.375=354^d\,9^h.$$

The number of days of the 8-year cycle is 2,835, which is 405 whole weeks. This means that after 8 years, the dates are repeated on the same calendar days, that is, the 8-year cycle is a perpetual calendar.

The order of intercalation of embolisms is calculated by (13). Since there are three possible values for β (0, 1 and 2) then there are three possible intercalation criteria. If $\beta=0$ the embolismic years are 3, 6 and 8. When $\beta=1$ the embolisms are years 2, 5 and 7 and finally if $\beta=2$ the abundant years are 1, 4 and 6.

To calculate the error in the calendar of the 8-year cycle, we use the same technique like the one described in section 20. The first is to express the synodic month on the periods of 8 years. Equation (40) is modified so that it is

$$T=\frac{JD-JD_0}{36\,525}=\frac{JD-JD_c}{2\,835}\frac{2\,835}{36\,525}+\frac{JD_c-JD_0}{36\,525}=\nu P+\mu$$

the numerical values are

$$\nu=0.077\,618; \quad \mu=-13.774\,264.$$

When P cycles of 8 years have elapsed since the Hijrah the synodic month is

$$m_s=p+qT=(p+q\mu)+q\nu P.$$

The calendar error in a cycle of 8 years or 2,835 days is

$$E=2\,835-96m_s=\left[2\,835-96(p+q\mu)\right]-96q\nu P \tag{51}$$

96 are the months in the 8-year cycle. Assuming we use the terrestrial time scale the error is

$$E=0^d.063\,756-1.611\,722\cdot 10^{-6}P.$$

The accumulated error in N periods of 8 years is

$$\begin{aligned}E(N)&=N\left[2\,835-96(p+q\mu)\right]-48q\nu N(N+1)=\\&=0^d.063\,756N-8.058\,611\cdot 10^{-6}N(N+1),\end{aligned}$$

in 15 cycles of 8 years or 120 years, the accumulated error is $22^h\,59^m\,55^s$, very close to 24 hours.

To calculate the error accumulated in the universal time scale, we use the (47)

$$\begin{aligned}E(N)&=N\left[2\,835-96(p'+q'\mu)\right]-48q'\nu N(N+1)=\\&=0^d.063\,090N+1.355\,769\cdot 10^{-6}N(N+1)\end{aligned} \tag{52}$$

in 15 cycles, the accumulated error from the Hijrah calendar and astronomy is $22^h\,42^m\,16^s.5$.

The results indicated above suggest that the 8-year cycle the correction be made after 15 periods of 8 years or 120 years, or every 128 years with a better fit to astronomy. By (52), we verify that the accumulated error is positive; that is, more days than necessary are placed on the calendar. Therefore, the correction is to remove one day from the calendar every 120 years.

We will derive the accumulated error formula according to the cycles of 120 years elapsed, but now using the universal time scale. In 120 years, there are 15 cycles of 8 years, 1,440 months and the number of days is

$$15 \times 2\,835 - 1 = 42\,524$$

that is 15 periods of 30 years minus one day, which corresponds to the correction that must be made in 120 years. The average lunation over 120 years is $29^d\,12^h\,41^m$, about three minutes smaller than the astronomical lunar year 2000.

To express the synodic month according to the 120-year P periods elapsed since the Hijrah, we have to express the Julian centuries in periods of 120 years, repeating the calculation (40)

$$T = \frac{JD - JD_c}{42\,524}\frac{42\,524}{36\,525} + \frac{JD_c - JD_0}{36\,525} = \nu P + \mu$$

with numerical values

$$\nu = 1.164\,244; \quad \mu = -13.774\,264,$$

then the synodic month is

$$m_s = p + qT = \left(p + q\mu\right) + q\nu P.$$

The error in 120 years in the period P counted from the beginning of the calendar is

$$E = 42\,524 - 1\,440 m_s = \left[42\,524 - 1\,440\left(p + q\mu\right)\right] - 1\,440 q\nu P =$$
$$= -0^d.043\,654 - 3.626\,294 \cdot 10^{-4} P,$$

the accumulated error at the end of N cycles of 120 years we deduce from (52)

