CHARACTERIZATIONS OF FUNCTIONS WITH STRONGLY $\alpha$-CLOSED GRAPHS

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Abstract

In this paper, we study some properties of functions with strongly $\alpha$-closed graphs by utilizing $\alpha$-open sets and the $\alpha$-closure operator.

1 Introduction and preliminaries

The notion of $\alpha$-open sets was introduced by O. Njastad [20] in 1965. Since then it has been widely investigated in the literature (see, [1], [2], [3], [9], [10], [11], [12], [15], [16], [17], [18], [19], [21], [23], [24], [26], [27], [28]). Functions with strongly closed graphs were introduced by Herrington and Long [7] to characterize $H$-closed spaces. Properties of such functions were further investigated by Long and Herrington [14] and Noiri [23]. In this paper, we study some properties of functions with strongly $\alpha$-closed graphs by utilizing $\alpha$-open sets and the $\alpha$-closure operator.

Throughout this paper, by $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) we always mean topological spaces. Let $A$ be a subset of $X$. We denote the interior, the closure and the complement of a set $A$ by $Int(A)$, $Cl(A)$ and $X \setminus A$ or $A'$ respectively. A subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open [20] (resp. semi-open [13]) if $A \subseteq Int(Cl(Int(A)))$ (resp. $A \subseteq Cl(Int(A))$). The complement of an $\alpha$-open (resp. semi-open) set is called $\alpha$-closed (resp. semi-closed [5]). By $\alpha O(X, \tau)$ (resp. $SO(X, \tau)$, $\alpha C(X, \tau)$), we denote the family of all $\alpha$-open (resp. semi-open, $\alpha$-closed) sets of $X$. We set $\alpha O(X, \tau) = \{ U \mid x \in U \in \alpha O(X, \tau) \}$,

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$O(X, x) = \{ U \mid x \in U \in \tau \}$ and $\alpha C(X, x) = \{ U \mid x \in U \in \alpha C(X, \tau) \}$. The intersection of all $\alpha$-closed (resp. semi-closed) sets containing $A$ is called the $\alpha$-closure (resp. semi-closure [4]) of $A$, denoted by $\alpha Cl(A)$ (resp. $sCl(A)$). A set $U$ in a topological space $(X, \tau)$ is an $\alpha$-neighborhood [16] of a point $x$ if $U$ contains an $\alpha$-open set $V$ such that $x \in V$.

**Lemma 1.1** The intersection of an arbitrary collection of $\alpha$-closed sets in $(X, \tau)$ is $\alpha$-closed.

**Corollary 1.2** [15]. Let $A$ be a subset of $X$. Then, $x \in \alpha Cl(A)$ if and only if for any $\alpha$-open set $U$ in $X$ containing $x$, $A \cap U \neq \emptyset$.

**Lemma 1.3** Let $A$ and $B$ be subsets of a space $(X, \tau)$, then the following properties hold:
(1) $A \subset \alpha Cl(A)$.
(2) If $A \subset B$, then $\alpha Cl(A) \subset \alpha Cl(B)$.
(3) $\alpha Cl(A)$ is $\alpha$-closed.
(4) $\alpha Cl(\alpha Cl(A)) = \alpha Cl(A)$.
(5) $A$ is $\alpha$-closed if and only if $A = \alpha Cl(A)$.

**Corollary 1.4** Let $A_i (i \in I)$ be a subset of a space $(X, \tau)$, then the following properties hold:
(1) $\alpha Cl(\cap\{ A_i : i \in I \}) \subset \cap\{ \alpha Cl(A_i) : i \in I \}$.
(2) $\alpha Cl(\cup\{ A_i : i \in I \}) \supset \cup\{ \alpha Cl(A_i) : i \in I \}$.

**Definition 1** A topological space $(X, \tau)$ is said to be:
(1) $\alpha$-$T_1$ [17], if for any pair of distinct points $x$ and $y$ in $X$, there exist an $\alpha$-open set $U$ in $X$ containing $x$ but not $y$ and an $\alpha$-open set $V$ in $X$ containing $y$ but not $x$.
(2) $\alpha$-$T_2$ [15], if for any pair of distinct points $x$ and $y$ in $X$, there exist $U \in \alpha O(X, x)$ and $V \in \alpha O(X, y)$ such that $U \cap V = \emptyset$.

