Refutation of description logics with cardinality constraint

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Abstract: We evaluate two definitions and one lemma for set and cardinality constraints of description logics (DLs) and cardinality constraints of logic ALCSC. These are not tautologous, and form a non tautologous fragment of the universal logic $VŁ4$.

We assume the method and apparatus of Meth8/$VŁ4$ with Tautology as the designated proof value, $F$ as contradiction, $N$ as truthity (non-contingency), and $C$ as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬; + Or, $∨$, $∪$; - Not Or; & And, $∧$, $∩$, ·, $⊗$; \ Not And;
> Imply, greater than, $⇒$, $↦$, $≻$, $⊃$; < Not Imply, less than, $€$, $⊂$, $∉$, $∉$, $≤$;
= Equivalent, $≡$, $⇔$, $↔$, $≜$, $≃$; @ Not Equivalent, ≠;
% possibility, for one or some, $∃$, $∃!$, $◊$, $M$; # necessity, for every or all, $∀$, $□$, L;
(z=z) $T$ as tautology, $T$, ordinal 3; (z@z) $F$ as contradiction, $Ø$, Null, $⊥$;
(%z>$\#z$) $N$ as non-contingency, $Δ$, ordinal 1; (%z<$\#z$) $C$ as contingency, $∇$, ordinal 2;
~( y < x) ( x ≤ y), ( x ⊆ y), ( x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

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Abstract We introduce and investigate the expressive description logic (DL) ALCSCC++, in which the global and local cardinality constraints introduced in previous papers can be mixed. On the one hand, we prove that this does not increase the complexity of satisfiability checking and other standard inference problems. On the other hand, the satisfiability problem becomes undecidable if inverse roles are added to the languages. In addition, even without inverse roles, conjunctive query entailment in this DL turns out to be undecidable. We prove that decidability of querying can be regained if global and local constraints are not mixed and the global constraints are appropriately restricted. The latter result is based on a locally-acyclic model construction, and it reduces query entailment to ABox consistency in the restricted setting, i.e., to ABox consistency w.r.t. restricted cardinality constraints in ALCSCC, for which we can show an ExpTime upper bound.

1 Introduction Description Logics (DLs) .. are a well-investigated family of logic-based knowledge representation languages, which are frequently used to formalize ontologies for application domains such as biology and medicine .. . To define the important notions of such an application domain as formal concepts, DLs state necessary and sufficient conditions for an individual to belong to a concept. These conditions can be Boolean combinations of atomic properties required for the individual (expressed by concept names) or properties that refer to relationships with other individuals and their properties (expressed as role restrictions). ...

2 The logic ALCSCC++

The substitution $σ$ satisfies the set constraint $s = t (s ⊆ t)$ if $σ(s) = σ(t)$ ($σ(s) ⊆ σ(t)$). (2.1.1)

LET $q, r, s$: $σ$, $t$ or $k$, $s$ or $ℓ$.  

\[((q \& s) = (q \& r)) \& \sim ((q \& r) < (q \& s))) \Rightarrow (s = (r \& \sim (r < s)))\;\]

\[
\text{TTTT TTTT FTTT TTTT} \quad (2.1.2)
\]

… The substitution \(\sigma\) satisfies the cardinality constraint \(k = \ell\) if \(\sigma(k) = \sigma(\ell)\), … \( (2.2.1) \)

\[((q \& r) = (q \& s)) \Rightarrow (r = s)\;\]

\[
\text{TTTT FTTT FTTT TTTT} \quad (2.2.2)
\]

**Remark 2:** Eqs. 2.1.2 and 2.2.2 are not tautologous, hence refuting set and cardinality constraints of description logics.

### 4 Restricted cardinality constraints and ABoxes in ALCSCC

**Lemma 11.** *Proof:* (1) … In addition, we have \(A \cdot (c + d) = A \cdot c + A \cdot d \geq b + b \geq b\), where the first inequality holds since \(c, d\) are solutions of \(\varphi\), and the last inequality holds since the components of \(b\) are non-negative. \( (4.11.1) \)

LET \( p, q, r, s: A, b, c, d. \)

\[(p \&(r+s)) = ((p \& r) + \sim (q > (p \& s)) + \sim (q < q)))\;\]

\[
\text{FFFF FTTT FTTT FTTT} \quad (4.11.2)
\]

**Remark 4.11:** Eq. 4.11.2 is not tautologous, hence refuting cardinality constraints in logic ALCSC.