Geometric description of Heisenberg's uncertainty principle

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Abstract Analysis of the simultaneous measurement stated in Heisenberg's uncertainty principle reveals its root in special relativity theory. Hence a natural extension of this principle with general relativity is shown, where a small gravitational correction term needs to be introduced. This gravitational term being small remains negligible in ordinary conditions but becomes significant at small Planck scale. At Planck scale the modified uncertainty relation leads to natural calculation of Planck scale parameters used in quantum gravity theories. A careful consideration of the gravitational term showed its connection with cosmological dark energy and could explain the apparent large discrepancy between cosmological observed value and vacuum energy estimate from quantum field theory.

Keywords: Heisenberg's Uncertainty principle; Special theory of relativity; General theory of relativity; Quantum gravity; Cosmological dark energy

Introduction

Heisenberg's uncertainty principle is one of the characteristic aspects of quantum mechanics. Often it is regarded as a distinctive feature which differentiate quantum physics from classical physics. For two non commuting quantum mechanical variables, like position and momentum, the uncertainty principle suggests one can not measure the simultaneous values of a physical system with arbitrary accuracies. In his original work Heisenberg derived the uncertainty relation heuristically on the basis of a supposed experiment based on observing an electron using a γ-ray microscope and showed that the product of highest possible accuracies Δx and Δp for position and momentum respectively can not be smaller than Planck's constant, that is $\Delta x. \Delta p \geq h$.[1, 2] Later on the uncertainty principle was derived under the quantum mechanical formalisms based on Schwarz inequality as $\Delta x. \Delta p \geq \hbar/_2$, where the uncertainties are defined as the statistical spread for an ensemble of similarly prepared systems. [3, 4, 5] Recently, while interpreting the uncertainty relation as the interplay between error and disturbance for a quantum state, claims have been made for refutation of Heisenberg's principle. [6-8] However, interpreting the uncertainty principle heuristically proposed by Heisenberg may not be very straightforward and necessarily require consideration of measurement processes in a greater detail. [9]

Although Erwin Schrödinger once suggested [10] that the uncertainty principle is incompatible with special theory of relativity, some efforts have been made to search for possible links. Rosen and Vallera considered an ideal experiment in which it could be shown that when the exact nature of interaction between an electron and light signal is not known, $\Delta x. \Delta p$ has an upper and lower bound which depends on the electron velocity. [11] But the upper and lower bounds could only merge to h for infinite velocity of the electron. However, here we will see that since the uncertainty principle restricts the accuracies for simultaneous measurements of two variables, the corresponding simultaneity of the measurements between the reference frames of the object under study and that of the observer or measuring device, needs to be analysed from the special relativity principles.[12] We will intend an answer of this fundamental question and will see that Heisenberg's uncertainty principle can be described as a consequence of the simultaneity principle of Einstein's special theory of relativity. The case of position and momentum will be primarily considered but the analysis presented should be generally applicable to other conjugate variables as well. Since the uncertainty relation could be seen to have origin in the special relativity theory, a natural extension of it will be derived from the general relativity principles. The modified uncertainty relation will be shown to have very significant implications on estimating cosmological dark energy and Planck scale quantum gravity theories, and offer satisfactory answers to the apparent paradoxes in a very certain manner.