Years of the cycle	Muharram	Safar	Rabi I	Rabi II	Jumanda I	Jumanda II	Rajab	Shaban	Ramadhan	Shawwal	Zul-Qida	Zul-Hijja
1	3	5	6	1	2	4	5	7	1	3	4	6
2	7	2	3	5	6	1	2	4	5	7	1	3
3	5	7	1	3	4	6	7	2	3	5	6	1
4	2	4	5	7	1	3	4	6	7	2	3	5
5	6	1	2	4	5	7	1	3	4	6	7	2
6	4	6	7	2	3	5	6	1	2	4	5	7
7	1	3	4	6	7	2	3	5	6	1	2	4
8	6	1	2	4	5	7	1	3	4	6	7	2

Table 11.- *Al-Biruni's perpetual calendar of the 8-year cycle. The first column is the order of the years in the 8-year cycle. The embolismic years are in italics. The remaining columns are the signatures of each month, that is, the weekly day of the first day of each month. Day 1 is Sunday.*

$$E(N) = N\left[42\ 524 - 1\ 440(p' + q'\mu)\right] - 720\,q'\nu N(N+1) = \\ = -0^d.053\ 650N - 3.050\ 412\cdot 10^{-4} N(N+1). \tag{53}$$

The accumulated error will be one day in cycle 17, that is when 2,040 years have passed since the Hijrah. Therefore a slightly higher error than in the 30-year cycle calendar, which reaches the error day after 2,580 years. Note that error (53) is negative; that is, the astronomical lunation is higher than that of the calendar, contrary to what happens with the 8-year calendar.

We know that the 8-year cycle calendar was used in Asia Centra (Table 11), Baghdad (in the ninth or tenth century), Tibet (fourteenth century), Turkey (eighteenth and nineteenth centuries), Cairo (nineteenth century) and in Southeast Asia until the 20th century.

Table 11 shows the perpetual calendar described by al-Biruni in the 10th century [29], whose intercalar years are located in positions 2, 5, and 7 of the cycle. The table gives the weekly day of each month of the year according to its order in the 8-year cycle. Al-Biruni says that to determine the position of the year in the cycle we have to use the formula

$$n = (Y_L + 4)\bmod 8.$$

that is to say, that the first year of the calendar, or year 1 H, should have number 5 of the 8-year cycle, because that day, as seen in Table 11, was Friday as corresponds to July 16, 622 CE. But the logical is that the first year of the calendar coincided with the first day of the 8-year cycle. This leads us to think that the perpetual calendar that al-Biruni gives is the result of several corrections in the past.

	Signature of the years						
Years of the cycle	1º period (1-120)	2º period 121-240	3º period (241-360)	4º period (361-480)	5º period 481-600	6º period 601-720	7º period 721-840
1	6	5	4	3	2	***1***	***7***
2	3	2	***1***	***7***	***6***	6	5
3	***7***	***6***	6	5	4	3	2
4	5	4	3	2	***1***	***7***	***6***
5	2	***1***	***7***	***6***	6	5	4
6	***6***	6	5	4	3	2	***1***
7	4	3	2	***1***	***7***	***6***	6
8	***1***	***7***	***6***	6	5	4	3

Table 12.- *Possible correction system by remodulation of the 8-year calendar. The numbers give the signature associated with each year of the 8-year cycle according to the 120-year period in which they are found. The periods are counted from the Hegira. In the first period, the first day of the cycle has the signature 6, which means that the first day of the year 1 H was Friday, as it should be. After 120 lunar years, one day must be removed from the calendar. We assume that the elimination of this day occurs in the last year of the 8-year cycle. The year 121 should start its first year by 6 (according to the sequence of the signature), but in reality, it will start a weekly day before, that is, at 5, since at the end of the year 120, the correction was made. And the same for the following 120 year periods. After the seventh period, the order of the signature is repeated. The sequence of the signature is only altered when the correction is made at the end of each period. But after this alteration, the signature continues in the same order, that is, 1, 6, 3, 7, 5, 2, 6, and 4. The embolismic years are always those with the signature 1, 7, and 6.*

