Global Optimization. Methods of *α* – functions. Estimations (Part 2).

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Abstract

Author develop ideas of Global Optimization offered in article “Methods of the Global Optimization by Deformation of Functions” [9]: A new method of optimization by means of a redefinition of the function over a wider set and a deformation of the function on the initial and additional sets is proposed.

The method (a) reduces the initial complex problem of optimization to series of simplified problems, (b) finds the subsets containing the point of global minimum and finds the subsets containing better solutions that the given one, and (c) obtains a lower estimation of the global minimum.

Key words: Global optimization, universal optimization, method of function deformation.

§1. *α* – functions over arbitrary set.

A). The special case of *β*-function is α-function [9] . It is defined over set *Z=X×Y* and has the following properties:

1) There exist subset $K \subset Z$ with projection *K* on X_i $pr_1K = X^*$.
2) $\widetilde{\alpha}(x,y) = 0$ on *K*.

Theorem 1.1. *Assume* $\widetilde{\alpha}(x,y)$ *is* $\widetilde{\alpha}$ *-function and exist the point of global minimum* $x^* \in X^*$.
Then the element $\bar{x}$ *is point of the global minimum of object function I(x) over set X* if and only if there exist* $\widetilde{\alpha}(x,y)$ *such that:*

1) $J(\bar{x},\bar{y}) = \inf[I(x)+\alpha(x,y)] \quad x,y \in Z; \quad 2)\ \bar{x},\bar{y} \in K$.

Proof: As $\bar{x},\bar{y} \in K$, then $\alpha(\bar{x},\bar{y}) = 0$ and

$$J(\bar{x},\bar{y}) = \inf_Z[I(x)+\widetilde{\alpha}(x,y)] = \inf_K[I(x)+\alpha(x,y)] = \inf_{X^*} I(x) .$$

Q.E.D.

One may made vice versa. Define set $K_1 = \{x,y : \widetilde{\alpha}(x,y) = 0, \quad x \in X, \quad y \in Y\}$. Find $X_1 = pr_1K_1$. Then $\bar{x}$ is the point of minimum *I(x)* over X_1, if $\bar{x},\bar{y} \in K_1$.

The special case of $\widetilde{\alpha}$ -function is *α*-function defined over *Z* and such that *α(x,y)* = 0 over X* for all $y \in Y$.

The following theorem is important:

Theorem 1.2. *Let us assume α(x,y) = 0 over* X^* *for all* $y \in Y$ *and there exist* $x^* \in X^*$.
The element $\bar{x}$ *will be the point of global minimum of objective function I(x) over X* if there exist function α (x,y) such that*

1) $J(\bar{x},y) = \inf[I(x)+\alpha(x,y)] \quad x,y \in Z; \quad 2)\ \bar{x} \in X^*$. (1.1)

Proof: As $\bar{x} \in X^*$, then $\alpha(x,y) = 0$ and

$$J(\bar{x},\bar{y}) = \inf_Z[I(x)+\alpha(x,y)] = \inf_X[I(x)+\alpha(\bar{x},y)] = \inf_{X^*} I(x) .$$ Q.E.D.

If *y* is not constant, one can use it (the function $\alpha(x,y)$ from *y*) for getting $\bar{x} \in X^*$.

Theorem 1.3. $\widetilde{\alpha}$ *and α – functions exist and their number is infinite.*

Theorem 1.4. *(Estimate). If in (1,1) $\bar{x} \notin X^*$, we have a lower estimation of the objective function I(x) on X^*:*

$$J(\bar{x}(y), y) \le I(x) \quad for \quad all \quad y \in Y.$$

Proof is same [9].

One can get this estimation from $\alpha(x, y) = 0$ on set X^* for all $y \in Y$ and **Principle of Extension**[1] [16], because $X^* \subseteq X$.

The Principle of extension state: any extension of set, which you find on a minimum of functional, can only decrease on a minimum of an objective function (can only decrease value of a minimum).

The dependence *J(x,y)* from y one may use for improving of estimation. In particular, one can take $\alpha = \alpha(x)$. Then from theorems 1.2, 1.3 one can get the following consequences:

Consequence 1. Assume $\alpha(x) = 0$ on X^* and exist $x^* \in X^*$. Element $\bar{x}$ is point of a minimum of the objective function *I(x)* on X^* if and only if the exist α(x) such, that

1) $J(\bar{x}) = \inf [I(x) + \alpha(x, y)] \quad x \in X; \quad 2)\ \bar{x} \in X^*$. (1.1')

Consequence 2. If $\bar{x} \in X^*, \quad \beta \equiv \alpha \quad then \quad \inf\limits_{X \times Y} J = \inf\limits_{X^*} I$.

As far as α-function is the particular case β-function consequently the theorem 1.1 of [9] is right in this case.

Theorem 1.5. Assume $\bar{x}$ is point of global minimum of Problem 2:

$$J(\bar{x}) = \inf[I(x) + \alpha(x)], \quad x \in X.$$

Then: 1) The points of global minimum of Problem 1 are in the set

$$M^* = M \cap X^*, \quad where \quad M = \{x : \alpha \ge \bar{\alpha}\};$$

2) Set $N^* = N \cap X^*, \quad where \quad N = \{x : J + I \le \bar{J} + \bar{I}\}$, contain same or better solution

that is in *N* the object function $I(x) \le I(\bar{x})$;

3) Set $P^* P \cap X^*, \quad where \quad P = \{x : \alpha \le \bar{\alpha}\}$ contains same or worse solutions (that is $I(x) \ge I(\bar{x})$ in P).

The same way for this case we can be formulated the Theorem 1.1

Since the set X^* is selected by equal $\alpha(x) = 0$ we get from Theorem 1.5 the consequences:

$Consequence\ 3:\quad If\ \alpha(\bar{x}) > 0,\ then \quad X^* \subseteq P.$

$Consequence\ 4:\quad If\ \alpha(\bar{x}) < 0,\ then \quad X^* \subseteq M.$

$Consequence\ 5:\quad If\ \alpha(\bar{x}) = 0,\ then \quad \bar{x} \in X^*.$

From Theorems 1.2 – 1.4 and Consequence 1 we get:

Algorithm 4. We take the bounded of below functional (objective function) defined on X*Y, find minimal $\bar{x} = \bar{x}(y)$ of Problem 2: $\inf (I + \alpha), x \in X$ or minimal in implicit form $\xi(\bar{x}, y) = 0$. We solve together the system equations (combining equations of α- function): $\inf(I + \alpha), \quad x \in X$. Then value $\bar{x}$ - root pf this system is the absolute minimal of Problem1: $\inf(I + \alpha), x \in X$.

Algorithm 4' (solution by choice of α-function).

We take the bounded of below functional α defined on *X* (or *X*Y*), Solve the Problem 2: $\inf(I+\alpha)$, $x \in X$. If $\bar{x} \in X^*$, we get minimal of Problem 1, if $\bar{x} \notin X^*$, we get the estimation below $J(x) \le I(x^*)$ of value of the objective function *I(x)* on set X^* and we get the sets *M, N, P*.

Comments: 1. If the admissible set X^* allocates by functional $F_i(x)=0$, you can find the α - functional in form $\alpha = \lambda_i(x)F_i(x)$ (here *i* means sum), where $\lambda_i(x)$ are some function of *x*.
2. If the admissible set allocate by functional $\Phi_j(x) \le 0$, you can find α – functional in form

$$\alpha = \omega_l(x)[\Phi_l(x) + |\Phi_l(x)|],$$

where $\omega_l(x)$ are some function of *x*, or in form

$$\alpha = \omega_l(x)\Phi_l(x),$$

where $\omega(x) \ge 0$ and it is fulfilled the condition $\omega_l(x)\Phi_l(x) \equiv 0$ on X^*.

3. Assume there is some *α* –functional and element $x \in X^*$ such $J(\bar{x}) = \inf[I(x)+\alpha(x)], \quad x \in X$. Then any element $x_1 \in X^*$ and is satisfying the condition

$$J(x_1) = \inf[I(x)+\alpha(x)], \quad x \in X. \tag{1.1”}$$

is point of the absolute minimum the functional *I(x)* on X^* and any point of the absolute minimum the functional *I(x)* on X^* satisfy the condition (1.1”).
This direct statement follows immediately from condition 1.

We proof the converse. Since the global minimal $x_1 \in X^*$, it means $\alpha(x_1)=0$, then

$$I(x_1) = \inf_{X^*} I(x) = J(x_1) = J(\bar{x}) = \inf_X [I(x)+\alpha(x)] .$$

Q.E.D.

Thus, if it is existing one element which satisfy (1.1) then all rest minimal elements of Problem 1 must satisfy it.
I illustrate the idea of *α*-functional the next sample.

Let us take some function *f(x)* definite on interval [*a, b*]. Digital values $n \in [a,b]$ are admissible for it. We want find the minimum of this function. The addition member (α –functional) do not change *f(n)* in points *n*, but deforms *f(x)* in gaps between *n* (see fig. 2.1).

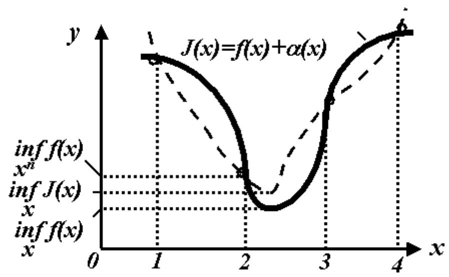

Fig. 2.1.

If *α* – functional is “good”, then $\inf_{x\in[a,b]}[f(x)+\alpha(x)] > \inf_{x\in[a,b]} f(x)$. If in addition $\bar{x} = n$, then we get the minimum of Problem 1.