Heisenberg's uncertainty principle and special relativity theory

First, we will see that Heisenberg uncertainty principle can be described in the special relativistic framework using simple semiclassical approach. Let first consider the case where an observer/measuring device is stationary in the laboratory frame of reference looking to measure the position and momentum of a particle moving with a precise velocity v ($v < c$, the velocity of light in vacuum) along the x-axis (single axis being considered for simplicity of description). We need to first assume that in the measurement process employed by the observer, the uncertainty principle is not applicable, that is position and momentum can be measured with arbitrary accuracy, although laws of relativity hold good. This seems a valid assumption since we are trying to see if the laws of relativity could lead to the uncertainty principle unknown to the observer. We further assume the observer to be equipped with a noiseless measurement apparatus and the measurements do not disturb the particle. It may be noted that such measurement methods although not presently available, the assumptions justify our aim to see whether Heisenberg's uncertainty principle is only a consequence of the disturbance caused by our measurement methods or it is an unavoidable natural consequence from the viewpoint of relativity theory. Now, to meet both position and momentum measurements, we need to carry out the measurement over an interval Δt, during which the particle moves by distance Δx. Let the measurement be described by its start and end points, which represent two events (x_1, ct_1) and (x_2, ct_2) in the Minkowski space [13], where $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$. The co-ordinates are expressed in the observer's frame. This should be generally applicable to any measurement scheme which produces in each run a position and a momentum value of the particle. The intervals Δt and Δx can be chosen arbitrarily small. It is worthwhile to mention here that $\Delta t \neq 0$ is not contradictory to our assumption of non-disturbing measuring method since v is assumed to remain unchanged during measurement and $\Delta t = 0$ implies no measurement performed.

For simultaneous knowledge of the observables, position and momentum, of the particle, at a definite instant of time we must have $\Delta t' = 0$ in the inertial frame attached to the particle. This implies the events (x_1, ct_1) and (x_2, ct_2) must occur simultaneously in the particle's frame. Under this simultaneity requirement we will try to find out the limits for values of Δx and Δt in the observer's frame. By this we agree to let $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$ get violated for meeting the simultaneity requirement. It is worthwhile to mention here that we have not assumed anything about the interaction of the particle and the observer's measurement device and it would likely employ quantum principles for measurements to be performed, but that should not change our results. The particle is essentially a quantum system where its velocity v is the corresponding group velocity of the associated de Broglie matter wave [14]. It is pertinent to mention here that the wave particle duality or the complementarity principle is an independent phenomenon described in quantum physics unrelated to measurement inaccuracies.

In Minkowsky flat space-time, the relative position and time intervals in the particle's frame are given as,

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix} \tag{1}$$

Where, $\beta = {}^{v}/_{c}$ and $\gamma = \sqrt{1 - {}^{v^2}/_{c^2}}$. The condition of simultaneity is obtained by letting $\Delta t' = 0$. Hence,

$$\beta \Delta x = c\Delta t \tag{2}$$

$$\text{or} \quad \Delta x . \Delta \omega = c/\beta \tag{3}$$

since the particle can be associated with matter wave with $\Delta\omega \sim \frac{1}{\Delta t}$ (for Δt is small), phase velocity $v_{ph} = {}^{c}/_{\beta} = \omega\lambda/2\pi$, where λ is the wavelength and ω is the angular frequency. The momentum of the particle is related to the wavelength by the de Broglie formula, $p = {}^{h}/_{\lambda} = {}^{\hbar\omega}/_{v_{ph}}$. Taking $\Delta p = \hbar\Delta\omega\beta/c$ (since we have assumed mean velocity, v to be precise), equation (3) therefore becomes,

$$\Delta x . \Delta p = \hbar \tag{4}$$

Considering this as the lower limit,

$$\Delta x . \Delta p \geq \hbar \tag{5}$$

Which is the Heisenberg's uncertainty relation describing limiting condition for measurement accuracies on a quantum particle. Recognizing the reciprocal nature of Lorentz transformations, equation (5) should also be valid when simultaneous values are sought in the observer's frame. Since, statistically the errors in measurements can be assumed to be similarly distributed between the events (x_1, ct_1) and (x_2, ct_2), we can write in terms of measurement precisions,

$$\overline{\Delta x}.\overline{\Delta p} \geq \hbar/2 \tag{6}$$

Since virtually in all quantum measurements we may work with the precision values, we may conveniently write,

$$\Delta x.\Delta p \geq \hbar/2 \tag{7}$$

Equation (7) is the more familiar expression of Heisenberg's uncertainty principle in terms of measurement precisions. The above derivation provides us with a valuable physical picture for the uncertainty principle. Since for any pointlike particle worldlines always lie within the timelike zone in the Minkowski space, relativity principles forbid simultaneous measurement of variables. Any simultaneous measurement will inevitable require a lightlike or spacelike worldline. Since a quantum particle can be represented by a corresponding matter wave, the associated position spread may allow the trajectories to be extended beyond timelike zone when we consider the infinitesimal time interval of measurement, hence allowing simultaneous joint measurement of variables. However, the measurement will be inevitably associated with inaccuracies. Heisenberg's principle may therefore be viewed as a concise mathematical description of the above process.