Fourth 120 year period					
Islamic day	Order in the 8-year cycle	Day weekly in the 8-year cycle	Julian day 30-year cycle	Day weekly in the 30-year cycle	Difference between calendars
1-1-378	2	7	2 082 036	7	0
1-1-388	4	2	2 085 580	2	0
1-1-398	6	4	2 089 124	4	0
1-1-408	8	6	2 092 667	5	+1
1-1-418	2	7	2 096 211	7	0
Fifth 120 year period					
Islamic day	Order in the 8-year cycle	Day weekly in the 8-year cycle	Julian day 30-year cycle	Day weekly in the 30-year cycle	Difference between calendars
1-1-498	2	6	2 124 560	6	0
1-1-508	4	1	2 128 104	1	0
1-1-518	6	3	2 131 648	3	0
1-1-528	8	5	2 135 191	4	+1
1-1-538	2	6	2 138 735	6	0
Seventh 120 year period					
Islamic day	Order in the 8-year cycle	Day weekly in the 8-year cycle	Julian day 30-year cycle	Day weekly in the 30-year cycle	Difference between calendars
1-1-738	2	5	2 209 608	4	+1
1-1-748	4	6	2 213 152	6	0
1-1-758	6	1	2 216 696	1	0
1-1-768	8	3	2 220 239	2	+1
1-1-778	2	5	2 223 783	4	+1

Table 13.- *The second column is the order of the date in the 8-year cycle of al-Biruni. The third column is the weekly day, according to the 120 year period (Table 12). The Julian day is calculated assuming that the Islamic date corresponds to the 30-year calendar and from which we obtain the weekly day. The last column is the difference between the weekly days of the two calendars. Sometimes the dates do not match, but as the weekly day must be the same, therefore the dates in both calendars are different.*

If we take into account that al-Biruni wrote around the year 1000, we must assume that your calendar corresponds to the fourth period of 120 years, since $3 \times 120 = 320$ and $320 + 622 \approx 1000$, therefore when writes al-Biruni, the calendar should be in the fourth 120-year period. Admitted this hypothesis, during the first period the first day of the 8-year cycle would begin with Friday; as it should be so that the first day of the year of the Hijrah is Friday.

When a correction is made to the 8-year calendar, one day must be removed, so the weekly days of the beginning of the years are reduced by one. Then the new perpetual calendar of the 8-year cycle is the same as the previous period, but with all the numbers decreased by one unit.

Table 12 shows the perpetual calendars valid for seven periods of 120 years, where the weekly days of the first day of each year are given. The year of the Hijrah was year 1 of the 8-year cycle and also the beginning of the first 120-year period, so, as seen in table 13, the first day of that year was 6, that is, Friday, since we follow the practice of starting the week with Sunday. At the end of the seventh period of 120 years, it returns to the initial calendar, in which the first year begins on Friday.

There is no certainty that this 8-year calendar will work, as we have indicated. That is to say; it is unknown if there was a defined rule to correct, or it was done when the observation noted that the calendar was not following astronomy.

We make a comparison between the calendars of 30 years and 8 years. As the criteria for making corrections in the 8-year cycle is unknown, we will follow the following procedure to compare your dates. We choose a date, for example, 1 Muharram 378 H (year 999 CE); we find the order of the year in the 8-year cycle, dividing 378 by 8 and seeing the rest, resulting in 2; then by table 11 (which we consider valid at that time) we find that the first day of the year is 7, that is Saturday. By (24), we find the Julian day of the date, finding $JD = 2{,}082{,}036$, and by (38), we determine the corresponding weekly day, which is 1 (Sunday). Therefore we check that the difference between the two calendars is one day. So the date of Muharram 1 of 378 H of the 30-year calendar corresponds to the 2 of Muharram of the year 378 H according to the al-Biruni calendar (which is Sunday, since the day before was Saturday).

In Table 13, we have collected the comparison between various dates in the 30 and 8 years calendars, where it can be seen that in some years, there is a difference of one day between the two dates.

Instead of a perpetual calendar, like the one in table 11, in other places where the 8-year calendar was used, another procedure was used to establish the calendar for a given year. Each month was associated with a number, which we have called regular monthly (see section 4 and Table 14). Each year it has a number called a signature (see section 4) that is used to determine the day of the week of the first day of the year.

The sequence of the cycle signature in the Southeast Asian calendar is 1, 5, 3, 7, 4, 2, 6, 4, somewhat different from the sequence of the al-Biruni calendar. The starting year of this sequence may change for reasons that we will see later. With the previous order, the abundant years are 5, 4, and 6, that is, the second, fifth, and seventh. Let us realize that after a typical year, the weekly day of the first day of the year shifts four days, the days that exceed 354 days an integer number of weeks. Still, if the year is embolismic, the displacement is five days, that is, the excess of 355 days of the abundant year over an entire number of weeks. We see that they increase in five days the second year

$$5\,(\text{signature}) + 5(\text{excess wek}) = 10 \equiv 3$$

for the fifth year $4 + 5 = 9 \equiv 2$ and for the seventh year $6 + 5 = 11 \equiv 4$, therefore years 2, 5 and 7 are embolismic.

