TRIGONOMETRIC IDENTITIES

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ABSTRACT: I present a proof of Trigonometric Identities involving Sin(x) and Cos(x).

1. INTRODUCTION

We are aware of the following identities such as $2SinxCosx = Sin2x$ and $2Cos^2x = 1 + Cos2x$ but the question is have we ever been aware of the fact that these two identities are special cases of some other identities? Well, they actually are, and we will see that.

2. NEW IDENTITIES

$$2^n Cos^n x Sin(n)x = \sum_{k=0}^{n} \binom{n}{k} Sin2kx$$

$$2^n Cos^n x Cos(n)x = \sum_{k=0}^{n} \binom{n}{k} Cos2kx$$

$$Sin(n)x \sum_{k=0}^{n} \binom{n}{k} Cos2kx = Cos(n)x \sum_{k=0}^{n} \binom{n}{k} Sin2kx$$

3. PROOF OF THE NEW IDENTITIES

Using binomial expansion, we see that;

$$\left(1 + e^{2ix}\right)^n = \sum_{k=0}^{n} \binom{n}{k} e^{2ikx} \tag{1}$$

Now, let's try to manipulate LHS and RHS of (1);

LHS;

$$\left(1 + e^{2ix}\right)^n = (e^{ix}(e^{-ix} + e^{ix}))^n$$

$$\left(1 + e^{2ix}\right)^n = (2e^{ix}(\frac{e^{ix} + e^{-ix}}{2}))^n$$

We know that;

$$Cosx = (\frac{e^{ix}+e^{-ix}}{2})$$

So,

$$\left(1 + e^{2ix}\right)^n = 2^n e^{inx} \cos^n x$$

We know that;

$$e^{inx} = Cosnx + i\,Sinnx$$

Therefore;

$$\left(1 + e^{2ix}\right)^n = 2^n (Cosnx + iSin(n)x) Cos^n x$$

$$\left(1 + e^{2ix}\right)^n = 2^n Cos^n x Cosnx + i(2^n Cos^n x Sin(n)x) \tag{2}$$

RHS;

$$\sum_{k=0}^{n} \binom{n}{k} e^{2ikx} = \sum_{k=0}^{n} \binom{n}{k} (Cos2kx + i\,Sin2kx)$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{2ikx} = \sum_{k=0}^{n} \binom{n}{k} Cos2kx + i \sum_{k=0}^{n} \binom{n}{k} Sin2kx \tag{3}$$

We see from (2) and (3) that;

$$2^n Cos^n x Cosnx + i(2^n Cos^n x Sin(n)x) = (\sum_{k=0}^{n} \binom{n}{k} Cos2kx) + i(\sum_{k=0}^{n} \binom{n}{k} Sin2kx) \tag{4}$$

Equating the real and imaginary parts of (4), we see that;

$$2^n Cos^n x Sin(n)x = \sum_{k=0}^{n} \binom{n}{k} Sin2kx \tag{5}$$

$$2^n Cos^n x Cos(n)x = \sum_{k=0}^{n} \binom{n}{k} Cos2kx \tag{6}$$

Dividing (6) by (5), we see that;

$$Sin(n)x \sum_{k=0}^{n} \binom{n}{k} Cos2kx = Cos(n)x \sum_{k=0}^{n} \binom{n}{k} Sin2kx \tag{7}$$

4. GENERALIZATION OF THE NEW IDENTITIES

- $2^n Cos^n ax Sin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Sin(2ak+m)x$
- $2^n Cos^n ax Cos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Cos(2ak+m)x$
- $Sin(an+m)x \sum_{k=0}^{n} \binom{n}{k} Cos(2ak+m)x = Cos(an+m)x \sum_{k=0}^{n} \binom{n}{k} Sin(2ak+m)x$

5. PROOF

We can see that;

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = (e^{\left(ai-ai+i\frac{m}{n}\right)x} + e^{\left(ai+ai+i\frac{m}{n}\right)x})^n$$
$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = (e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix})^n \tag{8}$$

Expanding the right hand side of (8) using binomial expansion, we can see that;

$$(e^{\left(\frac{m}{n}\right)ix} + e^{\left(a+\frac{m}{n}\right)ix})^n = \binom{n}{0}.(e^{n\left(\frac{m}{n}\right)ix}) + \binom{n}{1}.(e^{(n-1)\left(\frac{m}{n}\right)ix}).e^{\left(2a+\frac{m}{n}\right)ix} + \binom{n}{2}.(e^{(n-2)\left(\frac{m}{n}\right)ix}).e^{2\left(2a+\frac{m}{n}\right)ix} + \cdots + \binom{n}{n}.e^{n\left(2a+\frac{m}{n}\right)ix}$$
$$(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix})^n = \binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{\left(2ai+\left(\frac{im}{n}\right)+\left(\frac{inm}{n}\right)-\left(\frac{im}{n}\right)\right)x}) + \binom{n}{2}.(e^{\left(4ai+\left(\frac{2im}{n}\right)+\left(\frac{inm}{n}\right)-\left(\frac{2im}{n}\right)\right)x}) + \cdots + \binom{n}{n}.e^{(2an+m)ix}$$
$$(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix})^n = \binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{(2a+m)ix}) + \binom{n}{2}.(e^{(4a+m)ix}) + \cdots + \binom{n}{n}.e^{(2an+m)ix} \tag{9}$$

We can see clearly that;

$$\binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{(2a+m)ix}) + \binom{n}{2}.(e^{(4a+m)ix}) + \cdots + \binom{n}{n}.e^{(2an+m)ix} = \sum_{k=0}^{n} \binom{n}{k} e^{(2ak+m)ix}$$

