ON THE SIZE OF $\delta$–SEPARATED $\delta$–TUBES

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Abstract. In this preprint we will prove the Kakeya conjecture. Our proof is an Euclidean version of Dvir’s proof of the finite field Kakeya conjecture.

1. Introduction

The Kakeya maximal function conjecture and its variations have gained considerable interest especially after an influential paper by Bourgain [1]. For example, it would follow from the conjecture that the Kakeya sets and the Nikodym sets have necessarily full dimensions [10, 11, 7]. However, the Nikodym set conjecture is implied by the Kakeya set conjecture [7, 11]. The case $n = 2$ was proved by Davies see [4] and the finite field case by Dvir [5]. A Kakeya is a set that contains an unit line in every direction. For surveys see [15, 12, 2]. Almost all the necessary preliminaries for this paper can be found for example in [7], [10] and in [13]. Define the $\delta$ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_\delta^\omega(a) = \{x \in \mathbb{R}^n : |(x - a) \cdot \omega| \leq \frac{1}{2}, |\text{proj}_\omega (x - a)| \leq \delta\}.$$

Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. Define the Kakeya maximal function $f_\delta^* : S^{n-1} \rightarrow \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T_\delta^\omega(a)} \int_{T_\delta^\omega(a)} |f(y)| dy.$$

In this paper any constant can depend on dimension $n$. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

(1.1) $\|f_\delta^*\|_p \leq C_{\epsilon} \delta^{-n/p+1-\epsilon},$

for all $\epsilon > 0$. Remarkably, a bound of the form (1.1) follows from a bound of the form

(1.2) $\|\sum_{\omega \in \Omega} 1_{T_\omega(a, \omega)}\|_{p/(p-1)} \leq C_{\epsilon} \delta^{-n/p+1-\epsilon},$

for all $\epsilon > 0$, and for any set of $\delta$-separated of $\delta$-tubes. See for example [11] or [7]. We will prove that

**Theorem 1.1.** Let $K$ be any Kakeya set. Then there exists a set of

$$N \sim \delta^{-n} \ln(\frac{1}{\delta})$$

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δ-tubes in the δ-neighbourhood of $K$ s.t.
\[ | \bigcup_{i=1}^{N} T_i^{\delta} | \gtrsim \left( \frac{1}{\delta} \right)^{-1}. \]

Our theorem has a corollary:

**Corollary 1.2.** A Kakeya set has full dimension.

Our other result is a generalization of a lemma of Corbóda.

**Lemma 1.3.** For tube intersections of order $2^k$ it holds that
\[ | \bigcap_{i=1}^{2^k} T_i | \leq \delta | \omega_i - \omega_j | \]

2. **Previously known results**

We will use the following bound for the pairwise intersections of δ-tubes:

**Lemma 2.1** (Corbóda). For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have
\[ | T_{\omega_i}^\delta(a) \cap T_{\omega_j}^\delta(b) | \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}. \]

A proof can be found for example in [7].

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, define its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an δ-separated subset of $\Omega$.

**Lemma 2.2.** In the notation just defined
\[ 1 \leq N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}. \]

Again, a proof can essentially be found in [7]. A Kakeya set in the finite field $F_q^n$ is a set $K'$ that for every $v$ there exists $a$ s.t $a + vt \in F_q^n$ for all $t \in F_q$. Zeev Dvir has proved that in a finite field $F_q^n$ a Kakeya set $K'$ has a bound
\[ |K'| \geq \frac{q^n}{n^\nu}. \]

The constant was later improved to
\[ |K'| \geq \frac{q^n}{2^n}. \]

3. **A proof of the generalization of the lemma of Corbóda**

Let us suppose that $2^k = \delta^{-\beta}, 0 < \beta \leq n - 1$, and let’s suppose that tube $T_{\omega'}$ intersecting $T_{\omega} \cap E_{2^k}$ has it’s direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between $T_{\omega}$ and $T_{\omega'}$ is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 2.1 the intersection
\[ | \bigcap_{i=1}^{2^k} T_i | \leq | T_{\omega} \cap T_{\omega'} \cap E_{2^k} | \leq \delta^{n-1+\beta/(n-1)} = \delta^{n-1} \frac{2^k}{n^\nu}. \]

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_{\omega} \cap T_{\omega'}$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2.2 we get that
the cap can contain at most $\sim 2^k$ tube-directions. Thus, for any tube $T_\omega$ in the intersection there exists a tube $T_{\omega'}$, such that the angle between $T_\omega$ and $T_{\omega'}$ is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (3.1) is valid. Thus we proved the lemma 1.3.

4. An euclidean version of Dvir’s theorem

Let $q$ be prime and let

$$q\delta = 1.$$ 

Let us consider a grid $G$ of $q^n$ points scaled with $\delta^n$. We embed this grid to $[0, 1]^n$ so that the zeroes’s of the two sets meet. Let us consider the set $E'$ of algebraic lines of the grid $G$. Now each algebraic line $l \subset G$ is divided to maximally 2 parts in $G$, because as an arithmetic progression $l \subset \mathbb{N}/2q$. Our lines $a + vt \in E'$ are such that if $v \in tu$ then $a + tu = a + vt$. So we have maximally 2 parallel lines for every direction in $E'$. Now, a key question is the number of those lines. We need to consider only points in the lines where at least one coordinate is a prime, $1,0$, or $q - 1$. Because those generate our line set $E'$. We can choose those points $(x_1, \ldots, x_n)$ in $\sim \Pi(q)q^{n-1}$ different ways so we have that many different lines, where $\Pi(q)$ is the prime counting function. By Dvir’s theorem we have

$$|E'| \geq \frac{q^n}{2n}.$$ 

Now suppose we have a Kakeya set $K$ such that each line intersects $[0, 1]^n$. Because we have maximally 2 parallel line segments for every line in $E'$ they together contain $\geq \frac{q^n}{2n}$ points. We notice that even the shorter algebraic line in $E'$ contains at least one point. So we divide $E'$ to two sets $E$ and $E''$ so that we have $\sim \Pi(q)q^{n-1}$ lines in each. The next lemma show that the number of points in $E$ is greater than $\sim \Pi(q)q^{n-1}$.