Remark: There are different ways to solve problems by the *α*-functional:

a) You can take the known function as α-functional.
b) You can take α-functional as unknown function and find it together with the point of minimum.
c) You can take α-functional as function *α = α(x,y)* where *α* is known function but *y = y(x)* is unknown function of *x*. You must find it together with the point of minimum.
Let us consider the example. We take as example the non-good the functional which is difficult to solve by conventional method.

Example 1.1. Find the minimum of function

$$I=\frac{4x^2+4\pi x+4.1+\pi^2}{4(x^2+\pi x+1)+\pi^2}\cdot\frac{\sin^5 x-\sin^4 x\cdot\cos x+\sin^2 x\cdot\cos^3 x}{(\sin x-\cos x)(\sin^3 x+\cos^3 x)}\quad in\ X^*=\{x=0.5\pi n: n=0,\pm1,\pm2,...\}\ (1.2)$$

It is difficult to apply the known methods here because the functional is defined on digital set. The current methods offer only the calculation of all $x\in X^*$. But number of X^* equals infinity and calculation may be meaningless.
Let us to solve this example by the offered method. Take *α(x)* in form

$$\alpha=-\frac{4x^2+4\pi x+4.1+\pi^2}{4(x^2+\pi x+1)+\pi^2}\cdot\frac{0.5\sin 2x\cdot\cos x}{(\sin x-\cos x)(\sin^3 x+\cos^3 x)}.$$

You can see that *α(x)* = 0 in X^* because for *x* = 0.5π*n* $n=0,\pm1,\pm2,...,\quad \sin 2x=\sin\pi n=0$.

Let us to create the general functional

$$J=I+\alpha=-\frac{4x^2+4\pi x+4.1+\pi^2}{4(x^2+\pi x+1)+\pi^2}\cdot\frac{\sin^5 x-\sin^4 x\cdot\cos x+\sin^2 x\cdot\cos^3 x-0.5\sin 2x\cdot\cos x}{(\sin x-\cos x)(\sin^3 x+\cos^3 x)}.$$

Here the variable *x* is uninterrupted and - ∞ < *x* < ∞ (set *X*)

The additive *α(x)* allows to change the functional (1.2) to simple form

$$J=\frac{4x^2+4\pi x+4.1+\pi^2}{4(x^2+\pi x+1)+\pi^2}\cdot\frac{(\sin^2 x-\cos^2 x)(1-\sin x\cdot\cos x)\sin x}{(\sin^2 x-\cos^2 x)(\sin^2 x-\sin x\cdot\cos x+\cos^2 x)}=\left(\frac{0.1}{4+(2x+\pi)^2}+1\right)\sin x.$$

This general functional is simple. His minimum may be found the conventional method of theory the function one variable. Here $\bar{x}$ = - π/2 , $\bar{x}\in X^*$ for $\bar{n}=1,\quad \bar{I}=-1.025$. Consequently, that is absolute minimum (and sole) of initial functional (1.2).

We can apply an analogical method for finding of minimum on *x* the next function:

$$I=\cos^2\varphi+0.5\cos 2x\cos 2\varphi-2\cos x\cos\varphi\cos(x+\varphi)+0.5-0.1e^{-x^2},\quad X^*=\{x=0.5\pi n: n=0,\pm1,\pm2,...\}.$$

Here *φ* is given, *x* is digital. Let us take $\alpha=-0.5\sin 2x\sin 2\varphi$. After this we can change our functional *J = I + α* to simple form: $J=-0.1e^{-x^2}+\sin^2 x$. The point of absolute minimum this task (Problem 2) is $\bar{x}$ = 0. This point is in allowable set X^* for $\bar{n}=0$. That means $\bar{n}=0$ is point of the absolute minimum od the initial Problem 1.

The reader can think: if the allowable numerical set is limited we can use the conventional Lagrange's method [12]. Let us show: that is not correct.

Example 1.2. Find minimum of functional:

$$I=x^3-3x^2+x\quad on\quad X^*=\{x=0, x=3\}. \qquad (1.3)$$

Let us to write the Lagrange's function

$$F=x^3-3x^2+2x+\lambda_1 x+\lambda_2(x-3),$$

where λ_1, λ_2 are LaGrange's factors. Find the first derivative

$$F' = 3x^2 - 6x + 2 + \lambda_1 + \lambda_2 \ .$$

Substitute to here *x* = 0 , *x* = 3 and write the equations $F'(0) = 0, \quad F'(3) = 0.$ We find from these equations λ_1 , λ_2 . Find the second deviation $F'' = 6x - 6.$ When *x* = 0 the function $F''(0) = -6 < 0.$

When *x* = 3 the function $F''(3) = 12 > 0.$ Consequently *x* = 0 is the point of maximum, *x* = 3 is the point of minimum. Let us check up. Substitute *x* = 0 and *x* = 3 in (1.3). We find *I*(0) = 0, *I*(3) = 6 .

We see the LaGrange's method gives the opposed result: it declare the point of minimum as the point of maximum, but the point of maximum as the point of minimum. In here it is violating one condition of LaGrange's method: The number of additional equations is more of number of variables. This example is shows: this violation for LaGrange's method is unacceptable.

Let us to solve this example by the offered method. Take the *α(x)* in form

$$\alpha = x(x-3)(2/3-x) \ .$$

Then

$$J = I + \alpha = x^3 - 3x^2 + 2x + x(x-3)(2/3 - x), \quad J' = 4/3x = 0, \quad \bar{x} = 0 \in X^*, J'' = 4/3 > 0 .$$

From Consequence 1 the point $\bar{x} = 0$ is absolute minimum of functional (1.3). That shows the method of *α* – functional has more application then the LaGrange's method.

Example 1.3. Find minimum of integral

$$I = \int_{-10^{-3}}^{a} (\ln \mathrm{tg}\, t - 10^{-3}) dt \quad on \quad X^* = \{a = 10^{-3}\pi n : \ n = 1,2,..., 400\} \qquad (1.4)$$

Here the interval of integration is discrete. The direct search is difficult because integral (1.4) cannot be presented by simple function and it not have of tabulations.

Let us to find *α*-functional in form: $\alpha = -10^{-6} \sin 10^3 a$. You see on X^* the function *α*(*x*) = 0. Further

$$J = I + \alpha = \int_{10^{-3}}^{a} (\ln tg\ t - 10^{-3}) dt - 10^{-6} \sin 10^3 a \ ,$$

$$J_a^{'} = \ln tg\ a - 10^{-3} - 10^{-3} \cos 10^3 a = 0 \ , \quad \bar{x} = \pi/4 \in X^* \quad for \quad \bar{n} = 250 \ , \qquad (1.5)$$

$$J'' = \frac{2}{\sin 2a} + \sin 10^3 a \quad .$$

As $10^{-3} < x < 0.4\pi$, then *J''* > 0 into this interval. That means the root is single and $\bar{n} = 250$ is point of the absolute minimum.

Analogically we find the minimum of other integral which cannot be presented in simple functions

$$I = -\int_{0}^{a} [\sin(t^3) + 10^{-5}\sqrt{\pi}] dt \quad on \quad X^* = \{a = 10^{-3}\sqrt{\pi} n : n = 0, 1, ..., 1.5 \cdot 10^3\} \ . \qquad (1.6)$$

Here is $\alpha = 10^{-3} \sin 10^{-8} \sin 10^3 \sqrt{\pi} a \ ; \quad \bar{n} = 1000$.

Example 1.4 . Find the minimum of integral

$$I = \int_{\pi/2}^{\pi} \left(\frac{\cos at}{t} + 20a^3 \right) dt \quad on \quad X^* = \{a = 10^{-3} n : n = 0, \pm 1, \pm 2, ...\} \ . \qquad (1.7)$$

Here the under integral function is discrete. The integral from this function cannot be presented as elementary functions.

Let us take $\alpha = 10^{-3}\sin^2 10^3 \pi a, \quad J = I + \alpha$. Then

$$J_a^{'} = I_a^{'} + \alpha_a^{'} = \int_{\pi/2}^{\pi}(-\sin at + 40a)dt + 2\cdot 10^{-4}\pi \sin 2\cdot 10^3 \pi a =$$
$$= -\frac{2}{a}\sin\frac{3}{4}\pi a \cdot \sin\frac{\pi}{4}a + 20\pi a + 10^{-4}\pi \sin 2\cdot 10^3 \pi a \qquad (1.8)$$

This derivative not exist for $\bar{a} = 0 \in X^*$.

$For \quad a \geq 0, \quad J' > 0; \quad for \quad a < 0, \quad J' < 0; \quad (or \quad J'' > 0 \quad for \quad \forall a \neq 0)$.

Consequently $\bar{n} = 0$ is point of absolute minimum.

B) Consider the case when the point of optimum $x^* \in X^*$ not exist, but exist the sequence such that $\lim_{n\to\infty} I(x_n) = m$.

This sequence is named the minimizing sequence (see [9] of Ch.1).

Similarly point A we can show that consequence 1 can be generalized in this case.

Consequence 1'. Let us *α(x)* = 0 only on *X**, For minimizing sequence $\{x_n\} \subset X^*$ is necessary and sufficient the existing of function *α(x)* such that

$$\lim_{n\to\infty}[I(\alpha_n) + \alpha(x_n)] = \inf[I(x) + \alpha(x)], \quad x \in X. \qquad (1.9)$$

The sufficiency of this consequence is same the lemma in [2] and *J(x) = L* in [2].

We can generalize remark 3 of item 1 in this case: If exist *α* function and one sequence $\{x_n\} \subset X^*$ which satisfy (1.9), then the any sequence $\{x_n\} \subset X^*$ which satisfy (1.9) is the minimizing sequence. And on the contrary any the minimizing sequence satisfy the condition (1.9).

2. α – function in Banach space.

