A GENERALIZATION OF PYTHAGORAS' THEOREM

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Abstract: I present a proof of a theorem which is a generalization of Pythagoras' theorem. According to Wikipedia, the cosine rule is considered a general case of Pythagoras' theorem. However, it is known that the cosine rule includes an angle. The new theorem to be presented does not include any angle.

1. INTRODUCTION

Pythagoras theorem is one of the best results in mathematics and the theorem to be proved provides a generalization of Pythagoras theorem.

Theorem: If

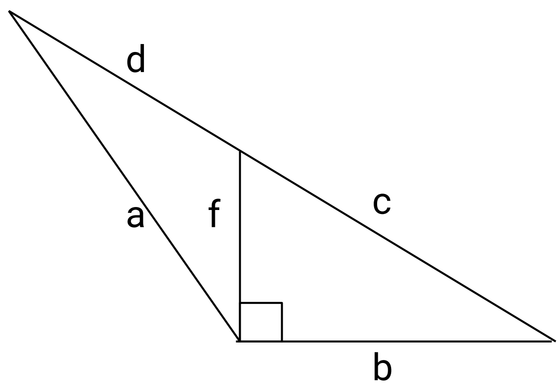

Fig 1.0

Then,

$$(c+d)^2 = a^2 + b^2\left(1+\frac{2d}{c}\right) \qquad (1)$$

We need to know that if |d| decreases to zero, we see that |a| decreases to |f| (i.e. if |d| = 0, |a| = |f|).

This means that if |d| = 0 such that |a| = |f|, (1) becomes;

$$(c+0)^2 = a^2 + b^2\left(1+\frac{2(0)}{c}\right)$$

$c^2 = a^2 + b^2$

But |a| = |f|

Therefore,

$c^2 = f^2 + b^2$, which is the Pythagoras theorem.

2. PROOF OF THEOREM

Let's take a look at the diagram below

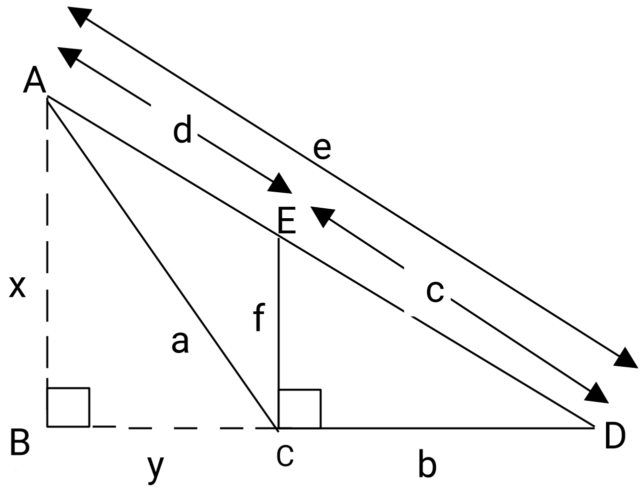

Fig 1.1

We see that,

Area of ABC $= \frac{1}{2}xy$ (Area of a triangle)

Also,

Area of ABCE $= \frac{1}{2}(x + f)y$ (Area of a trapezium)

But

Area of ABCE = Area of ABC + Area of ACE

Area of ACE = Area of ABCE - Area of ABC

Area of ACE $= \frac{1}{2}(x + f)y - \frac{1}{2}xy$

$= \frac{1}{2}xy + \frac{1}{2}fy - \frac{1}{2}xy$

Area of ACE $= \frac{1}{2}fy$ (2)

We know that;

$e^2 = x^2 + (y+b)^2$

$e^2 = x^2 + y^2 + 2yb + b^2$

But

$$a^2 = x^2 + y^2$$

$$e^2 = a^2 + 2yb + b^2$$

$$y = \frac{e^2 - a^2 - b^2}{2b} \qquad (3)$$

Putting (3) in (2),

$$\text{Area of ACE} = \frac{1}{2} f\left(\frac{e^2 - a^2 - b^2}{2b}\right)$$

We know that,

Area of ACD = Area of ACE + Area of CDE

But,

$$\text{Area of CDE} = \frac{1}{2} fb$$

So,

$$\text{Area of ACD} = \frac{1}{2} f\left(\frac{e^2 - a^2 - b^2}{2b}\right) + \frac{1}{2} fb \qquad (4)$$

Using Heron's formula, we see that

$$\text{Area of ACD} = \left(\frac{1}{4}\right)\sqrt{(a^2 + b^2 + e^2)^2 - 2(a^4 + b^4 + e^4)} \qquad (5)$$

By equating (4) and (5), we see that;

$$\left(\frac{1}{4}\right)\sqrt{(a^2 + b^2 + e^2)^2 - 2(a^4 + b^4 + e^4)} = \frac{1}{2} f\left(\frac{e^2 - a^2 - b^2}{2b}\right) + \frac{1}{2} fb$$

Multiplying both sides by 4, we see that;

$$\sqrt{(a^2 + b^2 + e^2)^2 - 2(a^4 + b^4 + e^4)} = f\left(\frac{e^2 - a^2 - b^2}{b}\right) + 2fb$$

Squaring both sides, we see that;

$$(a^2 + b^2 + e^2)^2 - 2(a^4 + b^4 + e^4) = \left[f\left(\frac{e^2 - a^2 - b^2 + 2b^2}{b}\right)\right]^2$$

$$= \left[f\left(\frac{e^2 - a^2 + b^2}{b}\right)\right]^2$$

$$2(a^2b^2 + a^2e^2 + b^2e^2) - (a^4 + b^4 + e^4) = f^2\left(\frac{e^4 + a^4 + b^4 - 2a^2e^2 + 2b^2e^2 - 2a^2b^2}{b^2}\right)$$