To complete determination of the calendar is necessary to know the calibration of the signature, that is, the relationship between numbers of signature and days of the week. For example, if the number 1 is associated with Sunday, then 2 would correspond to Monday and 7 to Saturday. But note that this association may vary for reasons that we will see later.

The table of the monthly regulars is unalterable and is always Table 14. To find the calendar for a year, we need to know the association between the years of the cycle and the signatures of the years and also know the number at which the week begins, that is the calibration or the number we associate with Sunday.

Regular monthly											
Muharram	Safar	Rabi I	Rabi II	Jumanda I	Jumanda II	Rajab	Shaban	Ramadhan	Shawwal	Zul-Qida	Zul-Hijja
7	2	3	5	6	1	2	4	5	7	1	3

Table 14.- *Regular monthly of the twelve months of the year. Adding the regular with the signature, we obtain the weekly day of the first day of each month. The sequence of the regular lunar indicates that the months are alternately 30 and 29 days, with the first month having 30 days.*

In the 8-year calendar that Ideler collected in Turkey [30] at the beginning of the 19th century, there are Table 14 of the monthly regulars, he identifies 1 as Sunday and to find the signature, table 15 is used, which was valid in the year 1824 CE or 1240 H. This calendar is supposed to have been established by Mehemed Efendi in the 16th century. This means that table 15, which gives us the signature, must have previously had two corrections, and therefore that table was valid in the 120-year period in which the year 1240 H was found.

In Southeast Asia, an 8-year calendar was also in use, with the same structure as those described previously [31]. Like the previous ones, this calendar required a correction approximately every 120 years or so. For these Southeast Asian calendars, the criteria for deciding when the correction was made is unknown, but most likely, it was due to observation and not a mathematical rule.

had been eliminated by the correction. One method was the one we have already referred to in the calendars already analyzed, which is called remodulation and which consists, as we have seen, in making a jump in the signature of the year, but maintaining the numerical value that is associated with each weekly day. But another procedure that was used when the correction was made was called recalibration, with this method the sequence of the signatures of the years was left unchanged, but the number that was given to the weekly days was modified, that is to say, it changes the calibration of the signature For example, if 1 corresponds to Sunday, when recalibration 1 becomes Saturday.

Years of the cycle	Signature of the year
1	3
2	7
3	***4***
4	2
5	***6***
6	4
7	1
8	***5***

Table 15.- *Signature of the year according to its position in the 8-year cycle in a 19th-century Turkish calendar. The embolismic years are those in italics. The table was valid for the year 1824 CE or 1240 H. We note that the sequence is different from al-Biruni's calendar.*

26. Appendix.- Non-regular tabular Islamic calendars. Semi-regular calendars

We have considered regular arithmetic or computational calendars, in the sense that the embolismic years are regularly distributed, with this technique the dispersion of the dates of the tabular calendar with respect to the observational or astronomical calendar is small.

However, it is possible to devise calendars in which the abundant years are not regularly distributed in the cycle. If we limit ourselves to a 30-year cycle of which 11 are abundant, the number of different intercalation criteria for embolisms are the combinations without repetition of 30 elements taken from 11 to 11, that is, the number of groups that can be formed of 11 numbers taken from the 30 of the cycle, without repeating any of them. The numbers in these series of 11 numbers represent the positions of the years that are embolismic in the 30-year cycle.

Using the formula of the combinations without repetition of 30 elements taken from 11 to 11 we find

$$C_{30,11} = \binom{30}{11} = \frac{30!}{11!(30-11)!} = 54\ 627\ 300$$

this is the total number of different calendars that can be formed with a 30-year cycle, with 11 of them embolismic.

But to avoid calendars that deviate significantly from astronomy because they are very irregular, we limit ourselves to those calendars in which the embolismic years follow one another in intervals of two or three years. We will call these calendars semiregular because the interval between embolismic years is regular. In section 8 we have exposed that the distribution of the embolismic years according to the separation between them in the case of a regular distribution follows the pattern 233-2333-2333, from which we deduce that there are two sub-cycles in the 30-year cycle (the 233 and 2333).