Let us to apply Theorem 1.2 to optimal problem is described in Banach space by equation

$$\frac{dx}{dt} = f(x,u), \quad t_1 \leq t \leq t_2, \quad x(t_1) = x_1, \quad x(t_2) = x_2, \qquad (1.10)$$

where *x, f(x,u)* – element complete linear normed space X_1 and X_2 respectively and $X_2 \subset X_1, \quad t \in = [t_1, t_2] = T$ is segment of real axis.

Let us name the permissible control the measurable limited function (in term [1], p.85)with value $u \in U$, where *U* is set in arbitrary topological space. In particular the set *U* may be metric, closed and limited. Let us assume that for any control *u(t)*the equation (1.10) has single solution *x(t)* with $x(t) \in X_1$ for almost all $t \in [t_1, t_2]$, where *x(t)* is continuous almost everywhere differentiable on function on $t \in [t_1, t_2]$.

Operator *f(x,u)* is defined on the direct product *X×U* . One is continuous and bounded. Boundary conditions are given t_1, t_2, $x(t_1) = x_1$, $x(t_2) = x_2$.

State the problem: Find the admissible control which transfers the system from given initial state in given final state with function

$$I = \int_{t_1}^{t_2} f_0(x, y)dt \qquad \textbf{(1.11)}$$

has a minimum.

Let us the set of the measurable functions *u (t)* is denoted *V*: set of the continuous, almost everywhere differentiable on (t_1,t_2) the functions *x (t)* is denoted *D*. Set of couple x(t), u(t) having named over properties and almost all satisfied the equation (1.10), we name *admissible* and denote Q. It is obvious $Q \subset D \times V$.

Assume $\psi = \psi(t,x)$ is the some unequivocal continuous differential function defined on *X×T* . We name it the *characteristic function*. We will find the α – function in form

$$\alpha = \int_{t_1}^{t_2} \psi_x * [\dot{x} - f(x,u)]dx \tag{1.12}$$

Here $\psi_x = \dfrac{\partial \psi}{\partial x}$ is particular deviation of Freshen. One is linear function. The * is sign of composition. Obvious that request of α-function is performed.

Compose the generalized function *I = J + α* and produce the function $\dot{\psi} = \psi_x \dot{x} + \psi_t$ we get

$$J = \psi[t_2, x(t_2)] - \psi[t_1, x(t_1)] + \int_{t_1}^{t_2} (f_0 - \psi_t - \psi_x \circ f)dt = \psi_2 - \psi_1 + \int Bdt \quad , \tag{1.13}$$

where $B = f_0 - \psi_t - \psi_x \circ f$. Because the set *Q* is different from the set *D×V* only that couple *x(t), u(t)* satisfy almost every where (1.10). For α-function in form (1.12) with according of Theorem 1.2 we can the initial Problem 1 (find the minimum (1.11) on *Q*) replace the Problem 2 – find minimum (1.13) on the broader set *D×V*. In this set the *x(t), u(t)* not bind the equation (1.10). So, we have

$$\bar{J} = \psi_2 - \psi_1 + \inf_{x(t) \in D, u(t) \in V} \int_{t_1}^{t_2} B(t,x,u)dt \, . \tag{1.14}$$

Theorem 1.6. If function $\bar{u}(t)$ getting from solution of problem $\inf\limits_{x(t) \in D, u(t) \in V} \int_{t_1}^{t_2} Bdt$ is $\bar{u}(t) \in V$, that it is same almost everywhere the function getting from solution the problem $\inf\limits_{\substack{x(t) \in D, \\ u(t) \in V}} \int_{t_1}^{t_2} Bdt$ and

$$\inf_{x(t) \in D, u(t) \in V} \int_{t_1}^{t_2} Bdt = \inf_{x(t) \in D} \int_{t_1}^{t_2} \inf_{u \in V} Bdt \tag{1.15}$$

Proof. Assume the contrary: $B(u^*) \neq \inf\limits_{u \in U} B(u)$ on subset of interval [t_1, t_2] with measure not equal zero. In this case $B(u^*) > B(\bar{u}) \quad \text{i.e.} \quad \int_{t_1}^{t_2} B(u^*)dt > \int_{t_1}^{t_2} B(\bar{u})dt$ on the subset. This contradict: the function $u^*(t)$ made the minimum for integral $\int_{t_1}^{t_2} Bdt,$.

From requirement (1.14) and Theorem 1.6 we have

$$\bar{J} = \psi_2 - \psi_1 + \inf_{x(t) \in D} \int_{t_1}^{t_2} \inf_{u \in V} B(t,x,u)dt \tag{1.16}$$

If function $\alpha[x(t),u(t)]$ is such that absolute minimum of Problem (1.16): $\vec{x}(t),\ \bar{u}(t) \in Q$, then ααaccording to Theorem 1.1 functions $\bar{x}(t), \bar{u}(t)$ are absolute minimum of the initial Problem.

So, we proofed

Theorem 1.7. To couple function were the absolute minimum the function *I* , it is sufficient the existing the characteristic function $\psi(t,x)$ such that

$$1)\ B(t,x,\bar{u}) = \inf_{u \in U} B(t,x,u); \quad 2)\ \int_{t_1}^{t_2} B(t,x,\bar{u})dt = \inf_{x(t) \in D} \int_{t_1}^{t_2} B(t,x,\bar{u})dt; \quad 3)\ \vec{x}(t),\ \bar{u}(t) \in Q \tag{1.17}$$

In particular, if take $\psi = p(t) \circ h$, where p(t) is linear function $h \in X_1$, then from item 1 and stationary condition item 2 [1.17] we get

$$H(t,x,\bar{u}) = \sup_{u \in U} \sup H(t,x,u)\,, \quad \dot{p}(x) = -\frac{\partial H}{\partial x}\,, \qquad \textbf{(1.18)}$$

where $H = p(t) \circ f(x,u) - f_0(x,u).$

Assumed $\partial H / \partial x$ is Fréchet derivative, which is continuous. As we see the necessary condition of Problem 2 following from (1.17) is same the necessary condition of Pontriagin principal of maximum generalized in Banach spaces.

3. Design of α-function for allowable subset of two function connected by logical conditions

Assume two functions $F_1(x)$ and $F_2(x)$ are refinished on the set X. Allowable are only points $x \in X$ and functions F_1 and F_2 which are connected the logical conditions. Assume $F_1(x) = 0$ is “true” and $F_2(x) \neq 0$ is “false’. The five main logical connections (↔, y, v, ∧, ~) $(\leftrightarrow, y, \vee, \wedge, \sim)$ are presented in next tables:

F_1	F_2	$F_1 \leftrightarrow F_2$
t	t	t
t	f	f
f	t	f
f	f	t

Double implication

F_1	F_2	F_1 y F_2
t	t	f
t	f	t
f	t	t
f	f	f

disjunction in the exclusive sense

F_1	F_2	$F_1 \vee F_2$
t	t	t
t	f	t
f	t	t
f	f	f

disjunction in the sense of a non-exclusive

F_1	F_2	$F_1 \wedge F_2$
t	t	t
t	f	f
f	t	f
f	f	t

Conjunction

F	*p*
t	*f*
f	*t*

Denial

We will use the symbol:

$$\operatorname{sign} F = 1 \quad if \quad F > 0,$$
$$\operatorname{sign} F = 0 \quad if \quad F = 0,$$
$$\operatorname{sign} F = -1 \quad if \quad F < 0,$$

In this case the α-function we can search in form:

$$1)\ X^* = \{x : F_1(x) \leftrightarrow F_2(x)\}, \quad \alpha = (p_1F_1 + p_2F_2)[1 - |\operatorname{sign}(F_1F_2)|],$$
$$2)\ X^* = \{x : F_1(x)\ y\ F_2(x)\}, \quad \alpha = p_1F_1F_2 + p_2[1 - |\operatorname{sign}(F_1^2 + F_2^2)|],$$
$$3)\ X^* = \{x : F_1(x) \vee F_2(x)\}, \quad \alpha = pF_1F_2\,,$$
$$4)\ X^* = \{x : F_1(x) \wedge F_2(x)\}, \quad \alpha = p_1F_1 + p_2F_2\,,$$
$$5)\ X^* = \{x : F_1(x) \sim F_2(x)\}, \quad \alpha = (p[1 - |\operatorname{sign}\ F|],$$

Here p, p_1, p_2 are some function x.

It is using these five connections we can create all other complex logic statements.

§2. The general principle of reciprocity the optimization problems

Let us suppose we want to solve the optimal problem Ch.1 §4 [9]:

$$I = f_0(x)\,, \quad f_i(x) = 0, \quad i = 1,2,\ldots,m\,, \tag{2.1}$$

Design general function in form

$$J = \sum_{i=0}^{i=n} \lambda_i(x,y) f_{i,}(x)\ , \tag{2.2}$$

where $\lambda_i(x,y)$ arbitrary functions of x, y.

Assume $\bar{x}(y)$ is absolute minimum (2.2) on X.

The general principle of reciprocity the optimization problems.

1. For any $y \in Y$ the point of an absolute minimum of the function J (2.2) is the point of the absolute minimum any function

$$\lambda_j(x.y)f_j(x)\,,\quad j=0.1,...,m \quad (\text{no sum for } j) \quad , \tag{2.3}$$

for limits in form

$$\lambda_i(x,y)=\lambda_i(\bar{x}(y),y)f_i(\bar{x}(y))\,,\quad i=0,1,...,m\,,\quad i\neq j\,,\quad (\text{no sum for } i)\,. \tag{2.4}$$

Any numbers of equality (2.4) you can change by non-equalities

$$\lambda_i(x,y)\leq\lambda_i(\bar{x}(y),y)f_i(\bar{x}(y)\,)\,. \tag{2.5}$$

2. For any $y\in Y$ the point of the absolute minimum of the function *J* (2.2) is point of the absolute minimum any sum the functions

$$\sum_j \lambda_j(x,y)f_j(x) \tag{2.3$'$}$$

for restrictions absent in sum (2.3)

$$\lambda_i(x,y)=\lambda_i(\bar{x}(y),y)f_i(\bar{x}(y)\,,\quad i=0,1,...,m\,,\quad i\neq j\,,\quad (\text{no sum for } i)\,. \tag{2.4$'$}$$

Any numbers of equality (2.4)’ you can change by non-equalities (2.5).

