Refutation of single axioms for group theory and Abelian groups
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Abstract: We evaluate single, “simplest” axioms defining group theory and Abelian groups from McCune, Tarski, and Higman-Neuman which are not tautologous. This means group theory is no longer exact and bivalent but effectively reduced to an inexact vector space (read as probabilistic guess). These results form a non tautologous fragment of the universal logic $VŁ4$.

We assume the method and apparatus of Meth8/$VŁ4$ with Tautology as the designated proof value, $F$ as contradiction, $N$ as truthity (non-contingency), and $C$ as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

ftp.mcs.anl.gov/pub/tech_reports/reports/P270.pdf

Abstract

This paper summarizes the results of an investigation into single axioms for groups, both ordinary and Abelian, with each of following six sets of operations: \{product, inverse\}, \{division\}, \{double division, identity\}, \{double division, inverse\}, \{division, identity\}, and \{division, inverse\}. In all but two of these twelve corresponding theories, we present either the first single axiom known to us or single axioms shorter than those previously known to us. The automated theorem-proving program Otter was used extensively to construct sets of candidate axioms and to search for and find proofs that given candidate axioms are in fact single axioms.

Theorem 1. The theory of groups can be defined by the single axiom

\[(x \cdot (y \cdot (((z \cdot z^{-1}) \cdot (u \cdot y)^{-1}) \cdot x))^{-1}) = u.\] (3.1)

Let $p, q, r, s: \quad u, x, y, z,$

\[(q \& ((%s>@z)\&(r\&(((s\&((%s>@z)s))\&((%s>@z})(p\&r)))\&(q))))=p ;
\[\begin{array}{cccc}
T & F & F & T
\end{array}
\]

(3.1.2)

Theorem 2. The theory of Abelian groups can be defined by the single axiom

\[(((x \cdot y) \cdot z) \cdot (x \cdot z)^{-1}) = y.\] (3.7)

\[(((q\&r)s\&((%s>@z)(q&s)))=r ;
\[\begin{array}{cccc}
T & T & T & T
\end{array}
\]

(3.7.2)
Remark 0: Professor McCune was prematurely deceased.

This work ... is toward showing that there are no single axioms shorter or simpler than those previously known for group theory in terms of \{product, inverse\} and in terms of \{division\}.

**Abelian groups**

The answer for abelian groups is easy. The known single axioms are the shortest possible. Tarski's axiom for abelian groups in terms of division has size (11,3)

\[
x/ (y/(z/(x/y)))=z
\]

\[
(q'(r'(s'(q'r'))))=s ; \quad FFTT \quad FFFF \quad TTFF \quad TTTT
\]

**Group theory**

Here are the simplest known axioms. In terms of division, we have the size (19,3) axiom

\[
x/(((x/x)/y)/z)/(((x/x)/x)/z))=y
\]

\[
(q'(((q'q)r)s)/(((q'q)q)s))=r ; \quad FFFF \quad TTTT \quad FFTT \quad TFFF
\]

Eqs. 3.1.2 and 3.7.2 are not tautologous and refute respectively the theory of groups and Abelian groups as defined by single axioms.

Eqs. 10.2 and 11.1 are not tautologous and refute respectively the theory of Abelian groups and groups as defined by the simplest known axioms.

This means group theory is no longer exact and bivalent but effectively reduced to an inexact vector space (read as probabilistic guess).