Refutation of the cancellation law, Basic Arithmetic (BA), and the MRDP theorem

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Abstract: We evaluate the left- and right-cancellation laws as both not tautologous. This denies the group, semi- and commutative-group, and ring as integral domain. We are lead to evaluate the MRDP theorem in Basic Arithmetic (BA) and its extensions to harbor and justify the cancellation law. We find BA and MRDP (despite its adulation) as not tautologous. These results form a non tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, F as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

\begin{enumerate}
    \item \textit{left-cancellation law:} \quad x \cdot y = x \cdot z \Rightarrow y = z \quad (1.1.1)
    \item \textit{right-cancellation law:} \quad x \cdot y = z \cdot y \Rightarrow x = z \quad (1.2.1)
\end{enumerate}

\textbf{Remark 1:} Eqs. 1.1.2 and 1.2.2 as rendered are \textit{not} tautologous, refuting the left- and right-cancellation laws. This denies the group, semi- and commutative-group, and ring as integral domain, among many other things.

From: Arde\v{r}šir, M.; et al. (2020). Provably total recursive functions and MRDP theorem in Basic
Abstract We study Basic Arithmetic, BA introduced by W. Ruitenburg. BA is an arithmetical theory based on basic logic which is weaker than intuitionistic logic. We show that the class of the provably recursive functions of BA is a proper sub-class of primitive recursive functions. Three extensions of BA, called BA+U, BAc and EBA are investigated with relation to their provably recursive functions. It is shown that the provably recursive functions of these three extensions of BA are exactly primitive recursive functions. Moreover, among other things, it is shown that the well-known MRDP theorem doesn’t hold in BA, BA+U, BAc, but holds in EBA.

1 Introduction Basic Arithmetic, BA is an arithmetical theory introduced by W. Ruitenburg in .. , based on Basic Predicate Calculus, BQC, as Heyting Arithmetic, HA is based on Intuitionistic Predicate Calculus, IQC and Peano Arithmetic, PA based on Classical Predicate Calculus, CQC. BQC is a weaker logic than IQC, in which the rule of Modus Ponens is weakened. Although the arithmetical axioms of BA are essentially the same as Peano axioms, BA is weaker than HA. For instance BA does not prove the cancellation law [later named arithmetical axiom], i.e.,

\[ x + y = x + z \Rightarrow y = z. \]  

Remark 1: Eq. 2.1.2 is not tautologous, refuting the cancellation law as an arithmetical axiom. The form of the cancellation law differs from Eqs. 1.1.2 and 1.2.2, using the + Or connective rather than the & And connective, and with different truth table results.

If Eq. 2.1.1 is taken as a right-cancellation law, then it can be rewritten as a right-cancellation law:

\[ x + y = z \Rightarrow x = z. \]

Remark 4: Logic extension EBA is BA extended with the cancellation law to have the MRDP theorem. Regarding the MRDP theorem, see below.

4 MRDP theorem in BA and some of its extensions In this section we consider the well-known MRDP theorem in BA and its extensions defined in the last section. We show that in BA and in two of its extensions, i.e., BA augmented by the cancellation law and also BA augmented by the cut-off function, the MRDP theorem does not hold. However, EBA is strong enough to have the MRDP theorem.

Remark 4: Logic extension EBA is BA extended with the cancellation law to have the MRDP theorem. Regarding the MRDP theorem, see below.


Hilbert’s Tenth Problem took seventy years to resolve. Important work towards a solution was done by Julia Robinson and Martin Davis, with a contribution from Hilary Putnam. The final, and key, step however was made by a young Russian mathematician, Yuri Matiyasevich. Putting everything together, we get the MDRP theorem, settling the Tenth Problem in the negative: provably, there is no algorithmic way of determining whether some arbitrary diophantine equation has a solution.

8 A final teaser . . .

Theorem 8.1. Let T be any axiomatizable \( \omega \)-consistent theory containing Robinson Arithmetic. Then there is an \( n \) (different for different theories) such that the following sentence is undecidable in T:
\[\exists a \exists b \forall (i \leq \bar{n}) \exists s \exists w \exists p \exists q \forall v \forall e \forall g ( (s+w)^2 + 3w + s = 2i \land ([j = w \land v = q] \lor [j = 3i \land v = p + q] \lor [j = s \land (v = p \lor (i = \bar{n} \land v = q + n))] \lor [j = 3i + 1 \land v = pq] \rightarrow a = v + e + ejb \land v + g = jb) \}. \quad (8.1.1)\]

One can but gasp in wonderment . . .

**Remark 8.1.1:** To limit Eq. 4.1.1 to wieldy 11-variables, we take the necessity \((i \leq \bar{n})\) and the only other occurrence of \((i = \bar{n})\) as the instance for replacement of the necessity of \(\bar{n}\) as equivalent to the necessity of \(i\), to rewrite 4.1.1 as:

\[\exists a \exists b \forall (i \leq \bar{n}) \exists s \exists w \exists p \exists q \forall v \forall e \forall g ( (s+w)^2 + 3w + s = 2i \land ([j = w \land v = q] \lor [j = 3i \land v = p + q] \lor [j = s \land (v = p \lor (i = \bar{n} \land v = q + n))] \lor [j = 3i + 1 \land v = pq] \rightarrow a = v + e + ejb \land v + g = jb) \}. \quad (8.2.1)\]

**Remark 8.2.2:** Eq. 8.2.2 is *not* tautologous, refuting the conjecture that MRDP is a theorem to prove undecidability. (Eq. 8.2.2 is quite decidable contrary to the claim.) In Eq. 8.2.1, expansion of the term of \((s+w)^2\) into \(s^2 + 2sw + w^2\) does not affect the truth table result in 8.2.2.