**Lemma 1.5** A topological space $(X, \tau)$ is $\alpha$-$T_2$ if and only if it is $T_2$.

**Proof.** This is shown in [27] and a simple proof is given in [[24], Corollary 4.7].
Definition 2 A function \( f : X \to Y \) is said to be
(1) \( \alpha \)-continuous \([19]\) if \( f^{-1}(V) \in \alpha O(X) \) for each open set \( V \) of \( Y \);
(2) weakly \( \alpha \)-continuous \([23]\) if for each \( x \in X \) and each \( V \in O(Y, f(x)) \), there exists \( U \in \alpha O(X, x) \) such that \( f(U) \subseteq Cl(V) \).

Lemma 1.6 Let \((X, \tau)\) be a topological space. Then \( \alpha Cl(V) = Cl(V) \) for each \( V \in SO(X) \).

Proof. For any \( V \in SO(X) \), \( \alpha Cl(V) = V \cup Cl(Int(Cl(V))) = V \cup Cl(Int(V)) = V \cup Cl(V) = Cl(V) \).

Lemma 1.7 A function \( f : X \to Y \) is weakly \( \alpha \)-continuous if and only if for each \( x \in X \) and each \( V \in \alpha O(Y, f(x)) \), there exists \( U \in \alpha O(X, x) \) such that \( f(U) \subseteq \alpha Cl(V) \).

Proof. Necessity. Let \( x \in X \) and \( V \in \alpha O(Y, f(x)) \). Then \( f(x) \in V \subseteq Int(Cl(Cl(V))) \) and there exists \( U \in \alpha O(X, x) \) such that \( f(U) \subseteq Cl(Int(Cl(Cl(V)))) \). By Lemma 1.6, we have \( Cl(Int(Cl(Cl(V)))) = Cl(Int(Cl(V))) = Cl(V) = \alpha Cl(V) \). Therefore, \( f(U) \subseteq \alpha Cl(V) \).

Sufficiency. Let \( x \in X \) and \( V \in O(Y, f(x)) \). There exists \( U \in \alpha O(X, x) \) such that \( f(U) \subseteq \alpha Cl(V) \). By Lemma 1.6, we obtain \( f(U) \subseteq Cl(V) \).

2 Strongly \( \alpha \)-closed graphs

If \( f : (X, \tau) \to (Y, \sigma) \) is any function, then the subset \( G(f) = \{(x, f(x)) : x \in X\} \) of the product space \((X \times Y, \tau \times \sigma)\) is called the graph of \( f \) \([8]\).

Definition 3 A function \( f : X \to Y \) has a strongly \( \alpha \)-closed (resp. strongly closed \([7]\)) graph if for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exist \( U \in \alpha O(X, x) \) (resp. \( U \in O(X, x) \)) and \( V \in O(Y, y) \) such that \( (U \times Cl(V)) \cap G(f) = \emptyset \).

Lemma 2.1 For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties are equivalent:
(1) \( G(f) \) is strongly \( \alpha \)-closed;
(2) For each \( (x, y) \in (X \times Y) \setminus G(f) \), there exist \( U \in \alpha O(X, x) \) and \( V \in O(Y, y) \) such that
$f(U) \cap Cl(V) = \emptyset$;

(3) For each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X,x)$ and $V \in \alpha O(Y,y)$ such that $(U \times \alpha Cl(V)) \cap G(f) = \emptyset$;

(4) For each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X,x)$ and $V \in \alpha O(Y,y)$ such that $f(U) \cap \alpha Cl(V) = \emptyset$.

Proof. It is obvious that $(1) \Leftrightarrow (2)$ and $(3) \Leftrightarrow (4)$.

$(1) \Rightarrow (3)$: Since $\tau \subset \alpha O(X) \subset SO(X)$, by Lemma 1.6 the proof is obvious.