<u>Proof.</u>

1) For any function (2.3) for conditions (2.4) the Theorem 1.2 is made. The point $\bar{x}(y)$ is point of its absolute minimum. As every function reaches the global minimum, obvious, the change equality (2.4) by restrictions (2.5) not influence to minimum. The point 2 is proofed similarly. Principle is proved.

<u>Consequence 1.</u>

Magnitude $J(\bar{x}(y),y)$ is the lower estimation of any function from (2.3), (2.3)’ if part or all equalities (2.4), (2.4)’ change equalities in form

$$\lambda_i(x,y)f_i(x)=0 \tag{2.6}$$

<u>Consequence 2.</u> In case corresponded (2.6) the absolute minimum of any functions (2.3) are located in set

$$M_j(y)=\{x:\sum\nolimits_{\substack{i=1\\ i\neq j}}^m \lambda_i(x,y)f_i(x)\geq\sum\nolimits_{\substack{i=1\\ i\neq j}}^m \lambda_i((\bar{x}(y),y)f_i(\bar{x}(y)) \tag{2.7}$$

<u>Consequence 3</u>. If possible the solution of Problem (2.1) by Algorithm 4, there are *y* such that

$$\lambda_i((\bar{x}(y),y)f_i(\bar{x}(y)\leq 0 \quad (\text{no sum for } i) \tag{2.8}$$

From the existence of solutions (2.1) follows that $f_i(x)=0$. So $\bar{\lambda}_i\bar{f}_i$ is minimum, than (2.8) is obvious.

§3. Applications α-function to well-known Problems of optimization

1. Problem the searching of conditional extreme the function of the limited number variables.

It is given

$$I=f_0(x)\,,\quad f_i(x)=0\,,\quad i=1,2,...,m<n \tag{3.1}$$

Here *x* is *n*-dimensional vector given in some numerical open region of *n*-dimensional space X^*.

Let us take the α-function in form

$$\alpha = p_i(x) f_i(x), \quad i = 1,2,...,m \tag{3.2}$$

(repeated indexes mean summarization). Here $p_i(x)$ are functions x, given on X:

$$X^* = \{x : \sum_{i=1}^{m} | f_i(x) | = 0\}, \ X^* = X.$$

Let us to design generalized functional $J(x) = f_0(x) + \alpha(x)$ take some $p_i(x)$ and sole the problem $\inf J(x), \quad x \in X$. From this solution the Problem 2, according Theorems §1, we can get the following information about Problem 1:

1) If $\bar{x} \in X^*$, than $\bar{x}$ is absolute minimum of Problem 1 (consequence 1, §1).

2) If $\bar{x} \notin X^*$, then:

a) $J(\bar{x})$ is the lower estimation of function $f_0(x)$ on X^* (Theorem 1.4).

b) For $\alpha(\bar{x}) > 0$ x^* is located in set $P = \{x : \alpha(x) \le \alpha(\bar{x})\}$ (consequence 3, §1).

c) For $\alpha(\bar{x}) < 0$ x^* is located in set $M = \{x : \alpha(x) \ge \alpha(\bar{x})\}$ (consequence 4, §1).

d) Set $N^* = N \cap X^*$ where $N = \{x : 2f_0 + \alpha \le 2\bar{f}_0 + \bar{\alpha}\}$ contains the equal or worse solutions (Theorem 1.5).

As we see even, if $\bar{x} \notin X^*$ our computation is useful. We received the lower estimation and narrow the region for searching of the optimal solution. Take row of α_i we can get the solution one of the Problems ***a, b, c, d*** or facilitate the solution of Problem ***a*** (see Ch, 1, §1 [9]).

Look your attention: the offered method does not require continuity and differentiability of the functions $f_0(x), f_i(x)$ in contrast to the classical method of Lagrange multipliers. The method can be applied to non-analytical function, for example, to the functions definite on the discrete set and extremal problems of the combinatorics (see Ch. 10).

2. Application the Theorems §1 to optimal problems described the conventional differential equations.

Assume the moving of object is described by system of the differential equations

$$\dot{x}_i = f_i(t, x, u), \quad i = 1,2,...,n, \quad t \in T = [t_1, t_2], \tag{3.3}$$

where $x(t)$ –n-dimensional continuous piecewise differentiable function, $x \in G(t)$; $u(t)$ – r-dimensional functions continuous everywhere on T, except limited number of points where one can have discontinuity of the first kind $u \in U(t)$. Boundary values t_1, t_2 are given, $x(t_1), x(t_2) \in R$.

Optimal function is

$$I = F(x_1, x_2) + \int_{t_1}^{t_2} f_0(t, x, u) dt, \quad x_1 = x(t_1), \quad x_2 = x(t_2). \tag{3.4}$$

Functions $F(x_1, x_2)$, $f_i(x,u,t)$, $i = 0,1,…,n$ are continuous, $F(x_1,x_2) > -\infty$. Set of the continuous almost everywhere differentiable functions $x(t)$ with $x \in G(t)$ we designate D. Set of the piecewise continuous (they can have the discontinuity of the first kind) functions u(t) such that $u \in U(t)$ we designate V. Couple $x(t), u(t)$ have named over properties and almost everywhere satisfy the equations (3.3) we name allowable and designate Q, $Q \subset D \times V$.

Enter in our research n single-valued functions $\lambda_i(t.x)$ $i = 1,2,…,n$. which are continuous and have continuous derivatives on T$\times$G. Let us to take the α-function in form

$$\alpha = \int_{t_1}^{t_2} \lambda_i(t,x)[\dot{x} - f_i(t,x,u)]dt \qquad (3.5)$$

It is obvious $\alpha = 0$ on Q. Let us design the general function $J = I + \alpha$, integer the term $\lambda_i \dot{x}_i$ by part and exclude $\dot{x}_i$ by (3.3). We get

$$J = F + \lambda_i x_i |_{t_1}^{t_2} + \int_{t_1}^{t_2} [f_0 - (x_j \frac{\partial \lambda_j}{\partial x_i} + \lambda_i) f_i - x_i \frac{\partial \lambda_i}{\partial t}]dt \qquad (3.6)$$

Designate

$$a = F + \lambda_i x_i |_{t_1}^{t_2}, \quad B = f_0 - (x_j \frac{\partial \lambda_j}{\partial x_i} + \lambda_i) f_i - x_i \frac{\partial \lambda_i}{\partial t}$$

Apply to (3.6) Consequence 1 §1. Here the Q is X^* and D x V is X (see Consequence 1 §1). Since now the couple of functions $x(t)$, $u(t)$ from DxV (having ends in R for condition $\bar{x}(t) \in D, \bar{u}(t) \in V, x_1 = x(t_1), x_2 = x(t_2)$) are not connected by the equations (3.3) we can write

$$\inf_{D\times V} (A + \int_{t_1}^{t_2} Bdt) = \inf_{x_1, x_2 \in R} A + \int_{t_1}^{t_2} \inf_{x \in G, u \in U} Bdt$$

and final

$$\bar{J} = \inf_{x_1, x_2 \in R} A + \int_{t_1}^{t_2} \inf_{x \in G, u \in U} Bdt \qquad (3.7)$$

So we proofed the **Theorem 3.**1:
The couple vector-function $\bar{x}(t), \bar{u}(t)$ will be point of absolute minimum of function (3.4) if it is exist n differentiable $\lambda_i(t,x)$ such that:

$$1)\ \bar{B} = \inf_{x \in G, u \in U} B, \quad 2)\ \bar{A} = \inf_{x_1, x_2 \in R} A > -\infty, \quad 3)\ \bar{x}, \bar{u} \in Q \qquad (3.8)$$

<u>Note</u>: That is sufficient condition only. That cannot be a necessary condition because we don't know advance about an existence of $\lambda(t,x)$.

From (3.8) it is follow: if we find at least one solution of an equation in particular derivations having n-unknown functions $\lambda_i(t,x)$:

$$\inf_{u \in U} [f_0 - (x_j \frac{\partial \lambda_j}{\partial x_i} + \lambda_i) - x_i \frac{\partial \lambda_i}{\partial t}] = 0, \qquad (3.9)$$

for boundary condition A = const, then points 1, 2 of the Theorem 3.1 will be executed. Any unsuccessful $\lambda_i(t,x)$ (if $\bar{x}(t), \bar{u}(t) \notin Q$) with according Theorem 1.4 gives the lower estimation of the global minimum.

Assume, for example, $x_n \neq 0$*. Substitute them in (3.7), we get the result published in work [9]**, (condition Bellman-Piconet):

$$\bar{J} = \inf_{x_1 \in G_1,\ x_2 \in G_2} \Phi - \int_{x_1}^{x_2} \sup_{x \in G, u \in U} R(t,x,u)dt, \qquad (3.10)$$

Here $\Phi = F + \varphi_{t_1}^{t_2},\ R = \varphi_t + \varphi_{x_i} f_i - f_0 = -B.$

----

* This limitation is not important because any $x_i \neq 0$ in $[t_1, t_2]$.

** Note: in given method (in difference from [2]) not request a priory assumption about existing the single potential function $\varphi(t,x)$ such that $\varphi_{xi} = \lambda_i$.

Sometimes it is more comfortable take function $\varphi(t,x)$ or in other terms (see [4]) $\psi(t,x)$. Then *A, B* are written:

$$A = F + \psi_2 - \psi_1, \quad B = f_0 - \psi_{x_i} f_i - \psi_t, \tag{3.11}$$

And Theorem 3.1 is same with [9], (see also [4]).

