Three-Dimensional Flow Impinging Obliquely on a Plane Wall

A . Dowgialo

Tuvinian Institute for Exploration of Natural Resources
SD RAS, Russia

An exact solution of the Navier-Stokes equation is given which represents steady three-dimensional flow of a viscous fluid impinging on a plane wall obliquely. Numerical discussions of the relevant functions as well as the structure of the flow field are made. A comparison with an existing theory is also given.

Key words. Navier-Stokes equations, exact solutions.
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Unlike a two-dimensional flow obliquely running against a flat wall, a three-dimensional flow occurs when the flow tilts not only in the direction perpendicular to the flow, but also in its own plane. This article gives an exact solution to the Navier-Stokes equation, which generalizes the Hiemenz flow [1] to the case of a stationary three-dimensional flow running on a flat wall.

For reasons related to the nonlinearity of the Navier-Stokes equation, only two true three-dimensional flows are currently known, which their exact solutions are. This is the Homan flow [2], finalized by Karman [3] for the case of a rotating disk and the Slezkin conical jet [4], and generalized to the case of swirling flow by Gol'dshtik [5] and Yih [6].

Although the Karman solution was obtained in cylindrical coordinates and the Slezkin solution in spherical coordinates, both of these solutions can be assigned to the same class of solutions of the Navier-Stokes equation. In both cases, to construct the solution, the corresponding ideal fluid flow is used as a basis, which is also a solution of the Navier-Stokes equation, which is nonlinear, and of a simpler, linear vortex-free equation

(1) $$\vec{\Omega} = \vec{\nabla} \wedge \vec{u}^{\,0} = \vec{0},$$

in which the velocity field $\vec{u}^{\,0}$ represented as a vector product of the gradients of its integral surfaces $\psi_i^0, i = 1,2$

(2) $$\vec{u}^{\,0} = \vec{\nabla}\psi_1^0 \wedge \vec{\nabla}\psi_2^0$$

With a special view of surfaces ψ_i^0

(3) $$\psi_1^0 = f_0^0(x_1) + f_1^0(x_1) f_2^0(x_2), \psi_2^0 = x_3 - f_3^0(x_1)$$

the variables in equation (1) are separated, which makes it possible to reduce it to a system of ordinary differential equations.

Further, in order to extend the ideal solution (3) to the case of a viscous flow, the form of the function $f_2^0(x_2)$ is preserved, and the remaining functions are searched again, assuming their asymptotic desire for their "ideal" analogues:

(4) $$\psi_1 = f_0(x_1) + f_1(x_1) f_2^0(x_2), \psi_2 = x_3 - f_3(x_1)$$

As a result, the solution of the Karman problem in cylindrical coordinates $[x_1, x_2, x_3] \to [r \cdot \cos(o), r \cdot \sin(o), z]$ is representable in the form

(5) $$\psi_1 = f_1(z) \cdot r^2, \psi_2 = o - \int \frac{f_3(z)}{f_1(z)} dz$$

and the Slezkin's solution in spherical coordinates $[x_1, x_2, x_3] \to [r \cdot \cos(o), r \cdot \sin(o), z]$

(6) $$\psi_1 = f_1(o) \cdot r, \psi_2 = o_1 - \int \frac{f_3(o)}{f_1(o)} do$$

We choose the Cartesian coordinates (x, y) in the plane of the wall and the axis z in the perpendicular direction. The inviscid version of the current stream is very simple, and given in terms of stream $\psi_i^0, i = 1,2$ functions in the form (3), after substituting into the stationary Navier-Stokes equation

(7) $\qquad \vec{\Omega} = \vec{\nabla} \wedge \vec{u} = \vec{0}$

differential equations for $f_0^0(z), f_1^0(z), f_2^0(x), f_3^0(z)$

$$\frac{d^2 f_2^0}{dx^2} = 0,\ f_0^{0''} = 0,\ f_1^{0''} = 0,\ f_3^{0'} = 0.$$

As a result, for an ideal fluid flow we have

(3') $\qquad \psi_1^0 = az^2 + x \cdot z, \psi_2^0 = y - b\ln(z)$

where a and b are scale constants. The flow consists of a vortex-free flow on the stagnation line and a uniform shear flow parallel to the wall (Fig. 1)