$(3) \Rightarrow (1)$: Let $(x,y) \in (X \times Y) \setminus G(f)$. There exist $U \in \alpha O(X,x)$ and $V \in \alpha O(Y,y)$ such that $(U \times \alpha Cl(V)) \cap G(f) = \emptyset$. Put $G = Int(Cl(Int(V)))$. Then $y \in V \subset G \in \sigma$ and $Cl(G) = Cl(V) = \alpha Cl(V)$. Therefore, we obtain $((U \times Cl(G)) \cap G(f) = (U \times \alpha Cl(V)) \cap G(f) = \emptyset$. This shows that $G(f)$ is strongly $\alpha$-closed.

**Theorem 2.2** If $f : X \to Y$ is a function with the strongly $\alpha$-closed graph, then for each $x \in X$, $f(x) = \cap \{\alpha Cl(f(U)) : U \in \alpha O(X,x)\}$.

Proof. Suppose the theorem is false. Then there exists a $y \neq f(x)$ such that $y \in \cap \{\alpha Cl(f(U)) : U \in \alpha O(X,x)\}$. This implies that $y \in \alpha Cl(f(U))$ for every $U \in \alpha O(X,x)$. So $V \cap f(U) \neq \emptyset$ for every $V \in \alpha O(Y,y)$. This, in its turn, indicates that $\alpha Cl(V) \cap f(U) \cap V \cap f(U) \neq \emptyset$ which contradicts the hypothesis that $f$ is a function with strongly $\alpha$-closed graph. Hence the theorem holds.

**Theorem 2.3** If $f : X \to Y$ is $\alpha$-continuous and $Y$ is $T_2$, then $G(f)$ is strongly $\alpha$-closed.

Proof. Let $(x,y) \in (X \times Y) \setminus G(f)$. The $T_2$-ness of $Y$ gives the existence of a set $V \in O(Y,y)$ such that $f(x) \notin Cl(V)$. Now $Y \setminus Cl(V) \in O(Y,f(x))$. Therefore, by the $\alpha$-continuity of $f$ there exists $U \in \alpha O(X,x)$ such that $f(U) \subset Y \setminus Cl(V)$. Consequently, $f(U) \cap Cl(V) = \emptyset$ and therefore $G(f)$ is strongly $\alpha$-closed.

It is shown in ([14], Theorem 3) and ([22], Theorem 2) that if $f : X \to Y$ is surjective and $G(f)$ is strongly closed, then $Y$ is Hausdorff. The following theorem is a slight improvement of this result.
Theorem 2.4 If $f : X \to Y$ is surjective and has a strongly $\alpha$-closed graph $G(f)$, then $Y$ is both $T_2$ and $\alpha$-$T_1$.

Proof. Let $y_1, y_2 (y_1 \neq y_2) \in Y$. The surjectivity of $f$ gives a $x_1 \in X$ such that $f(x_1) = y_1$. Now $(x_1, y_2) \in (X \times Y) \setminus G(f)$. The strongly $\alpha$-closedness of $G(f)$ provides $U \in \alpha O(X, x_1)$, $V \in O(Y, y_2)$ such that $f(U) \cap Cl(V) = \emptyset$, whence one infers that $y_1 \notin Cl(V)$. This means that there exists $W \in O(Y, y_1)$ such that $W \cap V = \emptyset$. So, $Y$ is $T_2$ and $T_2$-ness always guarantees $\alpha$-$T_1$-ness. Hence $Y$ is $\alpha$-$T_1$.

Theorem 2.5 A space $X$ is $T_2$ if and only if the identity function $id : X \to X$ has a strongly $\alpha$-closed graph $G(id)$.

Proof. Necessity. Let $X$ be $T_2$. Since the identity function $id : X \to X$ is continuous, it follows from Theorem 2.4 that $G(id)$ is strongly $\alpha$-closed.

Sufficiency. Let $G(id)$ be a strongly $\alpha$-closed graph. Then the surjectivity of $id$ and strong $\alpha$-closedness of $G(id)$ together imply, by Theorem 2.4, that $X$ is $T_2$.

Theorem 2.6 If $f : X \to Y$ is an injection and $G(f)$ is strongly $\alpha$-closed, then $X$ is $\alpha$-$T_1$.