To analyze the non-regular calendars obtained with the previous hypothesis, we assume that any of these calendars is formed by three sub-cycles and in each of these sub-cycles there is a single two-year interval between embolisms, so the possible sub-cycles are

$$\begin{aligned} &S_1 \equiv 2 \\ &S_2 \equiv 23 \\ &S_3 \equiv 233 \\ &S_4 \equiv 2333 \\ &S_5 \equiv 23333 \\ &\ldots\ldots. \\ &S_9 \equiv 233333333 \end{aligned}$$

S_1 is a cycle with two years, the second of which is embolismic; S_2 is 5 years, the second and the fifth being embolismic, and so on. The embolismic year patterns are formed by three of these subcycles, and must always be formed by 11 embolismic years.

Of the multiple possibilities that can occur, the closest to regularity is the calendars formed with the S_3, S_3, and S_5 subcycles, that is, the intervals between embolisms have the pattern 233-233-23333.

The following types of calendars are derived from the previous sequence of embolismic years

$$\begin{aligned} &2\,3\,3-2\,3\,3-2\,3\,3\,3\,3 \rightarrow I \\ &3-2\,3\,3-2\,3\,3-2\,3\,3\,3 \rightarrow II \\ &3\,3-2\,3\,3-2\,3\,3-2\,3\,3 \rightarrow III \\ &3\,3\,3-2\,3\,3-2\,3\,3-2\,3 \rightarrow IV \\ &3\,3\,3\,3-2\,3\,3-2\,3\,3-2 \rightarrow V \\ &2\,3\,3\,3\,3-2\,3\,3-2\,3\,3- \rightarrow VI \\ &3-2\,3\,3\,3\,3-2\,3\,3-2\,3 \rightarrow VII \\ &3\,3-2\,3\,3\,3\,3-2\,3\,3-2 \rightarrow VIII \\ &2\,3\,3-2\,3\,3\,3\,3-2\,3\,3- \rightarrow IX \\ &3-2\,3\,3-2\,3\,3\,3\,3-2\,3 \rightarrow X \\ &3\,3-2\,3\,3-2\,3\,3\,3\,3-2 \rightarrow XI \end{aligned}$$

eleven types in total, each of them admits three subtypes, according to the year of the cycle in which the first embolism is, whether it is the first, second, or third year of the cycle.

A calendar with this pattern was proposed by Ibn Futuh "Sevillian" in the 13th century, with the intercalation criteria 2, 5, 8, 10, 13, 16, 18, 21, 24, 26 and 29, which is type *IV-a* [11]. The calendar proposed by Rashed, Moklof, and Hamra, which we will call RMH [33] [34], is a semi-regular calendar of the *III-b* style. The sequence of the embolismic years is 2, 5, 7, 10, 13, 15, 18, 21, 23, 26, and 29 [36]. The RMH calendar is the same as that of al-Battani except that instead of 16 and 24, are embolismic the years 15 and 23.

To check the regularity of a calendar and therefore its adjustment to astronomy, we calculated the standard deviation of the sample formed by the deviations of the calendar from its average year, that is, the subtraction of the days elapsed in the 30-year cycle and the days that would have elapsed if the years had had the same duration as the average calendar year.

$D(n)$ represents the days elapsed until the end of year n of the 30-year cycle, and $12m_n$ is the average duration of the calendar year, with m_n being the calendar lunation, so we define the parameter

$$\Gamma(n) = D(n) - 12nm_m \tag{54}$$

the standard deviation is

$$\sigma = \sqrt{\frac{\sum_{n=1}^{n=30} \left\{ \left[\Gamma(n) \right] - \overline{\Gamma} \right\}^2}{30}}. \tag{55}$$

Another procedure to determine the regularity of a calendar is the mean of the absolute deviation

$$d = \frac{\sum_{n=1}^{n=30} \left| \Gamma(n) \right|}{30}. \tag{56}$$

The numerical results are

$$\overline{\Gamma}_{RMH} = 0.05 \quad \Rightarrow \quad \sigma_{RMH} = 0.292197; \quad d_{RMH} = 0.254\,444$$
$$\overline{\Gamma}_{B} = -0.01\overset{\frown}{6} \quad \Rightarrow \quad \sigma_{B} = 0.289\,907; \quad d_{B} = 0.249\,999$$

the RMH subscript refers to the Rashed, Moklof, and Hamza calendar, and the B subscript corresponds to the al-Battani calendar. The previous calculation shows us that the dispersion is greater in the first calendar, which is indicative that it has less regularity [37].