Additionally, if we consider small uncertainty in the particle mass (Δm), from equation (2) the semi classical approximation shows,

$$\Delta t = \Delta x \frac{\Delta m \beta c}{\Delta m c^2} \text{ or } \Delta E.\Delta t = \Delta x.\Delta p \tag{8}$$

Where $\Delta E = \Delta mc^2$ is the energy uncertainty and $\Delta p = \Delta m\beta c$. Comapring equation (7) and (8),

$$\Delta E.\Delta t \geq \hbar/2 \tag{9}$$

Modification in uncertainty relation due to gravity

It would be worthwhile to consider the scenario for non-inertial frames like in the Rindler coordinates [15] or in particular gravitational spacetime [16, 17], that is when a far away observer and the particle are placed in regions that have a gravitational potential difference between them. We may arrive at a first order adjustment of equation (7) by assuming that over a very small measurement time the particle (with small gravitational mass) undergoes uniform motion having the gravitation modified local spacetime as background. Further, we consider the complete measurement consists of two steps. First, the observer is local, that is experiencing the same gravitational field as that of the particle. Since the observer is stationary in the local spacetime and particle is moving with constant v, the observer will measure the uncertainties as given by equation (7), say $\Delta r'.\Delta p' \geq \hbar/2$. Where r' or r is the radial coordinate, used to more suitably describe the space around a spherical gravitational mass. Next, we consider the observer moves to a distant place where the gravitational potential becomes small and checks how he needs to correct earlier measurements in terms of his new space time co-ordinates. In that case, according to general theory of relativity, taking into account time dilation and space warping by the gravitational field, the observer will find his earlier measured $\Delta r'$ and $\Delta p'$ changed to Δr and Δp in such a way that $\Delta r < \Delta r'$ and $\Delta p < \Delta p'$ (or $\Delta t > \Delta t'$). Therefore, considering the case of a spherically symmetric gravitational mass (M) described by the Schwarzschild metric,

$$ds^2 = -c^2 d\tau^2 = -\left(1 - 2\frac{GM}{rc^2}\right) c^2 dt^2 + \left(1 - 2\frac{GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\emptyset^2) \qquad (10)$$

the uncertainty relation given in equation (7) needs to be corrected as,

$$\Delta r. \Delta p \gtrsim \frac{\hbar}{2} \frac{\left(1-2\frac{GM}{rc^2}\right)}{\left(1-2\frac{GM}{Rc^2}\right)} \qquad (11)$$

where, r and R represents the distance of the particle and the observer respectively from the spherical mass. Considering $R \to \infty$,

$$\Delta r. \Delta p \gtrsim \frac{\hbar}{2}\left(1 - 2\frac{GM}{rc^2}\right) \qquad (12)$$

$$\text{or } \Delta r. \Delta p \geq \frac{\hbar}{2}\left(1 - \frac{r^*}{r}\right)^2 \qquad (13)$$

where the Schwarzschild radius given as, $r^* = {}^{2GM}/_{c^2}$. Equation (12) or (13) is the modified uncertainty relation under the assumption that since one can always choose Δt arbitrarily small and thereby consider a locally flat spacetime for the particle's frame. Under ordinary situations, like for an observer carrying out measurements on phenomena occurring on earth's surface from a space station, the change in equation (7) will be only about $h \times 10^{-10}$, and therefore can be easily ignored. But this small correction term may have significant effects in specific situations.