Proof. Since $f$ is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Then $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Since $G(f)$ is strongly $\alpha$-closed, there exist $U \in \alpha O(X, x_1)$, $V \in O(Y, f(x_2))$ such that $f(U) \cap Cl(V) = \emptyset$. Therefore $x_2 \notin U$. Pursuing the same reasoning as before we obtain a set $W \in \alpha O(X, x_2)$ such that $x_1 \notin W$. Hence $Y$ is $\alpha$-$T_1$.

Theorem 2.7 If $f : X \to Y$ is a bijection with the strongly $\alpha$-closed graph, then both $X$ and $Y$ are $\alpha$-$T_1$.

Proof. The proof is an immediate consequence of Theorems 2.4 and 2.6.

Theorem 2.8 If a function $f : X \to Y$ is a weakly $\alpha$-continuous injection with the strongly $\alpha$-closed graph $G(f)$, then $X$ is $T_2$. 


Proof. Since \( f \) is injective, for any pair of distinct points \( x_1, x_2 \in X \), \( f(x_1) \neq f(x_2) \). Therefore \((x_1, f(x_2)) \in (X \times Y) \setminus G(f)\). Since \( G(f) \) is strongly \( \alpha \)-closed, there exist \( U \in \alpha O(X, x_1) \), \( V \in O(Y, f(x_2)) \) such that \( f(U) \cap Cl(V) = \emptyset \); hence \( U \cap f^{-1}(Cl(V)) = \emptyset \). Consequently, \( f^{-1}(Cl(V)) \subset X \setminus U \). Since \( f \) is weakly \( \alpha \)-continuous, there exists \( W \in \alpha O(X, x_2) \) such that \( f(W) \subset Cl(V) \). From this and the foregoing it follows that \( W \subset f^{-1}(Cl(V)) \subset X \setminus U \); hence \( W \cap U = \emptyset \). Thus for the pair of distinct points \( x_1, x_2 \in X \), there exist \( U \in \alpha O(X, x_1) \), \( W \in \alpha O(X, x_2) \) such that \( W \cap U = \emptyset \). By Lemma 1.5, this guarantees the \( T_2 \)-ness of \( X \).

**Corollary 2.9** If a function \( f : X \to Y \) is an \( \alpha \)-continuous injection with the strongly \( \alpha \)-closed graph, then \( X \) is \( T_2 \).

**Proof.** The proof follows from Theorem 2.9 and the fact that every \( \alpha \)-continuous is weakly \( \alpha \)-continuous.

**Remark 2.10** If \( f \) is not \( T_2 \) in Corollary 2.9, then even \( \alpha \)-continuity need not imply a strongly \( \alpha \)-closed graph. For example, let \( X \) be a topological space containing more than one point with the indiscrete topology and let \( \text{id} : X \to X \) the identity function. Then \( \text{id} \) is certainly \( \alpha \)-continuous, but the graph of \( \text{id} \) is not strongly \( \alpha \)-closed because \( X \times X \) has the indiscrete topology and hence the graph of \( \text{id} \) being the diagonal set, which is different from the whole space, is not strongly \( \alpha \)-closed.

**Theorem 2.11** If \( f : X \to Y \) is a weakly \( \alpha \)-continuous bijection with the strongly \( \alpha \)-closed graph, then both \( X \) and \( Y \) are \( T_2 \).

**Proof.** The proof follows from Theorems 2.8 and 2.4.

**Lemma 2.12** Every clopen subset of a quasi \( H \)-closed space \( X \) is quasi \( H \)-closed relative to \( X \).
Proof. Let $B$ be any clopen subset of a quasi $H$-closed space $X$. Let $\{O_\lambda : \lambda \in \Omega\}$ be any cover of $B$ by open sets in $X$. Then the family $F = \{O_\lambda : \lambda \in \Omega\} \cup \{X \setminus B\}$ is a cover of $X$ by open sets in $X$. Because of quasi $H$-closedness of $X$ there exists a finite subfamily $F^* = \{O_{\lambda_i} : 1 \leq i \leq n\} \cup \{X \setminus B\}$ of $F$ whose closure covers $X$. So, because of clopeness of $B$ we now infer that the family $\{Cl(O_{\lambda_i}) : 1 \leq i \leq n\}$ covers $B$. Therefore, $B$ is quasi $H$-closed relative to $X$.