(2') $\qquad \vec{u}^0 = [-2 \cdot az - x, b, z]$

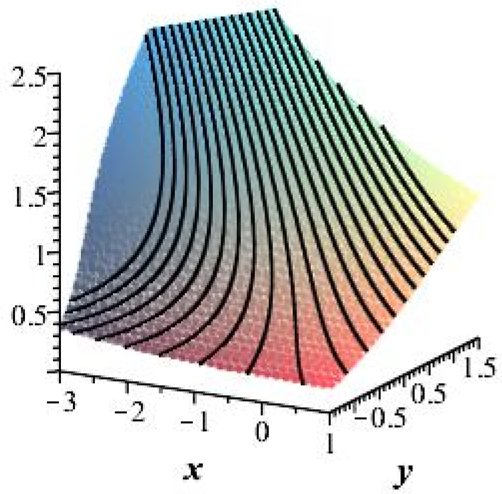


Fig. 1

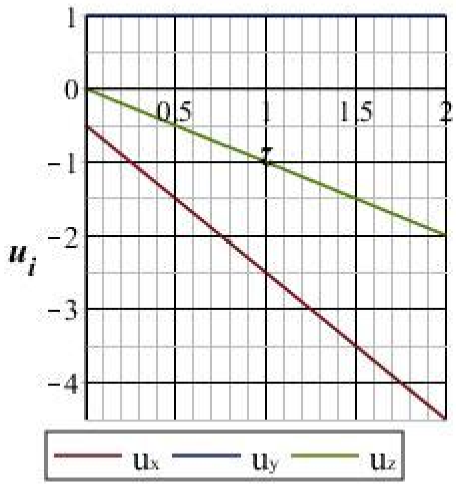


Fig. 2

If fluid viscosity is taken into account, a boundary layer appears along the wall. We assume a generalization of (3') in the form of (4), assuming $f_2^0 = x$:

(4') $\qquad \mathrm{Re} \cdot \psi_1 = f_0(z) + f_1(z) \cdot x, \psi_2 = y - \int \frac{f_3(z)}{f_1(z)} dz,$

where $\mathrm{Re} = \frac{u_s l}{\nu}$.

Then the stationary Navier-Stokes equation

(7') $\qquad \vec{\nabla} \wedge \left(\mathrm{Re} \cdot \vec{\Omega} \wedge \vec{u} + \vec{\nabla} \wedge \vec{\Omega}\right) = \vec{0}, \vec{\Omega} = \vec{\nabla} \wedge \vec{u}$

gives ordinary differential equations for $f_0(z), f_1(z), f_3(z)$.

Thus

(8) $$f_1''' - f_1 f_1'' + (f_1')^2 - 1 = 0\,,$$

(9) $$f_0'' - f_1 f_0' + f_1' f_0 + c_0 = 0\,,$$

(10) $$f_3''' - f_3'' f_1 - f_3' f_1' = 0\,,$$

where the dashes denote differentiation by z. Suitable boundary conditions follow from the expression for the flow rate

(11) $$\mathrm{Re}\cdot\vec{u} = \left[-f_0 - x\cdot f_1', f_3, f_1\right],$$

and have the form:

$$f_0(0) = 0, f_1(0) = 0, f_1'(0) = 0, f_3(0) = 0$$

$$f_0(\infty) = v_0, f_1(\infty) \to z, f_1'(\infty) = 1, f_3(\infty) = v_3.$$

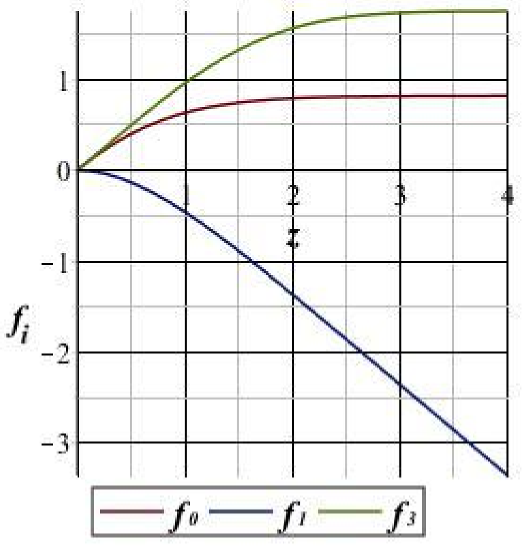
Equation (8) is the familiar Himenz equation. Equation (9) is a linear homogeneous Stuart equation [7] for $f_0(z)$. The function $f_3(z)$ satisfying the linear differential equation (10) gives a three-dimensional character to the flow

$$f_3(z) = c_3^0 + c_3^1 \cdot \int e^{-\int -f_1 dz} dz + c_3^2 \cdot \int\left(\int e^{\int -f_1 dz} dz\right) e^{-\int -f_1 dz} dz\,.$$

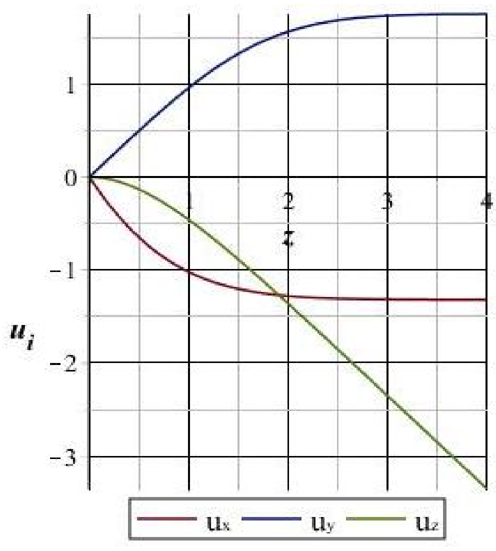
Since it is known that $f_1(z) \to z, z \to \infty$, equations (8) and (9) can be approximated for large z as

$$f_0(\infty) \to z\left(1 - erf\left(\frac{z}{\sqrt{2}}\right)\right) + v_0 \quad f_3(\infty) \to v_3 erf\left(\frac{z}{\sqrt{2}}\right)$$

The solutions of equations (8) - (10) and the components of the velocity field are presented in Figs. 3 and 4.



Фиг.3



Фиг.4.

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