Lemma 4.1. The number of points $\#(P)$ in $N$ algebraic lines is greater than $N$.

Proof. W.L.O.G we can consider only intersection points. Let us enumerate the points with $j \in \mathbb{N}$. We choose the lines from $E$ recursively. In stage $j = 0$ we choose a line containing a point $P_j$. In the stage $j > 0$ we choose a line containing a point $P_j$ not chosen before. Because every line intersects some point $P_j$, every line gets chosen once. So the number of lines $N$ is exactly the number of choices that is less than $P$. 

We can assume that each line segment in $E$ contains at least two points, because the line segments in $E''$ that contain $q - 1$ points belong to two hyperplanes in $[0, 1]^n$. So the number of lines in $E$ that are just points are $Oq^{n-1}$. So we have $\sim \Pi(q)q^{n-1}$ different lines in $E$. It follows that the lines in $\mathbb{R}^n$ obtained from the algebraic lines cannot be $\delta$-separated. The lines are actually more crammed together, but the points in the lines are $\delta$-separated and that is important for our proof. Let us suppose that we have more than $\sim \Pi(q)q^{n-1}$ points. Multiplying the bound for the number of $\delta$-separated points $\frac{q^{n-1}n(q)}{2n}$ by $\delta^n$ we have

$$|\bigcup_{l \in E} T_l^\delta| \geq |E|\delta^n \geq \frac{\Pi(q)q^{n-1}\delta^n}{2n} \geq \frac{1}{\delta} \frac{1}{(\delta^n)^{-1}},$$

So we have theorem 1.1, because we have

$$\Pi(q) \sim \frac{q}{\ln(q)}.$$
via the prime number theorem.

5. The Kakeya Conjecture is true

In this section we will prove the corollary 1.2. Let $K$ be a Kakeya set, that is, a compact set that contains an unit line in every direction. Let $\bigcup_{j=1}^{\infty} B_j$ be a cover of $K$ with balls of diameters less than $1 > r > r_j > 0$. Let $n > n - \alpha > 0$ be such that

$$\sum_{j=1}^{\infty} r_j^{n-\alpha} < 1. \quad (5.1)$$

If the $n - \alpha$-dimensional Hausdorff measure is zero that kind of cover exists. By compactness of the Kakeya set we can take a subcover with diameters such that $1 > r > r_j \geq \delta = q^{-1} > 0$, where at least one $r_j = \delta$ and $q$ is a prime. Now, we have proved that there exists $N$ s.t

$$\sum_{j=1}^{M} r_j^n \geq \left| \bigcup_{j=1}^{M} B_j \right| \geq \left| \bigcup_{i=1}^{N} T_i \right| \geq 1. \quad (5.2)$$

The second inequality above follows because the balls cover the middle lines of the tubes, so there exists a constant such that the second inequality above is valid. This can be seen by considering the $\delta$-neighbourhood of the cover. Using inequality (5.1) and (5.2) we obtain

$$C_{\alpha/k} \delta^{-\alpha/k} \sum_{j=1}^{M} r_j^n > \sum_{j=1}^{M} r_j^{n-\alpha}. \quad (5.3)$$

Thus,

$$\sum_{j=1}^{M} r_j^n (C_{\alpha/k} \delta^{-\alpha/k} - r_j^{-\alpha}) > 0. \quad (5.4)$$

It follows that for the average value of a power of diameters it holds that

$$C_{\alpha/k} \delta^{-\alpha/k} > \frac{1}{M} \sum_{j=1}^{M} r_j^{-\alpha} \geq \frac{1}{M - \alpha} \left( \sum_{j=1}^{M} r_j \right)^{-\alpha}, \quad (5.5)$$

where we used Jensen’s inequality. Thus,

$$c_{\alpha} \frac{1}{M} \sum_{j=1}^{M} r_j > \delta^{1/k}. \quad (5.6)$$

From above it follows that

$$\frac{(c_{\alpha})^n}{M} \left( \sum_{j=1}^{M} r_j^n \right) \geq \left( \frac{c_{\alpha}}{M} \right)^n \left( \sum_{j=1}^{M} r_j \right)^n > \delta^{n/k},$$

where we used Jensen’s inequality again. Thus, from above and inequality (5.1)

$$C_\alpha > M \delta^{n/k}. \quad (5.7)$$

It follows from above that

$$\delta^{-n/k} C_\alpha > M$$
We can do the steps (5.3), (5.4) and (5.5) again for \( \epsilon = \alpha/2 \) and obtain
\[
C_{\alpha/2} \delta^{-\alpha/2} > \frac{1}{M} \sum_{j=1}^{M} r_j^{-\alpha}. \tag{5.8}
\]
Let \( k \) and a small \( \delta \) be such that
\[
\delta^{-\alpha/3} > C_{\alpha} \delta^{-n/k}. \tag{5.9}
\]
From above and inequalities (5.7) and (5.8) we obtain
\[
C_{\alpha/2} \delta^{-\alpha/2} > \delta^{\alpha/3} \sum_{j=1}^{M} r_j^{-\alpha} > \delta^{\alpha/3} \delta^{-\alpha} = \delta^{-2/3}. \tag{5.9}
\]
which is a contradiction when \( \delta \) is small enough.

References


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