Function α for given task we can define also the next way. Take some function $\psi(t,x)$. Then

$$\alpha = \int_{t_1}^{t_2} \psi_{x_i} [\dot{x} - f_i(t,x,u)] dt$$

Integrate but parts the first member we get

$$\alpha = \psi |_1^2 - \int_{t_1}^{t_2} (\psi_{x_i} f_i + \psi_t) dt$$

<u>*Note:*</u> 1. Theorem 3.1 is corrected and in notations (3.8):

$$\int_{t_1}^{t_2} B dt = \inf_{x(t) \in B} \int_{t_1}^{t_2} \inf_{u \in U} B dt.$$

This form is offered in [4]. Difference between these forms is important in consideration the second variation, conditions in angle points and in some other cases. Let us take the last corrected form of V. Krotov optimization [8] (problem of speed):

<u>Example 3.1</u>. Find minimum t_2 in task:

$$I = \int_{t_1}^{t_2} dt, \quad \dot{x} = u, \quad |u| = 1, \quad x(0) = 1, \quad x(t_2) = 0.$$

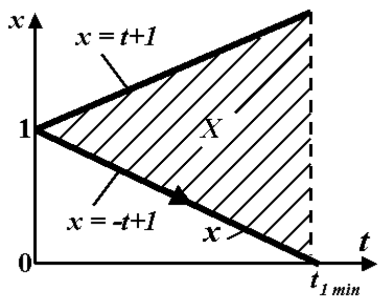

Fig.2.2.

If we take $\varphi = 0$, we get R = -1. Consequently $\sup\limits_{x,u} R$ is reached in ANY curve, for example, u = - 0.01 (I = 100).

In case when *min* forward integral for $\psi = 0$ we have

$$\inf_{x(t) \in D} \int_{t_1}^{t_2} dt = \inf_{x(t) \in B} t_2[x(t)].$$

Since the set all serves with bounded derivative $|\dot{x}| \leq 1$ for $x(0) = 1$ located between lines $x = t - 1$, $x = -t+1$ (Fig. 2.2), we get $\bar{x} = 1 - t, \quad \bar{u} = -1 \quad and \quad I = t_{1,\min} = 1.$

Notes: 1. As set B we can take a set {*x*(*t*)} with bounded derivative $\dot{x}_i \in X_i = \{f_i(t,x,u) : u \in U\}$. This narrowing can help in finding of optimal solution.

2. Note 3 §1 in given case has the following view: If exist the function *ψ(t,x)* and at list one allowable couple $\bar{x}(t), \bar{u}(t)$, satisfying (3.8). That any other couple satisfying (3.8) is minimum of problem 1 and any allowable minimum the problem 1 satisfy p.1, 2 (3.8).

3. If t1, t2 are not fixed, we can show that point 1, 2 (3.8) are:

$$1)\ \bar{B} = \inf_{x \in G,\, u \in U} B = 0, \quad 2)\ \bar{A} = \inf_{t_1, t_2, x_1, x_2 \in R} A > -\infty$$

We can satisfy the condition *inf B* = 0, if we take *ψ = φ(t,x)* + y_{n+1} and

$$\dot{y}_{n+1} = f_0 - \varphi_{x_i} f_i - \varphi_t \,.$$

4) Theorem 3.1 is particular case of more common theorem 2.1 considered in Chapter Ш.

Assume we take some $\lambda_i(t,x)$ (or *ψ(t,x)*).

Theorem 3.2. Assume *F* = 0 and solved the problem $\inf_{x,u} B$. Then:

1) Set $N = \{t,x,u : B + f_0 \le \bar{B} + \bar{f}_0,\, t \in T\}$ contains same and better solutions of Problem 1;
2) Set $P = \{t,x,u : B - f_0 \le \bar{B} - \bar{f}_0,\, t \in T\}$ contains same and worse solutions of Problem 1.

Proof: 1) Deduct $B \ge \bar{B}$ from inequality $B + f_0 \le \bar{B} + \bar{f}_0$. We get

$$f_0 \le \bar{f}_0 \quad \text{on} \quad T, \quad \text{i.e.} \quad \int_T f_0 dt \le \int_T \bar{f}_0 dt \,.$$

2) Deduct $B \ge \bar{B}$ from inequality $B - f_0 \le \bar{B} - \bar{f}_0$. We get

$-f_0 \le -\bar{f}_0 \quad \text{on} \quad T, \quad \text{i.e.} \quad \int_T f_0 dt \ge \int_T \bar{f}_0 dt$. Theorem is proved (QED).

Let us take instead function (3.4) simpler function $\int_T B_1(t,x,u)dt$ (here B1 is given function). Than

Theorem 3.3. Assume *F* = 0 and solved the problem $\bar{J}_1 = \inf \int_T B_1(t,x,u)dt \text{ on } Q$. Than:

3) Set $N = \{t,x,u : B_1 + f_0 \le \bar{B}_1 + \bar{f}_0,\, t \in T\}$ contains the same and better solutions of Problem 1;
4) Set $P = \{t,x,u : B_1 - f_0 \le \bar{B}_1 - \bar{f}_0,\, t \in T\}$ contains the same and worse solutions of Problem 1.

Proof: 1) From *N* we have the inequality $\int_T (f_0 + B_1)dt \le \int_T (\bar{B}_1 + \bar{f}_0)dt$. Deduct from this inequality the inequality $\int_T B_1 dt \ge \int_T \bar{B}_1 dt$. We get $\int_T f_0 dt \le \int_T \bar{f}_0 dt$.

2) From *P* we have the inequality $\int_T (B_1 - f_0)dt \le \int (\bar{B}_1 - \bar{f}_0)dt$. Deduct $\int_T B_1 dt \ge \int \bar{B}_1 dt$ from this inequality. We get $\int_T f_0 dt \ge \int_T \bar{f}_0 dt$. Theorem is proofed (QED).

Consequence. If set *P* cover the set $T \times G \times U$ (or reachability set) and $\bar{x}, \bar{u} \in Q, \quad \text{then} \quad \bar{x}, \bar{u}$ are absolute minimum of Problem 1.

Note. Delete part equation (3.1) or (3.2) [in case (3.2) x_i corresponded deleted equations became the control in the rest equations]. Then gotten solution is the low estimation of initial Problem as it is following from principle of expansion [16]: $I(x) \geq I(\bar{x})$ and $I(x,u) \geq I(\bar{x},\bar{u})$, where $\bar{x}(t), \bar{u}(t)$ are absolute minimum "truncated" task.

When right parts of equations (3.3), (3.4) do not depend clearly from x(t), we can stand out not only set *N,P* but the set *M*. It is correct the following theorem

Theorem 3.4. *Assume* $F \geq 0$, *ends x(t) is free, the right parts of equations (3.3), (3.4) depent only from t, u , i.e.:* $f_i = f_i(t,u)$ *i = 0,1,…,n. and solved task* $\inf\limits_{x,u} B_1(t,u)$. *Than:*

1) Set $M = \{t,u : B_1 - f_0 \geq \bar{B}_1 - \bar{f}_0, t \in T\}$ *contains the absolute minimum of Problem 1;*

2) Set $N = \{t,u : B_1 - f_0 \geq \bar{B}_1 - \bar{f}_0, t \in T\}$ *contains the same and better solutions of Problem 1;*

3) Set $P = \{t,x,u : B_1 - f_0 \leq \bar{B}_1 - \bar{f}_0, t \in T\}$ *contains the same and worse solutions of Problem1.*

Proof for sets *N*, *P* full equally with the proof of Theorem 3.2. Proof for *M* follows from discontinuity u*(t)* and depends the right parts of equation only from *u*.

3. Task the dynamic programming of Bellman

Assume there is physical system *S*. The control of this system separated in *m* steps. On every *i* step we have the control U_i. Using this control we transfer our system from allowable stand S_{i-1} getter in (*I* - 1) step in new allowable stand $S_i = S_i(S_{i-1}, U_i)$. This transfer is bounded by some conditions. The purpose is minimum function

$$W = \sum_{k=1}^{n} w_k$$

Let us to biuld the common function

$$J_i = W_i + \alpha, \text{ where } W_i = \sum_{k=1}^{n} w_k, i = 1,2,...,m$$

In this case we can change the task of the conditional minimum inf Wi in the task of direct minimum $\inf\limits_{V} J_i$. If the limitations are absent or they allow the select U_k in every step to make with associated conditions, then from $\alpha = 0$ in the admisseble elements we get the Bellman equation [11].

$$\overline{W}_i(S_{i-1}) = \min_{U_i}\{W_i(S_{i-1}, U_i\}, \quad i = 1,2,...,m.$$

3. Application *α*-function for solution the problems with distributed parameters

Let us consider about absolute minimum the Problem with distributed parameters

$$I(x,u) = \int_P f_0(t,x,u)dt + F(x(\tau)), \tag{3.12}$$

where $t = (t_1, t_2, ..., t_m), \quad x = (x_1, x_2, ..., x_n), \quad u = (u_1, u_2, ..., u_r)$ are elements of vector space *T, X, U* * respectively. *P* is closed area in space *T*, bounded continuous piecewise smooth, fixed hypersurface *S*. On *S* the *t* = *τ*. *P** is internal part this area, functions $x_i(t)$ on *P* are absolute-continuous, $u_\alpha(t)$ are measurable on *P* and have values from area *U*, which can be closed and bounded.

Functions *x(t), u(t)* satisfy almost everywhere the system *n·m* in depended differential equations with particular deviations

$$\frac{\partial x_i}{\partial t_j} = f_j^i(t,x,u), \quad i = 1,2,...,n; \quad j = 1,2,...,m \tag{3.13}$$

Funsions f_j^i, f_0 are continuously together with its particular derivatives the first order. The function *x(t), u(t)* we name allowable if they satisfy the named above conditions (set *Q*).