If we cross multiply, we see that;

$$b^2[2(a^2b^2+a^2e^2+b^2e^2) - (a^4+b^4+e^4)] = f^2(e^4 + a^4 + b^4 - 2a^2e^2 + 2b^2e^2 - 2a^2b^2)$$

$$2a^2b^4 + 2a^2b^2e^2 + 2b^4e^2 - a^4b^2 - b^6 - b^2e^4 = e^4f^2 + a^4f^2 + b^4f^2 - 2a^2e^2f^2 + 2b^2e^2f^2 - 2a^2b^2f^2$$

Collecting like terms, we see that;

$$(b^2+f^2)a^4 - (2b^4+2b^2e^2+2e^2f^2+2b^2f^2)a^2 + (b^6-2b^4e^2+b^2e^4+e^4f^2+b^4f^2+2b^2e^2f^2) = 0 \qquad (6)$$

We see that (6) is a bi-quadratic equation.

If we divide (6) by (b^2+f^2), we get;

$$a^4 - 2(b^2+e^2)a^2 + \left(\frac{b^6 - 2b^4e^2 + b^2e^4 + e^4f^2 + b^4f^2 + 2b^2e^2f^2}{b^2 + f^2}\right) = 0$$

(7)

We know that;

If $q^4+pq^2+s = 0$ then, $q^2 = \dfrac{-p \pm\sqrt{p^2 - 4s}}{2}$

Solving for **a** in (7) by setting $q = a$, $p = -2(b^2+e^2)$, and

$$s = \left(\frac{b^6 - 2b^4e^2 + b^2e^4 + e^4f^2 + b^4f^2 + 2b^2e^2f^2}{b^2 + f^2}\right),$$

We see that;

$$a^2 = \frac{2(b^2 + e^2) \pm \sqrt{4(b^2 + e^2)^2 - 4\dfrac{b^6 - 2b^4e^2 + b^2e^4 + e^4f^2 + b^4f^2 + 2b^2e^2f^2}{b^2 + f^2}}}{2}$$

$$a^2 = \frac{2(b^2 + e^2) \pm 2\sqrt{(b^2 + e^2)^2 - \dfrac{b^6 - 2b^4e^2 + b^2e^4 + e^4f^2 + b^4f^2 + 2b^2e^2f^2}{b^2 + f^2}}}{2}$$

$$a^2 = (b^2 + e^2) \pm \sqrt{(b^2 + e^2)^2 - \frac{b^6 - 2b^4e^2 + b^2e^4 + e^4f^2 + b^4f^2 + 2b^2e^2f^2}{b^2 + f^2}} \quad (8)$$

If we simplify the expression

$(b^2 + e^2)^2 - \frac{b^6 - 2b^4e^2 + b^2e^4 + e^4f^2 + b^4f^2 + 2b^2e^2f^2}{b^2 + f^2}$ in (8), we see that;

$$(b^2 + e^2)^2 - \frac{b^6 - 2b^4e^2 + b^2e^4 + e^4f^2 + b^4f^2 + 2b^2e^2f^2}{b^2 + f^2} = \frac{4b^4e^2}{b^2 + f^2}$$

Therefore,

$$a^2 = b^2 + e^2 \pm \sqrt{\frac{4b^4e^2}{b^2 + f^2}}$$

$$a^2 = b^2 + e^2 \pm \frac{2b^2e}{\sqrt{b^2 + f^2}} \quad (9)$$

From fig 1.1, since |e| = |c|+|d|, we know that;

$$c^2 = b^2 + f^2$$

Putting this in (9) gives;

$$a^2 = b^2 + e^2 \pm \frac{2b^2e}{c} \quad (10)$$

From fig 1.1, since |e| = |c|+|d|, we see that (10) becomes;

$$a^2 = b^2 + (c + d)^2 \pm \frac{2b^2(c + d)}{c}$$

$$a^2 = b^2 + (c + d)^2 \pm 2b^2 \pm \frac{2b^2d}{c} \quad (11)$$

In (11), the negative sign has to be true because if we choose the positive sign and set |d| to zero, we see that;

$$a^2 = b^2 + (c + 0)^2 + 2b^2 + \frac{2b^2(0)}{c},$$

$$a^2 = 3b^2 + c^2,$$

But if |d| = 0, |a| = |f|,

$$f^2 =$$

$3b^2 + c^2$, which is not the required pythagoras' theorem.

But in (11), if we choose the negative sign and set |d| to zero, we see that;

$$a^2 = b^2 + (c + 0)^2 - 2b^2 - \frac{2b^2(0)}{c},$$

$$a^2 = c^2 - b^2$$

$$c^2 = a^2 + b^2,$$

If |d| = 0, |a| = |f|,

$c^2 = f^2 + b^2$, which is the required pythagoras' theorem.

Therefore, simplifying (11) and choosing the negative sign, we get;

$$a^2 \quad b^2 + (c + d)^2 - 2b^2 - \frac{2b^2 d}{c}$$

$$a^2 = (c + d)^2 - b^2 - \frac{2b^2 d}{c}$$

$$(c+d)^2 = a^2 + b^2 + \frac{2b^2 d}{c}$$

$$(c+d)^2 = a^2 + b^2\left(1 + \frac{2d}{c}\right),$$

which completes the proof of the theorem.

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