**Theorem 2.13** If $Y$ is a quasi $H$-closed extremally disconnected space, then a function $f : X \to Y$ with the strongly $\alpha$-closed graph $G(f)$ is weakly $\alpha$-continuous.

**Proof.** Let $x \in X$ and $V \in O(Y, f(x))$. Take any $y \in Y \setminus Cl(V)$. Then $(x, y) \in (X \times Y) \setminus G(f)$. Now the strong $\alpha$-closedness of $G(f)$ induces the existence of $U_y(x) \in \alpha O(X, x)$, $V_y \in O(Y, y)$ such that $f(U_y(x)) \cap Cl(V_y) = \emptyset$.\textsuperscript{(*)}. Now extremal disconnectedness of $Y$ induces the clopenness of $Cl(V)$ and hence $Y \setminus Cl(V)$ is also clopen. Now $\{V_y : y \in Y \setminus Cl(V)\}$ is a cover of $Y \setminus Cl(V)$ by open sets in $Y$. By Lemma 2.13, there exists a finite subfamily $\{V_{y_i} : 1 \leq i \leq n\}$ such that $Y \setminus Cl(V) \subset \bigcup_{i=1}^{n} Cl(V_{y_i})$. Let $W = \bigcap_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are $\alpha$-open sets in $X$ satisfying (*). Also, $W \in \alpha O(X, x)$.

Now $f(W) \cap (Y \setminus Cl(V)) \subset f[\bigcap_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup_{i=1}^{n} Cl(V_{y_i})) \subset \bigcup_{i=1}^{n} (f[U_{y_i}(x)] \cap Cl(V_{y_i})) = \emptyset$, by (*). Therefore, $f(W) \subset Cl(V)$ and this indicates that $f$ is weakly $\alpha$-continuous.

Noiri [22] showed that if $G(f)$ is strongly closed then $f$ has the following property:

(P) For every set $B$ which is quasi $H$-closed relative to $Y$, $f^{-1}(B)$ is a closed set of $X$.

Analogously, we have the following theorem.

**Theorem 2.14** If a function $f : X \to Y$ has a strongly $\alpha$-closed graph $G(f)$, then $f$ enjoys the following property:

(P') For every set $F$ which is quasi $H$-closed relative to $Y$, $f^{-1}(F)$ is $\alpha$-closed in $X$.

**Proof.** Let $f^{-1}(F)$ be not $\alpha$-closed in $X$. Then there exists $x \in \alpha Cl(f^{-1}(F)) \setminus f^{-1}(F)$. Let $y \in F$. Then $(x, y) \in (X \times Y) \setminus G(f)$. Strong $\alpha$-closedness of $G(f)$ gives the existence of
$U_y(x) \in \alpha O(X, x)$ and $V_y \in O(Y, y)$ such that $f(U_y(x)) \cap Cl(V_y) = \emptyset$...(*).

Clearly $\{V_y : y \in F\}$ is a cover of $F$ by open sets in $Y$. Since $F$ is quasi $H$-closed relative to $Y$, there exist a finite number of open sets $V_{y_1}, V_{y_2}, ..., V_{y_n}$ in $Y$ such that $F \subset \bigcup_{i=1}^{n} Cl(V_{y_i})$.

Let $U = \bigcap_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are the $\alpha$-open sets in $X$ satisfying (*). Also $U \in \alpha O(X, x)$.

Now $f(U) \cap F \subset f[\bigcap_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup_{i=1}^{n} Cl(V_{y_i})) \subset \bigcup_{i=1}^{n} (f[U_{y_i}(x)] \cap Cl(V_{y_i})) = \emptyset$. But since $x \in \alpha Cl(f^{-1}(F))$, $U \cap f^{-1}(F) \neq \emptyset$; hence $f(U) \cap F \neq \emptyset$. This is a contradiction. Hence the result holds.

### 3 Additional properties

**Lemma 3.1** For a topological space $X$, the following properties are equivalent:

1. $X$ is Urysohn;
2. For every pair of distinct points $x, y \in X$, there exist $U \in \alpha O(X, x), V \in \alpha O(X, y)$ such that $Cl(U) \cap Cl(V) = \emptyset$;
3. For every pair of distinct points $x, y \in X$, there exist $U \in \alpha O(X, x), V \in \alpha O(X, y)$ such that $\alpha Cl(U) \cap \alpha Cl(V) = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): Since $\alpha Cl(U) = Cl(U)$ for each $U \in \alpha(X)$ by Lemma 1.6, this is obvious.