27. Conclusions

We investigated the arithmetic, computational or tabular Islamic calendars, that is, formed by lunar years, of alternative months of 30 and 29 days. We find in epigraphs 3, 4, 5, and 7 algorithms that define the regular lunar calendars, from which we choose the one formed by 30-year cycles, of which 11 are embolismic or abundant, which are those that have one day more than the usual years.

The pattern of embolismic years characterizes tabular calendars, that is, the sequence of the intervals between abundant years. We chose the pattern that makes the regular calendar, that is, the embolisms are more homogeneously located in the 30-year cycle. From this pattern, 11 types of intercalations are deduced, each one giving rise to three styles, according to the weekly day in which they begin: the first, second, or third of the cycle. For the regular 30-year calendar, we find in epigraph 9 thirty different calendars, each one characterized by the sequence of the embolismic years.

In epigraphs 10 and 11, we investigate the pre-Islamic calendar, which was in use in Arabia until the 10th year of the Hijrah. In that year, the Prophet prohibited the intercalation of months. Therefore, the lunisolar calendar became lunar.

To do the conversion between the Islamic arithmetic calendar and the Julian or Gregorian calendar and vice versa, we use the chronological Julian day, which allows us to find algorithms that are easy to use and that can be applied to computer programs. The computational Islamic lunar calendar is very close to astronomy, and we found that in ancient times it is a maximum of one day away from the first observation of the crescent moon. In epigraphs 20 and 21, we calculate the error accumulated by the tabular calendar, using both terrestrial time and universal time.

As the Islamic calendar is exclusively lunar, the seasons dates are not fixed on the calendar. In section 12, we find that the seasons can move 10, 11, or 12 days each year.

In section 25, we study the Islamic calendar with an 8-year cycle. It is astronomically less accurate than the 30-year cycle calendar, so it requires frequent corrections; approximately every 120 years, it is necessary to remove one day from the 8-year calendar. Two methods have we used to correct, remodulation (the most frequent procedure) and recalibration.

In the appendix, we extend our research to non-regular calendars. The total number of 30-year cycle calendars exceeds fifty-four million. But the number is reduced for semi-regular calendars, in which the embolismic years follow each other after two or three years.

To evaluate the degree of regularity of a calendar we have used two techniques: the standard deviation of the dispersions of the calendar with respect to the average year and the absolute deviation of these same values [35].

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$$q = a_L - 11.8371 + 6.3226\,\omega_{max} - 0.7319\omega_{max}^2 + 0.1018\,\omega_{max}^3$$

if *q* is positive, the crescent will be visible. The above formula applies at the time of sunset.
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$$J_1 = \text{int}\left(\frac{11\,340Y_I' + 12}{32}\right) \tag{57}$$

something different from (21). To find the coefficient *R* that multiplies Y_I' we apply the formula

$$R = \frac{1}{n}\text{cint}\left(nr\right) \tag{58}$$

where *r* is the slope of the adjustment line of the numerical values obtained by subtracting the days elapsed in the cycle and the days that would have passed if the duration of the year had its average length, and *n* is an integer. To apply (57), we previously found $r = 354.375\,059$ and we verified that with $n = 32$, the value of *R* found in (58) fulfills the conditions of Zeller's theorem. Wenceslao Segura González, *Hemerología. La ciencia de los calendarios,* op. cit., pp. 1257-258.

With (57) the Julian day of a date of the Islamic auxiliary calendar is calculated, using a formula equivalent to (24)

$$JD = \mathrm{int}\left(\frac{11\,340 Y_I' + 12}{32}\right) + \mathrm{int}\left(\frac{59 M_I' + 1}{2}\right) + D_I' - 7\,664,$$

Y_I', M_I', D_I' is the auxiliary calendar date calculated by (20).

[37] The standard deviation varies from a calendar to another calendar. In Table 16, we make the statistical calculations for regular calendars of types *a, b*, and *c* (section 8).

Calendar type	$\overline{\Gamma}$	d	σ
VII-a	0.343	0.366	0.273
VI-b	0.083	0.257	0.288
VII-c	-0.383	0.397	0.321

Table 16.- *In the first column is the regular calendar type. The second column is the mean value of the differences in the days elapsed in the cycle and the days that would have passed if the years had the same duration as the average year (54). The third column is the absolute deviation of the previous values (56), and the last column is the standard deviation (55).*