Statement of Problem: Find couple function *u(t), x(t)*, which give the function *I* (3.12) the minimal value.

Add to system (3.13) the integrability condition:

$$\varphi^\gamma = \frac{\partial f_j^i}{\partial t_k} - \frac{\partial f_{ki}^i}{\partial t_j} = 0, \quad i = 1,2,...n; \quad j,k = 1,2,...,m; \quad k > j. \tag{3.14}$$

Not difficult to calculate, that number of difficult equation (3.14) may be $0.5(m-1)mn, \quad i.e. \quad \gamma = 1,2,...,0.5(m-1)mn$ (number of combinations $C_m^2 n)$. For simplicity we will assume: all functions φ^γ in (3.14) contain *u* and these *u* may be find from (3.14) Assume the number of in depended equations (3.14) are less *r*.

Let us lead to consider *m*-dimensional function $\psi(t,x) = \{\psi^1,\psi^2,...,\psi^m\}$. The components of this function $\psi^j(t,x)$ $j = 1,2,...,m$ are continuous and have the continuous partial derivatives almost everywhere in *T*.

Name this function – characterlike function. Let us lead also the integrable vector-function

$\lambda_1(t), \lambda_2(t),...,\lambda_p(t)$.

Let us take α- function in form

$$\alpha = \int_S \psi^j(\tau,x)\cos(n,t^j)d\tau - \int_P (\psi_{t_j}^j + \psi_{x_i}^j f_j^i + \lambda_\gamma \varphi^\gamma)dt, \tag{3.15}$$

Where *n* is outer normal to surface *S, dτ* is element surface *S*. We present the function J = I + α in form

$$J = A + \int_P B dt, \quad where \quad A = \int_s \psi^j(\tau,x)\cos(n,t^j)d\tau, \quad B = f_0 - \psi_{t_i}^j + \psi_{x_i}^j f_j^i + \lambda_\gamma \varphi^\gamma. \tag{3.16}$$

Theorem 3.5. *Assume* $u(t) \in V$. *In order to couple u(t), x(t) will be the absolute minimum the purpose function (3.12) it is sufficiently* exicting of α-function (3.15) such that*

$$1)\ \overline{B} = \inf_{x,u\in U} B(t,x,u), \quad 2)\ \overline{A} = \inf_{x(\tau)} B > -\infty, \quad 3)\ \overline{x}(t), u(t) \in Q. \tag{3.17}$$

The proof is identical [2] №7, but in difference from [2] the theorem 3.5 contain the integrability condition.

If $\overline{x}(t), \overline{u}(t) \notin Q, \quad \text{than} \quad \bar{J}$ is the lower estimation the function (3.12).

If exist the functions *ψ, λ* and at least one pair $\overline{x}(t), \overline{u}(t)$ satisfying (3.17), then any other pair satisfying (3.17) is minimum of the function (3.12) and any allowable minimum the function (3,12) is satisfying the points 1, 2 (3.17) (consicvently remark 3 §1). The set contains the same or better solution, then $\overline{x}(t), \overline{u}(t)$ is

$$N = \{t,x,u : B(t,x,u) + f_0(t,x,u) \geq \overline{B} + \bar{f}_0\} \quad on \quad P^* \times U,$$

Assume, functions $f_j^i(t,x,u), \varphi^\gamma(t,x,u)$ are continuous and differentiable. Let us take ψ^j in form $\psi^j = p_{ij}(t)x_i$. Let us denote:

$$H = p_{ij}(t) f_{ij}(t,x,u) - f_0(t,x,u) + \lambda_\gamma \varphi^\gamma(t,x,u).$$

Then p.1 (3.17) of theorem 3.4 we can rewrite: $H(\overline{u}) = \sup_{u \in U} H$ and necessary condition of minimum (stationarity condition) following from p.2 (3.17) gives:

$$\frac{\partial B}{\partial x_i} = -\frac{\partial p_{ij}}{\partial t_i} - \frac{\partial H}{\partial x_i} = 0, \quad i = 1,2,\ldots,n. \tag{3.18}$$

§4. Inverse substitution method

A. From previous paragraph we have: if we know the minimum any function on acceptable set, we can get information about solution the Problem 1 and solve one from Problem a, b, c, g the §1.

It is known , that the most direct Problems inf $f_0(x)$ on X^* or

$$\inf \int_{t_1}^{t_2} f_0 dt$$

on Q (i.e. finding the minimum of main Problem) are difficult or do not have the satisfaction solution. However, if purpose function is not in advance definitized, the solution for this non-diminished purpose is finding easy. This is not surprising. In mathematics it has long been known that many inverse problems are solved more easily than direct problems. An example, let us consider the problem of finding the roots of an algebraic equation. In the general case for $n>$ 5 it is solved with difficulty and her decision (roots) not to be expressed in terms radicals. If the roots are given, then the corresponding algebraic equation may be found easy. On the basis of this idea below it is given method to build function for which an admissible element would be the point of absolute minimum on an admissible set. Since we thus have to solve a problem back to the original problem (not find the minimum given function, but find the function for given the minimum or for given field). This method is called **the method of reverse lookup.** The method is presented for two cases: problems of the theory of extrema of functions of a finite numbers of variables (p.B) and optimization problems described by ordinary differential equations (p.C).

A. Let us consider usual Problem of minimum the function of finite variables

$$I = f_0(x), \quad f_i(x) = 0, \quad i = 1,2,\ldots,m < n \,. \tag{4.1}$$

Let us convert this Problem. Select *m* components *x* and name them main (base). Suppose for definiteness that this is the first components *m* of the vector *x*. The rest of components *n* - *m* = *r* denote u_j (j = 1,2,…,r).

Granted Problem (4.1) we can re-write

$$I = f_0(x,u), \quad f_i(x,u) = 0, \quad i = 1,2,\ldots,m < n \,. \tag{4.2}$$

where *x* – *m* - dimensional vector, $x \in X$, *u* – *r* - dimensional vector, $u \in U$.

Let us take more simple purpose function $J_1(x,u)$ and find it's the absolute minimum on $X \times U$. This solution may be used for building of sets *M, N, P*:

$$M = \{x,u : J_1 - f_0 \geq \bar{J}_1 - \bar{f}_0\}, \qquad (4.3)$$
$$N = \{x,u : J_1 + f_0 \leq \bar{J}_1 + \bar{f}_0\}, \qquad (4.4)$$
$$P = \{x,u : J_1 - f_0 \leq \bar{J}_1 - \bar{f}_0\}. \qquad (4.5)$$

Disadvantage this method is next: the some of these sets cannot have the admissible elements (i.e. *x, u* satisfaction f_i = 0).

Assume, the limitations $f_i(x,u) = 0$ in (4.2) may be solved about *x*:

$$x_i = x_i(u), \quad i = i = 1,2,\ldots,m \qquad (4.6)$$

and $x \in X$ for any $\vee u \in U$.

Assume we take simple function $J_1(x,u)$. Substitute in it's the (4.6) and find $\inf\limits_U J_1(x(u),u)$, $\bar{u}$, and (4.6) $\bar{x}$. This solution is analog (4.3)-(4.5). One may be used for finding sets *M, N, P*. The intersection of these sets with admissible set is not empty. You can take $J_1(x,y,u)$, than $\bar{u} = \bar{u}(y)$. You can use the dependence of *M, N, P* from *y* for changing the "size" of these sets. It is clear assessment

$$\Delta = \inf_y \sup_u [J_1(x(u),u) - I(x(u)),u] \cdot$$

C. In point 2 §3 we considered the optimization Problem described by conventional differentials equations

$$I = \int_{t_1}^{t_2} f_0(t,x,u)dt, \quad \dot{x} = f_i(t,x,u), \quad i = 1,2,...,n, \quad u \in U. \tag{4.7}$$

We was shown: if we take some function *ψ(t,x)* and find minimum of $\inf\limits_{x,u} B$ in (t1, t2) and $\inf\limits_{x_1,x_2} A$, we get the minimum of Problem 1 or the its lower estimation.

Statement of the Problem. Let us to state the Problem 1 the other way: the find the function which matches the function *ψ(t.x)* and minimum of this function of the admissible set.

Note. Let us note: the offered statement very different from the back problem of variation calculation. In variation calculation, the back-problem states next: we have a curve. Find the function, which gives the minimum in this curve. In common case this problem is more difficult than a direct problem.

In our case the minimum curve not given. We find it by given function *ψ(t,x).*

Theorem 4.1. The minimum function corresponding function *ψ(t,x)* is

$$J_1 = \int_{t_1}^{t_2} B_1(t,x)dt = \int_{t_1}^{t_2} -\inf_{u \in U}[-\psi_{x_i} f_i(t,x,u) - \psi_t]dt \quad \cdot \tag{4.8}$$

And correcponding to it the minimum curve is given by equations

$$\dot{x}_i = f_i[t,x,\bar{u}(t.x,\psi_{x_i}\psi_t)], \quad i = 1,2,...,n, \tag{4.9}$$

where $\bar{u} = \bar{u}(t,x,\psi_{x_i},\psi_t)$ we find from (4.8).

Proof. Write the expression *B* (see (3.11)) for problem (4.7) and checkup condition (3.8) of theorem 3.1:

$$B_2(t) \equiv \inf_{x,u} \inf [B_2(t,x) - \psi_{x_i} f_i(t,x,u) - \psi_t] \cdot \tag{4.10}$$

Obviosly, the (4.10) identically equals zero for *ψ* = *ψ(t,x*) from (4.8) and $\bar{x}, \bar{u}$ satisfaction (4.7). If we take as $x(t_2)$ the value *x(t)*, received from (4.9) for t_2, then the point 2 (3.8) disappear and all condition (3.8) of theorem is executed. Theorem is proofed.