Now,

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = \sum_{k=0}^{n} \binom{n}{k} e^{(2ak+m)ix} \tag{10}$$

Now, let's try to manipulate LHS and RHS of (10);

LHS;

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = (2e^{\left(a+\frac{m}{n}\right)ix}(\frac{e^{iax}+e^{-iax}}{2}))^n$$

We know that;

$$Cos(a)x = (\frac{e^{iax}+e^{-iax}}{2})$$

So,

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = 2^n e^{n\left(a+\frac{m}{n}\right)ix} \cos^n ax$$
$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = 2^n e^{(an+m)ix} \cos^n ax$$

We know that;

$$e^{(an+m)ix} = Cos(an+m)x + i\, Sin(an+m)x$$

Therefore;

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = 2^n(Cos(an+m)x + iSin(an+m)x)Cos^n ax$$
$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = 2^n Cos^n ax Cos(an+m)x + i(2^n Cos^n ax Sin(an+m)x) \tag{11}$$

RHS;

$$\sum_{k=0}^{n} \binom{n}{k} e^{(2ak+m)ix} = \sum_{k=0}^{n} \binom{n}{k} (Cos(2ak+m)x + i\, Sin(2ak+m)x)$$
$$\sum_{k=0}^{n} \binom{n}{k} e^{(2ak+m)ix} = \sum_{k=0}^{n} \binom{n}{k} Cos(2ak+m)x + i\sum_{k=0}^{n} \binom{n}{k} Sin(2ak+m)x \tag{12}$$

Since (11) equals (12), then;

$$2^n Cos^n ax Cos(an+m)x + i(2^n Cos^n ax Sin(an+m)x) = \sum_{k=0}^{n} \binom{n}{k} Cos(2ak+m)x + i\sum_{k=0}^{n} \binom{n}{k} Sin(2ak+m)x \tag{13}$$

Equating the real and imaginary parts of (13), we see that;

$$2^n Cos^n ax Sin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Sin(2ak+m)x \tag{14}$$
$$2^n Cos^n ax Cos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Cos(2ak+m)x \tag{15}$$

Dividing (14) by (15), we see that;

$$Sin(an+m)x\sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x = Cos(an+m)x\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x \quad (16)$$

6. SOME OTHER NEW IDENTITIES

$$2^nCos^n\left(\frac{2a+1}{2}\right)xSin\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}Sin((2a+1)k+m)x$$

$$2^nCos^n\left(\frac{2a+1}{2}\right)xCos\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}Cos((2a+1)k+m)x$$

$$Sin\left(\frac{n(2a+1)+2m}{2}\right)x\sum_{k=0}^{n}\binom{n}{k}Cos((2a+1)k+m)x = Cos\left(\frac{n(2a+1)+2m}{2}\right)x\sum_{k=0}^{n}\binom{n}{k}Sin((2a+1)k+m)x$$

$$2^nCosh^n\left(\frac{2a+1}{2}\right)xSinh\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}Sinh((2a+1)k+m)x$$

$$2^nCosh^n\left(\frac{2a+1}{2}\right)xCosh\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}Cosh((2a+1)k+m)x$$

$$Sinh\left(\frac{n(2a+1)+2m}{2}\right)x\sum_{k=0}^{n}\binom{n}{k}Cosh((2a+1)k+m)x = Cosh\left(\frac{n(2a+1)+2m}{2}\right)x\sum_{k=0}^{n}\binom{n}{k}Sinh((2a+1)k+m)x$$

$$2^nSin^naxSin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSin(2ak+m)x \quad (n\ is\ even)$$

$$2^nSin^naxSin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCos(2ak+m)x \quad (n\ is\ odd)$$

$$2^nSin^naxCos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSin(2ak+m)x \quad (n\ is\ odd)$$

$$2^nSin^naxCos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCos(2ak+m)x \quad (n\ is\ even)$$

$$2^nSin^n\left(\frac{2a+1}{2}\right)xSin\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSin((2a+1)k+m)x \quad (n\ is\ even)$$

$$2^nSin^n\left(\frac{2a+1}{2}\right)xSin\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCos((2a+1)k+m)x \quad (n\ is\ odd)$$

$$2^nSin^n\left(\frac{2a+1}{2}\right)xCos\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSin((2a+1)k+m)x \quad (n\ is\ odd)$$

$$2^nSin^n\left(\frac{2a+1}{2}\right)xCos\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCos((2a+1)k+m)x \quad (n\ is\ even)$$

$$2^nSinh^naxSinh(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSinh(2ak+m)x \quad (n\ is\ even)$$

$$2^nSinh^naxSinh(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCosh(2ak+m)x \quad (n\ is\ odd)$$

$$2^nSinh^naxCosh(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSinh(2ak+m)x \quad (n\ is\ odd)$$

$$2^nSinh^naxCosh(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCosh(2ak+m)x \quad (n\ is\ even)$$

$$2^nSinh^n\left(\frac{2a+1}{2}\right)xSinh\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSinh((2a+1)k+m)x \quad (n\ is\ even)$$

$$2^nSinh^n\left(\frac{2a+1}{2}\right)xSinh\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCosh((2a+1)k+m)x \quad (n\ is\ odd)$$

$$2^nSinh^n\left(\frac{2a+1}{2}\right)xCosh\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kSinh((2a+1)k+m)x \quad (n\ is\ odd)$$

$$2^nSinh^n\left(\frac{2a+1}{2}\right)xCosh\left(\frac{n(2a+1)+2m}{2}\right)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^kCosh((2a+1)k+m)x \quad (n\ is\ even)$$