ON SOME PROPERTIES OF WEAKLY
LC-CONTINUOUS FUNCTIONS

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Abstract

M. Ganster and I.L. Reilly [2] introduced a new decomposition of continuity called
LC-continuity. In this paper, we introduce and investigate a generalization LC-
continuity called weakly LC-continuity.

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continuity, LC-compact spaces and LC-connected spaces.

1 Introduction and Preliminaries

M. Ganster and I.L. Reilly in [2] introduced three types of continuity, that is, LC-irresoluteness,
LC-continuity and sub-LC-continuity based on a notion, namely locally closed sets, implicitly introduced in Kuratowski and Sierpinski’s work [4]. They have further investigated
LC-continuity in [3]. In this paper, we introduce and investigate the class of LC-continuous
functions.

In what follows \((X, \tau)\) and \((Y, \sigma)\) (or \(X\) and \(Y\)) denote topological spaces. Let \(A\) be a subset of \(X\). We denote the interior, the closure and the complement of a set \(A\) by \(\text{Int}(A)\), \(\text{Cl}(A)\)
and \(X \setminus A\), respectively.

Definition 1 A subset \(A\) of a topological space \(X\) is said to be locally closed [1] in \(X\) if it
is the intersection of an open subset of \(X\) and a closed subset of \(X\). The complement of a
locally closed set is said to be locally open.
The family of all locally closed sets of \( X \) containing a point \( x \in X \) is denoted by \( \text{LC}(X,x) \). The family of all locally closed (resp. locally open) sets of \( X \) is denoted by \( \text{LC}(X) \) (resp. \( \text{LO}(X) \)). Similarly, we denoted by \( O(X,x) \) (resp. \( C(X,x) \)) the family of all open (resp. closed) sets of \( X \) containing a point \( x \in X \).

**Remark 1.1** The following properties are well-known.

(i) A subset \( A \) of \( X \) is locally closed if and only if its complement \( X \setminus A \) is locally open, it is the union of an open set and a closed set.

(ii) Every open (resp. closed) subset of \( X \) is locally closed.

(iii) The complement of a locally closed set need not be locally closed.

**Definition 2** [2] A function \( f : (X,\tau) \to (Y,\sigma) \) is said to be

1. \( \text{LC} \)-continuous if \( f^{-1}(V) \in \text{LC}(X,\tau) \) for each \( V \in \sigma \).
2. \( \text{LC} \)-irresolute if \( f^{-1}(F) \in \text{LC}(X,\tau) \) for each \( F \in \text{LO}(Y,\sigma) \).

**2 Some fundamental properties**

We introduce the following notions.

**Definition 3** A point \( x \in X \) is called a \( \text{LC} \)-cluster point of a subset \( A \) of \( X \) if \( U \cap A \neq \emptyset \) for every \( U \in \text{LC}(X,x) \). The set of all \( \text{LC} \)-cluster points of \( A \) is called the \( \text{LC} \)-closure of \( A \) and is denoted by \( [A]_{\text{LC}} \). A subset \( A \) is said to be \( \text{LC} \)-closed if \( A = [A]_{\text{LC}} \).

The complement of a \( \text{LC} \)-closed set \( A \) is said to be \( \text{LC} \)-open.

**Remark 2.1** For a subset \( A \) of a space \( X \), \( [A]_{\text{LC}} = \bigcap \{ V : A \subseteq V, V \in \text{LO}(X) \} \).

Observe that as an example for Definition 3, take \( X = \{ a, b, c \} \) with topology \( \tau = \{ X, \emptyset, \{ a \}, \{ a, b \} \} \). Then \( \{ a, c \} \) is \( \text{LC} \)-closed but not locally closed.

**Definition 4** A function \( f : X \to Y \) is said to be weakly \( \text{LC} \)-continuous at \( x \in X \) if for each open set \( V \) of \( Y \) containing \( f(x) \), there exists a locally closed set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). If \( f \) is weakly \( \text{LC} \)-continuous at every point of \( X \), then it is called weakly \( \text{LC} \)-continuous on \( X \).
It should be noticed that:
continuity ⇒ LC-irresolute ⇒ LC-continuity ⇒ weak LC-continuity by ([2], p. 421) and
Example 3 of [2]. Example 3 is an example of a weakly LC-continuous function which is not
LC-continuous.

**Theorem 2.2** For a function \( f : X \to Y \), the following are equivalent:

1. \( f \) is weakly LC-continuous;
2. \( f([A]_{LC}) \subseteq Cl(f(A)) \) for every subset \( A \) of \( X \);
3. \( [f^{-1}(B)]_{LC} \subseteq f^{-1}(Cl(B)) \) for every subset \( B \) of \( Y \);
4. \( f^{-1}(F) \) is LC-closed for every closed set \( F \) of \( Y \);
5. \( f^{-1}(V) \) is LC-open for every open set \( V \) of \( Y \).