Planck scale in quantum gravity

Equation (12) can be similarly expressed in terms of energy time uncertainty as,

$$\Delta E. \Delta t \gtrsim \frac{\hbar}{2}\left(1 - 2\frac{GM}{rc^2}\right) \qquad (14)$$

Since, at $r = r*$, the uncertainty product vanishes, we can find corresponding limiting value for energy, by taking $r = r* = c\Delta t_{min}$ and $\Delta E_H = Mc^2$,

$$\Delta E_H = \frac{c^5 \Delta t_{min}}{2G} \tag{15}$$

Equating (15) to the energy value in absence of gravitational correction, $\Delta E_Q = \hbar/2\Delta t_{min}$, we get $\Delta t_{min} = \sqrt{\hbar G/_{c^5}}$, which is Plank time ($t_p \approx 5.4\text{x}10^{-44}$ s). Therefore, at t_p, $\Delta E = 0$, afterwards tor $\Delta t < t_p$, ΔE becomes negative. Although physically understanding negative ΔE is not straightforward, existence of such energy has been suggested by models of black hole physics.[18, 19]

We may also define a minimum length $\Delta l_{min} = r^* = c\Delta t_{min} = \sqrt{\hbar G/_{c^3}} \approx 1.6 \times 10^{-35}$, that is Planck length (l_p). Clearly below l_p the internal area becomes invisible as per the laws of general relativity.

For quantum fluctuations of spacetime the uncertainties can be assumed to be equal to the corresponding absolute values of the variables. Hence, ignoring higher order terms, we may write equation (14) as,

$$\Delta E \approx \frac{\hbar}{2\Delta t} - \frac{G\hbar^2}{2c^5 \Delta t^3} \tag{16}$$

This has similarity with generalized uncertainty relations proposed in several forms in recent time, based on heuristic arguments or gedanken experiments.[20] Equation (16) gives a limiting value of energy as, $\Delta E_{max} = \frac{1}{3\sqrt{3}}\sqrt{\frac{hc^5}{G}} \approx 2.3 \times 10^{18}$ GeV, for $\Delta t = \sqrt{3}t_p$. A plot of fluctuation energy with Δt is shown in Figure 1, where a smooth merging of quantum fluctuation energy with gravitational energy at the Planck time scale can be seen. At t_p the gravitational correction just cancels the dominant quantum mechanical first term in equation (16) making total fluctuation energy zero. Beyond the Planck scale (below t_p) the estimated

