Lower bound on a special type of cyclic sums

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"Entia non sunt multiplicanda praeter necessitatem" (Ockam, W.)

"Dios no juega a los dados con el Universo" (Einstein, Albert)

"Te doy gracias, Padre, porque has ocultado estas cosas a los sabios y entendidos y se las has revelado a la gente sencilla" (Mt 11,25)

Abstract

In this brief paper it is proved a theorem regarding the relative value of the cyclic sums of $f(x) = \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}}$ and the sum of its variables.

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1 Introduction

We define a cyclic sum $\sum_{cyc} f(a_1, a_2, ..., a_n)$ as equal to $f(a_1, a_2, ..., a_n) + f(a_2, a_3, ..., a_n, a_1) + f(a_3, a_4, ..., a_1, a_2) + ... + f(a_n, a_1, ..., a_{n-1})$. Therefore, all the variables are cycled through.

We are interested in studying the sum

$$\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} = \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} + \frac{a_2^{\alpha+1}}{k_1 a_2^{\alpha} + k_2 a_3^{\alpha}} + ... + \frac{a_n^{\alpha+1}}{k_1 a_n^{\alpha} + k_2 a_1^{\alpha}}$$

And its relative value compared to

$$\sum_{k=1}^{n} a_k$$

In this regard, in this paper it is proposed and proved the following theorem:

Theorem.

$$\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \geq \frac{\sum_{k=1}^{n} a_k}{k_1 + k_2}$$

2 Proof

2.1 Previous Lemmas

We will need firstly the following

Lemma 1.

$$\sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^{\alpha}$$

Proof.

Applying the Rearrangement inequality on $a_1, a_2, ..., a_n$ and $a_1^{\alpha}, a_2^{\alpha}, ..., a_n^{\alpha}$, we have that $\sum_{cyc} a_1 a_2^{\alpha}$ is maximized when $a_1, a_2, ..., a_n$ and $a_1^{\alpha}, a_2^{\alpha}, ..., a_n^{\alpha}$ are similarly sorted.

Therefore, we can affirm that

$$\sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^{\alpha}$$

Other hand, we need the following

Lemma 2.

$$\sum_{cyc} \frac{a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} = \frac{\sum_{k=1}^{n} a_k}{k_1}$$

Proof.

If we establish that

$$\frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} + \frac{n}{m} = \frac{a_1}{k_1}$$

Operating, we get that

$$\frac{a_1^{\alpha+1}m + n\left(k_1a_1^\alpha + k_2a_2^\alpha\right)}{m\left(k_1a_1^\alpha + k_2a_2^\alpha\right)} = \frac{a_1}{k_1}$$

$$k_1\left(a_1^{\alpha+1}m + n\left(k_1a_1^\alpha + k_2a_2^\alpha\right)\right) = a_1m\left(k_1a_1^\alpha + k_2a_2^\alpha\right)$$

$$k_1a_1^{\alpha+1}m + k_1n\left(k_1a_1^\alpha + k_2a_2^\alpha\right) = a_1mk_1a_1^\alpha + a_1mk_2a_2^\alpha$$

$$k_1n\left(k_1a_1^\alpha + k_2a_2^\alpha\right) = a_1mk_2a_2^\alpha$$

$$\frac{n}{m} = \left(\frac{k_2}{k_1}\right)\frac{a_1a_2^\alpha}{k_1a_1^\alpha + k_2a_2^\alpha}$$

Therefore, we have that

$$\frac{a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right)a_1a_2^\alpha}{k_1a_1^\alpha + k_2a_2^\alpha} = \frac{a_1}{k_1}$$

And subsequently, repeating the process for each variable, we get that

$$\sum_{cyc}\frac{a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right)a_1a_2^\alpha}{k_1a_1^\alpha + k_2a_2^\alpha} = \frac{\sum_{k=1}^n a_k}{k_1}$$

2.2 Proof

Applying Lemma 1, we derive that

$$\left(\frac{k_2}{k_1}\right)\sum_{k=1}^n a_k^{\alpha+1} \geq \left(\frac{k_2}{k_1}\right)\sum_{cyc} a_1a_2^\alpha$$

Therefore, substituting in the expression of Lemma 2 and operating, we have that

$$\sum_{cyc}\frac{a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right)a_k^{\alpha+1}}{k_1a_1^\alpha + k_2a_2^\alpha} \geq \frac{\sum_{k=1}^n a_k}{k_1}$$

$$\sum_{cyc}\frac{\left(\frac{k_1}{k_1}\right)a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right)a_k^{\alpha+1}}{k_1a_1^\alpha + k_2a_2^\alpha} \geq \frac{\sum_{k=1}^n a_k}{k_1}$$

$$\sum_{cyc} \frac{\left(\frac{k_1+k_2}{k_1}\right) a_k^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \geq \frac{\sum_{k=1}^{n} a_k}{k_1}$$

$$\sum_{cyc} \frac{a_k^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \geq \frac{\sum_{k=1}^{n} a_k}{\left(\frac{k_1+k_2}{k_1}\right) k_1}$$

$$\sum_{cyc} \frac{a_k^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \geq \frac{\sum_{k=1}^{n} a_k}{k_1 + k_2}$$

As we wanted to prove.