Consequence. If $B_1 = f_0(t,x)$, then x(t) getting from (4.10) give the set of the minimal curves for boundary condition $\psi_2 = \psi$. In particularly, if the end of curve *x(t)* from (4.9) match with given boundary conditions, that this curve is minimum curve of Problem 1.

Note. Boundary conditions in the left end can always be performed. For it we must start the integration from the given conditions (4.9). We can perform the boundary condition in the right end the next method. Take in form *ψ(t,x,c)* where *c – n –* dimensional constant. Substitute *ψ(t,x,c)* in (4.9) and select *c* such that to perform the given end condition in the right end.

Getting numerical function may be used for receiving the set *N, P* of Theorem 3.3 :

$$N - \{t,x : f_0 + B_1 \le \bar{f}_0 + \bar{B}_1\}, \quad P = \{t,x : B_1 - f_0 \le \bar{B}_1 - \bar{f}_0\},$$
$$\text{where} \quad f_0 = f_0[t,x,\bar{u}(t,x,\psi_x,\psi_t)], \quad \psi(t,x) \quad \text{is} \quad \text{given}.$$

If we find

$$\bar{J} = \psi_2 - \psi_1 + \int_{t_1}^{t_2} \inf_x (f_0 - B_1)dt \text{ '}$$

We get also the lower estimation.

Memo, the assignment $\psi(t,x)$ gives us not single no metical function and its point of minimum. One gives a set of minimums satisfaction the boundary conditions $\psi_2 - \psi_1 = c$.

Note: We can take $\psi(t,x,y)$. Then $B_1(t,x,y)$. If we can select such $\bar{y}(t)$ that $B_1(t,x,\bar{y}) = f_0(t,x)$ and boundary conditions is performed, then $\bar{u}(t,x,\bar{y})$ is the optimal synthesis of Problem 1.

D. We also show: how you can find the numerical function for given the syntes of control $u = u(t,x)$.

Equate the given $u = u(t,x)$ to the control fended from (4.8). We get the equation in particular derivities

$$u(t,x) = \bar{u}(t,x,\psi_{x_i},\psi_t)\,. \qquad (4.11)$$

Substitute its solution $\psi(t,x)$ and given u(t,x) in (4.8), we find the numerical corresponding function. If $B_1 = f_0(t,x)$ that is synthesis the Problem 1 for the bounded condition $\psi_2 = \psi$.

Possible the other method . We take $u = u(t,x,y)$. Substitute it in (4.8). Then $B_1 = B_1(t,x,c,y)$. We can try using y to reach the identify $f_0 \equiv B_1$ and using c to minimize the numerical function I.

Example 4.1. Let us consider the task of design the regulator

$$I = \int_{t_1}^{t_2} b_{ij}x_i x_j dt, \qquad (4.12)$$

$$\dot{x}_i = a_{ij}x_j + u, \quad 0 \le t \le \infty, \qquad (4.13)$$

$$x_i(0) = x_{i,o}, \quad x_i(\infty) = 0, \qquad (4.14)$$

where $f_0 = b_{ij}x_ix_j$ is the positive definite form.

Take $u = c_ix_i$, where c_i are constants. Let us to search ψ as the quadratic form $\psi = A_{ij}x_ix_j$ with unknown coefficients. Equate $f_0 \equiv \dot{\psi}$:

$$b_{ij}x_i x_j = A_{ij}x_i(a_{ij}x_j + c_j x_j).$$

Let us equate coefficient in same x_i, x_j in left and right of this equation. We get the set $n(n+1)/2$ the linear inhomogenius equations having the same number of unknown Aij. If the determinant of this system $\Delta \neq 0$, we find A_{ij}. We substitute $f_0 \equiv \dot{\psi}$ in (4.12), integrate and find $I = \psi(\infty,c) - \psi(0,c)$ or using (4.14) $I = -\psi(x_{io},c)$. When we find minimum of this expression for c, we get the optimal systems. If $-\psi(x,\bar{c})$ is the positive definite form then this function is the Lyapunov function (because $-\dot{\psi} \ge 0$ and the regulator is asymptotic stable.

§5. Method of combining extrema in problems of constrained minimum.

We will show in this paragraph that method combining extrema, considered in §2 the Chapter 1 [9], it is apply in tasks of theory the functions of a finite number of variables (point A) and tasks described the conventional difference equations.

A) Let us again consider the Problem of the theory the functions of a finite number of variables

$$I = f_0(x), \quad f_i(x) = 0, \quad i = 1,2,\ldots,m. \qquad (5.1)$$

Write the numerical function

$$J(x,c) = f_0(x) + \beta(x,c) + \alpha_1(x), \tag{5.2}$$

Here $\alpha_1(x)$ is α – function, c is n – dimensional constant.

From condition

$$\inf_{x\in X^*} J(x,c), \tag{5.3}$$

we find $\varphi_1(x^{(1)},c) = 0.$

From condition

$$\Phi(x,c) = \sup_{x\in X^*}[\beta(x,c) + \alpha_2(x)], \tag{5.4}$$

we find $\varphi_2(x^{(2)},c) = 0.$ Solve equations φ_1, φ_2 together with (5.1) (combining equations):

$$\varphi_1(x^{(1)},c) = 0, \quad \varphi_2(x^{(2)},c) = 0, \quad x^{(1)} = x^{(2)}, \tag{5.5}$$

we receive the absolute minimum the Problem 1. The additive $\beta(x,c)$ selects so that tasks (5.3), (5.4) are solved easier.

For example $\alpha_1 = \lambda_i f_i, \quad \alpha_2 = \nu_i f_i.$ Functions $f_i(x)$, $i = 0,1, \ldots,n$ are continuous and difference , the functions $J(x,c)$, $\Phi(x,c)$ have single minimum and maximum for any c. That we have system ($3n + 2m$) equations with same numbers of unknown magnitudes α(1), α(2), c, λ, ν.

Example is not including.

B) Let us to consider the task, described the conventional different equations:

$$I = \int_{t_1}^{t_2} f_0(t,x,u)dt, \quad \dot{x}_i = f_i(t,x,u), \quad i = 1,2,\ldots,n, \quad u \in U, \quad x(t_1) = x_1, \quad x(t_2) = x_2, \tag{5.9}$$

Take ψ in form $\psi^{(1)} = p_i^{(1)}(t)\alpha_i^{(1)}$ and create the function

$$B_1 = f_0 + \beta(t,x^{(1)},u^{(1)},z) - p_i^{(1)} f_i^{(1)} - \dot{p}_i^{(1)} x_i^{(1)} = -H^{(1)} - \dot{p}_i^{(1)} x_i^{(1)}.$$

Here $z(t)$ is r – dimensional function. One can have the limited gaps the first type.

From $\inf_{x,u} B_1$ and (5.9) we find

$$\dot{p}^{(1)} = -H_x^{(1)}, \quad \bar{u}^{(1)} = \bar{u}^{(1)}(t,x^{(1)},p^{(1)},z), \quad \dot{x}^{(1)} = f(t,x^{(1)},u^{(1)}). \tag{5.10}$$

Take $\psi^{(2)} = p_i^{(2)} x_i^{(2)}$ and create the function

$$B_2 = \beta(t,x^{(2)},u^{(2)},z) - p_i^{(2)} f_i^{(2)} - \dot{p}_i^{(2)} x_i^{(2)} = -H^{(2)} - \dot{p}_i^{(2)} x_i^{(2)}.$$

From $\inf_{x,u} B_2$ and (5.9) we find

$$\dot{p}^{(2)} = -H_x^{(2)}, \quad \bar{u}^{(2)} = \bar{u}^{(2)}(t,x^{(2)},p^{(2)},z), \quad \dot{x}^{(2)} = f(t,x^{(2)},u^{(2)}). \tag{5.11}$$

Using the combining equation: $x^{(1)} = x^{(2)}, \quad u^{(1)} = u^{(2)}$ we get final:

$$\dot{x}=f(t,x,u^{(1)}),\quad \dot{p}^{(1)}=-H_x^{(1)},\quad \dot{p}^{(2)}=-H_x^{(2)},\quad \overline{u}^{(1)}(t,x,p^{(1)},z)=\overline{u}^{(2)}(t,x,p^{(2)},z)\quad , \quad (5.12)$$

That is system 3*n + r* equations with 3*n + r* unknown *x, $p^{(1)}$, $p^{(2)}$, z* . Last equation in (5.12) is the combining equation. The additive function *β* selecting so that the solution task of finding *inf* and *sup* were simpler.

§6. Generalizing the Theorem 3.1 in case the bracken *ψ(t,x)*.

Theorem 6.1. *Assume there is numerical function ψ(t,x) defined on set $T\times G$, bounded below, piecewise differentiable and piecewise continuous. The function ψ(t,x) and its derivatives can have the breaks the first types on the limited set $\Phi_s(t_s,x),\ s=1,2,...,k-1$ zero measure. This function is such that there is:*

$$1)\ \inf_R\,(F+\psi_k-\psi_o),\quad 2)\ \inf_{t_s,x\in\Phi_s}(\psi_s^- -\psi_s^+),\quad \bar{t}_s\succ\bar{t}_{s-1},\quad t_k^{'}\succ\bar{t}_{k-1}^{'},\quad s=1,2,...,k-1,$$
$$3)\ \inf_{G\times T}B=0,\quad 4)\ \overline{x}(t),\overline{u}(t)\in Q.$$

Then $\overline{x},\overline{u}$ (are got from points 1 -3) is the absolute minimum the Problem 1.

Here ψ_s^-,ψ_s^+ are value *ψ* in left and right side (along $\overline{x}(t)$) of the breaks the function *ψ* and its derivatives.