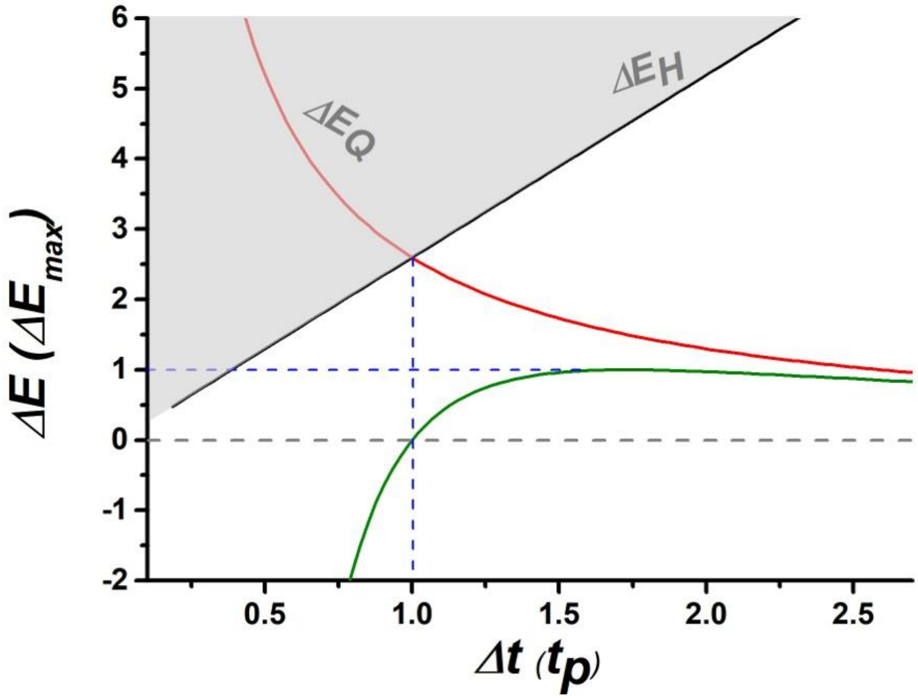

Fig. 1. The variations of fluctuation energy terms ΔE_H, ΔE_Q and ΔE with Δt are shown. The energy and time axes are scaled by ΔE_{max} and Planck time respectively. The shaded area indicate the energy values above ΔE_H, where the general relativity forbids physical phenomena to be seen from outside. ΔE_Q remains well under the limit of ΔE_H above Planck scale but below t_p it becomes larger suggesting unobservable energy levels from the viewpoint of general relativity. The total energy ΔE, shown in green, vanishes at t_p and asymptotically merges into quantum filed theory calculation (ΔE_Q) for larger Δt.

fluctuation energy becomes negative and the region becomes invisible to an outside observer by the effect of gravity. So it seems the Planck scale define a natural limit for the minimum values of the fundamental quantities. Away from the Plank scale the fluctuation energy asymptotically approaches the quantum mechanical estimate.

Dark energy problem and Einstein's cosmological constant

A careful consideration of equation (16) suggests, the first term in the right hand side, represents the fluctuation energy as seen by a local observer, whereas the second term represents a loss in the energy when seen by a distant observer. Therefore, it is reasonable to assume, the second term represents the fraction of energy gets entrapped or coupled to the space-time fabric and may contribute to large scale cosmological phenomena. Therefore, considering the ratio between the first and second term,

$$\left(\frac{G\hbar^2}{c^5 t_0^3}\right)\left(\frac{\hbar}{t_0}\right)^{-1} = \left(\frac{G\hbar}{c^5}\right).\frac{1}{t_0^2} \approx 1.57 \times 10^{-122} \tag{17}$$

Where, $\Delta t \sim t_0 \sim 13.7 \times 10^9$ years [21], the age of our universe in the present epoch given by,

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{aH} \tag{18}$$

H being the scale dependent Hubble parameter and *a* is the scale factor of the universe.

Equation (17) is in excellent agreement with the estimated ratio between the observed dark energy density in our present *epoch* [21] and vacuum energy density calculated from quantum field theory.

The left hand side of equation (17) may therefore be matched with the analytical ratio between the cosmological dark energy density and the vacuum energy density calculated from quantum field theory [22],

$$\left(\frac{G\hbar}{c^5}\right).\frac{1}{t_0^2} = \frac{\Lambda G\hbar}{2\pi c^3} \tag{19}$$

Hence we get the cosmological constant, $\Lambda = \frac{2\pi}{(ct_0)^2} \approx 3.76 \times 10^{-52}$ m^{-2}, or more generally as,

$$\Lambda = \frac{2\pi}{c^2 t(a)^2} \tag{20}$$

Where $t(a)$ is the age described in terms of the scale factor [22].

This is in excellent agreement with the observed value of Λ in our time.

Conclusions

A derivation of Heisenberg's uncertainty principle is presented based on the simultaneity concept of special relativity. The origin of Heisenberg's principle from the special relativity naturally suggests extension of this principle with the general theory of relativity. This led us to a modified uncertainty principle with an additional small term contributed by spacetime warping by gravitational field. While under usual circumstances this term remains small and can be neglected for most of the purposes, it has significant implications at small Planck scale. The modified uncertainty relation shows the existence of Planck scale as a limiting scenario for physical laws. Consideration of the additional term in the modified uncertainty relation shows its association with the cosmological dark energy and could explain the large difference between values estimated from cosmological data and quantum field theory calculations.

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