Lower bounds on a special type of cyclic sums

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"Entia non sunt multiplicanda praeter necessitatem" (Ockam, W.)

"Dios no juega a los dados con el Universo" (Einstein, Albert)

"Te doy gracias, Padre, porque has ocultado estas cosas a los sabios y entendidos y se las has revelado a la gente sencilla" (Mt 11, 25)

Abstract

In this brief paper it is proved a theorem regarding the relative value of the cyclic sums of \( f(x) = a_{\alpha+1}^{a_1} + a_{\alpha+1}^{k_1 a_1^\alpha + k_2 a_2^\alpha} + a_{\alpha+1}^{a_2} + a_{\alpha+1}^{k_1 a_2^\alpha + k_2 a_3^\alpha} + ... + a_{\alpha+1}^{a_n} + a_{\alpha+1}^{k_1 a_n^\alpha + k_2 a_1^\alpha} \) and the sum of its variables.

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1 Introduction

We define a cyclic sum \( \sum_{\text{cyc}} f(a_1, a_2, ..., a_n) \) as equal to \( f(a_1, a_2, ..., a_n) + f(a_2, a_3, ..., a_n, a_1) + f(a_3, a_4, ..., a_1, a_2) + ... + f(a_n, a_1, ..., a_{n-1}) \). Therefore, all the variables are cycled through.

We are interested in studying the sum

\[
\sum_{\text{cyc}} \frac{a_{\alpha+1}^{a_1}}{k_1 a_1^\alpha + k_2 a_2^\alpha} = \frac{a_{\alpha+1}^{a_2}}{k_1 a_2^\alpha + k_2 a_3^\alpha} + ... + \frac{a_{\alpha+1}^{a_n}}{k_1 a_n^\alpha + k_2 a_1^\alpha}
\]

And its relative value compared to

\[
\sum_{k=1}^{n} a_k
\]

In this regard, in this paper it is proposed and proved the following theorem:
Theorem.
\[ \sum_{cyc} \frac{a_1^{\alpha+1}}{k_1a_1^\alpha + k_2a_2^\alpha} \geq \frac{\sum_{k=1}^{n} a_k}{k_1 + k_2} \]

2 Proof

2.1 Previous Lemmas

We will need firstly the following

Lemma 1.
\[ \sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^\alpha \]

Proof.

Let us assume, without loss of generality,
\[ a_1 \geq a_2 \geq a_3 \geq ... \geq a_n \]

We have that
\[ \sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^\alpha \]

It could be noted that
\[ a_1 - a_n = -(a_2 - a_1) - (a_3 - a_2) - ... - (a_n - a_{n-1}) \]

Therefore, substituting, we have that
\[ \sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_2 (a_2 - a_1) + a_3 (a_3 - a_2) + ... + a_n (a_n - a_{n-1}) - a_1 (a_2 - a_1) - a_1 (a_3 - a_2) - ... - a_1 (a_n - a_{n-1}) \]

As
\[ a_1^\alpha \geq a_2^\alpha \geq a_3^\alpha \geq ... \geq a_n^\alpha \]

We have that
\[ a_2^\alpha (a_2 - a_1) - a_1^\alpha (a_2 - a_1) \geq 0 \]
\[ a_3^\alpha (a_3 - a_2) - a_1^\alpha (a_3 - a_2) \geq 0 \]
\[ ... \]
\[ a_n^\alpha (a_n - a_{n-1}) - a_1^\alpha (a_n - a_{n-1}) \geq 0 \]
Then, subsequently,

\[ \sum_{k=1}^{n} a_k^{\alpha+1} - \sum_{cyc} a_1 a_2^\alpha \geq 0 \]

\[ \sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^\alpha \]

Other hand, we need the following

**Lemma 2.**

\[ \sum_{cyc} a_1^{\alpha+1} + \left( \frac{k_2}{k_1} \right) a_1 a_2^\alpha \]

\[ \frac{n}{k_1 a_1^\alpha + k_2 a_2^\alpha} = \frac{\sum_{k=1}^{n} a_k}{k_1} \]

**Proof.**

If we establish that

\[ \frac{a_1^{\alpha+1}}{k_1 a_1^\alpha + k_2 a_2^\alpha} + \frac{n}{m} = \frac{a_1}{k_1} \]

Operating, we get that

\[ \frac{a_1^{\alpha+1} + n (k_1 a_1^\alpha + k_2 a_2^\alpha)}{m (k_1 a_1^\alpha + k_2 a_2^\alpha)} = \frac{a_1}{k_1} \]

\[ k_1 \left( a_1^{\alpha+1} + n (k_1 a_1^\alpha + k_2 a_2^\alpha) \right) = a_1 n \left( k_1 a_1^\alpha + k_2 a_2^\alpha \right) \]

\[ k_1 a_1^{\alpha+1} + k_1 n (k_1 a_1^\alpha + k_2 a_2^\alpha) = a_1 n k_1 a_1^\alpha + a_1 n k_2 a_2^\alpha \]

\[ k_1 n (k_1 a_1^\alpha + k_2 a_2^\alpha) = a_1 n k_2 a_2^\alpha \]

\[ \frac{n}{m} = \left( \frac{k_2}{k_1} \right) \frac{a_1 a_2^\alpha}{k_1 a_1^\alpha + k_2 a_2^\alpha} \]

Therefore, we have that

\[ \frac{a_1^{\alpha+1} + \left( \frac{k_2}{k_1} \right) a_1 a_2^\alpha}{k_1 a_1^\alpha + k_2 a_2^\alpha} = \frac{a_1}{k_1} \]
And subsequently, repeating the process for each variable, we get that

\[
\sum_{cyc} a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right) a_2^\alpha = \sum_{k=1}^n \frac{a_k}{k_1}
\]

2.2 Proof

Operating, we have that

\[
\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^\alpha + k_2 a_2^\alpha} = \sum_{k=1}^n \frac{a_k}{k_1} - \sum_{cyc} \left(\frac{k_2}{k_1}\right) a_1^\alpha
\]

Applying Lemma 1, we derive that

\[
\sum_{k=1}^n a_k^{\alpha+1} \geq \left(\frac{k_1}{k_1}\right) \sum_{cyc} a_1^\alpha
\]

As a result, we can affirm that

\[
\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^\alpha + k_2 a_2^\alpha} \geq \sum_{cyc} \left(\frac{k_1+k_2}{k_1}\right) a_1^\alpha - \sum_{cyc} \left(\frac{k_2}{k_1}\right) a_1^\alpha
\]

Thus,

\[
\sum_{cyc} \left(\frac{k_1+k_2}{k_1}\right) a_2^\alpha \leq \sum_{k=1}^n \frac{a_k}{k_1}
\]

\[
\sum_{cyc} \frac{a_1 a_2^\alpha}{k_1 a_1^\alpha + k_2 a_2^\alpha} \leq \sum_{k=1}^n \frac{a_k}{k_1 + k_2}
\]

And subsequently,

\[
\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^\alpha + k_2 a_2^\alpha} \geq \sum_{k=1}^n \frac{a_k}{k_1 + k_2}
\]

As we wanted to prove.