(3) $\Rightarrow$ (1): Suppose that (3) holds. For every pair of distinct points $x, y$, there exist $U \in \alpha O(X, x), V \in \alpha O(X, y)$ such that $\alpha Cl(U) \cap \alpha Cl(V) = \emptyset$. Now, put $G = Int(Cl(Int(U)))$ and $H = Int(Cl(Int(V)))$, then $G$ and $H$ are open sets containing $x$ and $y$, respectively. Furthermore, $Cl(G) \cap Cl(H) = Cl(U) \cap Cl(V) = \alpha Cl(U) \cap \alpha Cl(V) = \emptyset$. Therefore, $X$ is Urysohn.

Recall, that a function $f : X \to Y$ is said to be $\alpha$-open [19] if $f(A) \in \alpha O(Y)$ for all open set $A$ of $Y$.

**Lemma 3.2** Let a bijection $f : X \to Y$ be $\alpha$-open. Then for any closed set $B$ of $X$, $f(B) \in \alpha C(Y)$. 


Urysohn spaces remain invariant under certain bijective function as is shown in the next theorem.

**Theorem 3.3** If a bijection $f : X \to Y$ is $\alpha$-open and $X$ is Urysohn, then $Y$ is Urysohn.

**Proof.** Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Since $f$ is bijective, $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$. The Urysohn property of $X$ gives the existence of sets $U \in O(X, f^{-1}(y_1))$, $V \in O(X, f^{-1}(y_2))$ such that $Cl(U) \cap Cl(V) = \emptyset$. As $Cl(U)$ is a closed set in $X$, then by the bijectivity and $\alpha$-openness of $f$ together then indicate, by Lemma 3.2 that $f(Cl(U)) \in \alpha C(Y)$. Therefore by the injectivity of $f$, $\alpha Cl(f(U)) \cap \alpha Cl(f(V)) \subset f(Cl(U)) \cap f(Cl(V)) = f(Cl(U) \cap Cl(V)) = \emptyset$. Thus $\alpha$-openness of $f$ gives the existence of two sets $f(U) \in \alpha O(Y, y_1)$, $f(V) \in \alpha O(Y, y_2)$, with $\alpha Cl(f(U)) \cap \alpha Cl(f(V)) = \emptyset$. By Lemma 3.1, $Y$ is Urysohn.

**Theorem 3.4** If $f : X \to Y$ is weakly $\alpha$-continuous and $Y$ is Urysohn, then $G(f)$ is strongly $\alpha$-closed.

**Proof.** Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since $Y$ is Urysohn, there exist $V \in O(Y, y), W \in O(Y, f(x))$ such that $Cl(V) \cap Cl(W) = \emptyset$. Since $f$ is weakly $\alpha$-continuous, there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Cl(W)$. This, therefore, implies that $f(U) \cap Cl(V) = \emptyset$. So by Lemma 2.2, $G(f)$ is strongly $\alpha$-closed.

**Theorem 3.5** Let $X$ be a Urysohn space. Then any $\alpha$-open bijection $f : X \to Y$ has a strongly $\alpha$-closed graph.

**Proof.** Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and $y \neq f^{-1}(y)$, where $f^{-1}(y)$ is a singleton. Since $X$ is Urysohn, there exist open sets $U_x$ and $U_y$ such that $x \in U_x$, $f^{-1}(y) \in U_y$ and $Cl(U_x) \cap Cl(U_y) = \emptyset$. Since $f$ is $\alpha$-open, $f(U_x) \in \alpha O(Y, f(x))$, $f(U_y) \in \alpha O(Y, y)$ and $f(U_x) \cap \alpha Cl(f(U_y)) \subset \alpha Cl(f(U_x)) \cap \alpha Cl(f(U_y)) \subset f(Cl(U_x)) \cap f(Cl(U_y)) = \emptyset$. Therefore, by Lemma 2.2, $G(f)$ is strongly $\alpha$-closed.

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