Proof: From points 1 – 3 we have

$$\bar{J}=\inf_R(F+\psi_k-\psi_0)+\sum_{s=1}^{k-1}\inf_{t_s,x}(\psi_s^- -\psi_s^+)+\sum_{s=0}^{k-1}\int_{t_s}^{t_{s-1}}\inf_{x,u}Bdt\ .$$

On feasible curves (from *Q*) the $\bar{J}$ convert in function $I=F+\int_{t_1}^{t_2}f_0dt$. In this case if we apply the consequence 4 , §1, point 4 of the theorem statement is obviously. Theorem is proofed.

Note. The conditions 3 of Theorem 6.1 is sometimes difficult to check up. In this case the requirements 2 - 3 of theorem 6.1 we can change the damage

$$\inf_{t_s}[\inf_x(\psi_s^- -\psi_s^+)+\int_{s-1}^{s}\inf_{G\times U}Bdt+\int_s^{s+1}\inf_{G\times U}Bdt]\ .$$

One must be checked up in every point t_s, *s* = 1,2,…,k-1.

§7. Optimization the problems described the conventional differential equations having the limitations.

We find minimum *A, B* in Theorem 3.1, chapter II on the corresponding sets *R* and $U\times G$. The most widely method of separating the feasible sets is the separation of them from more widely set by equalities and inequalities. In this case, we can solve our problem by the methods the *α*- and *β*-functions.

Let us shortly consider the most common cases.

1. Limitations are the equalities

a) Assume the admissible set *R* is separated by equalities:

$$g_i(x_1,x_2)=0,\quad i=1,2,...,l<2n\ . \quad (7.1)$$

Then the task *inf A* we can change the task

$$\inf_{x_1,x_2}[A+\mu_i(x_1,x_2,z_i)g_i(x_1,x_2)]\cdot \tag{7.2}$$

Here μ_i is known functions, z is l-dimensional unknown vector. In particularly, we can take $\mu_i = z_i$.

b) Assume the admissible set $U \times G$ is separated by equalities

$$\varphi_i(t,x,u)=0, \quad i=1,2,...,l<r\cdot \tag{7.3}$$

Assume, we can find from (7.3) the l component the vector u. Than the problem $\inf\limits_{G\times U} B$ we can change the problem

$$\inf_{x,u}[B+\lambda_i(t,x,w)\varphi_i(t,x,u)], \tag{7.4}$$

Where λ_i are known function, w_i is l - dimensional unknown vector function. In particular, we can take $\lambda_i = w_i$.

c) Assume the admisseble set G is separated by the equalities

$$\varphi_i(t,x)=0, \quad i=1,2,...,l<r\cdot \tag{7.5}$$

Differentiate (7.5) full case for t and find

$$\varphi_i^{(1)}(t,x,u)\equiv\frac{\partial\varphi_i}{\partial x_j}f_j(t,x,u)+\frac{\partial\varphi_i}{\partial t}=0, \quad i=1,2,...,l<n\cdot \tag{7.6}$$

If in system (7.6) there is equations do not contain u, we differentiate them next time and so on whole we get the system where all l equation contains u. Assume, we can find all l components from this system ($l < r$).

Than the problem (7.5) is reduced to the tasks the point *a, b* in which (7.6) is (7.3), but (7.5) and all equations (7.6) not contain u, are (7.1).

2. Limitations are inequalities. (excerpt)

a) Feasible set R is allocated by inequalities:

$$g_i(x_1,x_2)\le 0, \quad i=1,2,...,l.$$

Then acording the Teorem 1.4 Chapter 1 we change the problem $\inf\limits_{R} A$ by problem (7.2) with the additional conditions:

$$\bar\lambda_i\bar g_i=0, \quad \bar\lambda_i\ge 0 \quad (\text{here} \quad i \quad \text{is} \quad \text{not} \quad \text{sum}) \tag{7.7}$$

b) Feasible set $U \times G$ is allocated by inequalities:

$$\varphi_i(t,x,u)\le 0, \quad i=1,2,...,l. \tag{7.8}$$

All inequalities contain *u.* Then the task $\inf\limits_{U\times G} B$ we change the task (7.4) with conditions

$$\bar\lambda_i\bar\varphi_i=0, \quad \bar\lambda_i\ge 0 \quad (\text{here} \quad i \quad \text{is} \quad \text{not} \quad \text{sum}) \tag{7.9}$$

Example 7.1. Assume in task

$$I=\int_{t_1}^{t_2} f_0(t,x,u)dt, \quad \dot{x}_i = f_i(t,x,u), \quad i=1,2,...,n,$$

Control *u* is scalar, the feasible set *U* limited inequality $a \le u \le b, \ (a<b)$. Compose (7.4):

$$\inf_U [B+\lambda_1(u-b)+\lambda_2(-u+a)].$$

According (7.9) on feasible u: $\bar{\lambda}_1(\bar{u}-b)=0, \quad \bar{\lambda}_2(-\bar{u}+a)=0.$ That way we have

$$\inf_u [B+\lambda_1(u-b)+\lambda_2(-u+a)] = \inf_{u_1,u_2\in U} B.$$

In right side we have one condition the Pontryagin method.

(Part of the text are missing)

§10. Note on the equivalence of different forms of variational problems

A) In §3 the next problem of minimization was considered

$$I=F(x_1,x_2)+\int_{t_1}^{t_2} f_0(t,x,u)dt, \tag{10.1}$$

on solution of equations

$$\dot{x}_i = f_i(t,x,u), \quad i=1,2,...,n. \tag{10.2}$$

In the theoretical analysis for the sake of simplicity, we often assume that in (10.1) $F\equiv 0 \quad or \quad f_0 \equiv 0.$ We show that it does not restrict the generality of our reasoning.

Take

$$I=\int_{t_1}^{t_2} f_0(t,x,u)dt$$

And differentiate it for the variable upper limited *t* and designate $\dot{x}_{n+1}=f_0$. We get the task

$$I=x_{n+1}(t_1), \quad \dot{x}_i = f_i, \quad \dot{x}_{n+1}=f_0. \tag{10.3}$$

B) Assume $I=F(x_1,x_2)$. Differentiate it by *t* and integrate, we get numerical function

$$I=\int_{t_1}^{t_2}(F_{x_i}f_i)dt \tag{10.4}$$

We can same way to convert (10.1) in (10.4) and in (10.3).

C) Let us to assume the (10.1) and (10.2) depend from constants c_k which must be optimal. Designate ck = x_i+k and add to (3.3) equation $\dot{x}_{n+k}=0$. We reduced the task having the optimizing constants to conventional task.

In practice it is comfortable to solve the problem (10.1), (10.2) with constant parameters. Than to change them (for example the gradient method) so, the function (10.1) decreases.

D) The problem with $f_i(t,x,u)$ which obviously depend from *t*, we can reduce to problem $f_i(x,u)$ do not depend obviously from *t*, if to designate $t = x_{n+1}$ and add to (10.1) the equation $\dot{x}_{n+1}=1$.

C) Let us to show how the task with the moving ends t_1 and t_2 we can reduce the task with fix interval of integrate. Take the new variable $t = c\tau$. Than task (10.1),(10.2) having variables t1 or t2 was reduced in task with fix interval (τ_1,τ_2):

$$I=F+\int_{t_1}^{t_2} cf_0(\tau,x,u)d\tau, \quad x'=cf_i(c\tau,x,u),$$

where the touch means the derivative for τ. The constant $c>0$ is selected from minimum *I*.

Application.

1. Theorem 3.1 and known methods of solution the problem described the ordinary differential equations.

From Theorem 3.1 we can to get the conditions which are same with known algorithms of optimal control, for example: Pontriagin principle [10], Bellman equation [11], classical calculus of variation [12],

Let us to request additional that function *f, ψ* have the need continuous derivatives.

a) Pontriagin principle. According [10] take *ψ(t,x)* in form $\psi = p_i(t)\Delta x_i$, where $p_i(t)$ are some differenciable functions *t*, $\Delta x_i = x_i - \bar{x}_i$. Create the Hamiltonian

$$H = p_i f_i(t,x,u) - f_0(t,x,u) \cdot \tag{1}$$

Then $B = - H - p_i x_i$. Necessary condition of the minimum *B* for *x*, which follows from p.1 (3.8) of Theorem 3.1 (stationarity condition) is

$$B_{x_i} \equiv -p_i - H_{x_i} = 0, i = 1,2,...,n. \tag{2}$$

Moreover of claim 1 (3.8) we have

$$B(t,x,\bar{u}) = \inf_{u\in U} B(t,x,u) \quad \text{or} \quad \inf_{u\in U}(-H) = -\sup_{u\in U} H \tag{3}$$

Terms and conditions (2), (3) together with (3.3) coincide with the corresponding terms and conditions of the Maximum principle* [1].

b) Belman equation. Assume $x_n \neq 0$. Take all $\lambda_i = 0$ $i = 1,2,\dots,n-1$ with exception $\lambda_n = \psi(t,x)/x_n$. Substitute them in (3.9) §3, we get the known Bellman equation [11]

$$\inf_{u\in U}(f_0 - \psi_{x_i} f_i - \psi_t) = 0 \tag{4}$$

Boundary condition for them is *A = const*. Solution of this equation is the field of all optimal trajectories.

c) Classical calculus of variation. From claims 1, 2 Theorem 3.1 easy to get the conditions of a relative minimum coinciding with the relevant terms of the calculus of variations [12].

Let us assume *U* is the open area, $\dot{x}(t), u(t)$ are continuously, $f_i(t,xu)$ have continuous partial derivatives up the third order. Take $\psi = p_i(t)\Delta x_i$. From (3) that at minimum

$$B_{u_i}(t,x,u) = -H_{u_i}(t,x,u) = 0, \quad i = 1,2,..,r\ , \tag{5}$$

Equations (2),(4) equal the conventional Euler-Lagrange equations [12] §2. From [12] also follow

$$-H_{u_i u_j}\delta u_i \delta u_j \geq 0, \quad i,j = 1,2,...,r\ . \tag{6}$$

That matches with Krebs condition.

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