**Proof.** (1) ⇒ (2): Let \( y \in f([A]_{LC}) \) and let \( V \) be any open set of \( Y \) containing \( y \). Then, there exists a point \( x \in [A]_{LC} \) such that \( f(x) = y \in V \). Since \( f \) is weakly LC-continuous, there exists \( U \in LC(X, x) \) such that \( f(U) \subseteq V \). Since \( x \in [A]_{LC} \), \( U \cap A \neq \emptyset \) holds and hence \( f(A) \cap V \neq \emptyset \). Therefore we have \( y = f(x) \in Cl(f(A)) \).

(2) ⇒ (3): Let \( B \) be an arbitrary set containing of \( Y \) and let \( A = f^{-1}(B) \). Then by (2), we have \( f([A]_{LC}) \subseteq Cl(f(A)) \subseteq Cl(B) \). This implies that \([A]_{LC} \subseteq f^{-1}(Cl(B)) \). That is \([f^{-1}(B)]_{LC} \subseteq f^{-1}(Cl(B)) \).

(3) ⇒ (4): Let \( F \) be any closed set of \( Y \). By (3), we have \([f^{-1}(F)]_{LC} \subseteq f^{-1}(Cl(F)) = f^{-1}(F) \). By Remark 2.1, \([f^{-1}(F)]_{LC} \subseteq f^{-1}(F) \) and hence \([f^{-1}(F)]_{LC} = f^{-1}(F) \). Therefore, \( f^{-1}(F) \) is LC-closed.

(4) ⇒ (5): Let \( V \) be any open set of \( Y \). We have \( f^{-1}(X \setminus V) = X \setminus f^{-1}(V) \) and by (4), \( f^{-1}(V) \) is LC-open.

(5) ⇒ (1): Let \( x \in X \) and \( V \in O(Y, f(x)) \). By (5), \( x \in f^{-1}(V) \) and \( f^{-1}(V) \) LC-open. Therefore, \( X \setminus f^{-1}(V) \) is LC-closed and \( x \notin [X \setminus f^{-1}(V)] \). Hence there exists \( U \in LC(X, x) \) such that \( U \cap (X \setminus f^{-1}(V)) = \emptyset \); hence \( U \subseteq f^{-1}(V) \). Therefore, we obtain \( f(U) \subseteq V \). This shows that \( f \) is weakly LC-continuous.

**Definition 5** Let \((X, \tau)\) be a topological space. Since \( LC(X) \) is closed under a finite intersection, \( LC(X) \) is a base of some topology for \( X \). We denote it by \( \tau_{LC} \).
Theorem 2.3 A function \( f : (X, \tau) \to (Y, \sigma) \) is weakly LC-continuous if and only if \( f : (X, \tau_{LC}) \to (Y, \sigma) \) is continuous.

Proof. Necessity. Let \( V \in \sigma \) and \( x \in f^{-1}(V) \). Then there exists \( U_x \in LC(X, x) \) such that \( f(U_x) \subseteq V \). Hence we obtain \( \bigcup \{ U_x : x \in f^{-1}(V) \} = f^{-1}(V) \in \tau_{LC} \). Therefore, \( f : (X, \tau_{LC}) \to (Y, \sigma) \) is continuous.

Sufficiency. Let \( x \in X \) and \( V \in O(Y, f(x)) \). Then \( x \in f^{-1}(V) \in \tau_{LC} \) and there exists \( U \in LC(X, x) \) such that \( x \in U \subseteq f^{-1}(V) \); hence \( f(U) \subseteq V \). This shows that \( f \) is weakly LC-continuous.

Definition 6 Let \( A \) be a subset of \( X \). A mapping \( r : X \to A \) is called a weakly LC-continuous retraction if \( r \) is weakly LC-continuous and the restriction \( r \mid_A \) is the identity mapping on \( A \).

Theorem 2.4 Let \( A \) be a subset of \( X \) and \( r : X \to A \) be a weakly LC-continuous retraction. If \( X \) is Hausdorff, then \( A \) is a LC-closed set of \( X \).

Proof. Suppose that \( A \) is not LC-closed. Then, there exists a point \( x \) in \( X \) such that \( x \in [A]_{LC} \) but \( x \notin A \). It follows that \( r(x) \neq x \) because \( r \) is weakly LC-continuous retraction. Since \( X \) is Hausdorff there exists disjoint open sets \( U \) and \( V \) in \( X \) such that \( x \in U \) and \( r(x) \in V \). Now let \( W \) be an arbitrary locally closed set containing \( x \). Then \( W \cap U \) is a locally closed set containing \( x \). Since \( x \in [A]_{LC} \), we have \( (W \cap U) \cap A \neq \emptyset \). Therefore, there exists a point \( y \) in \( W \cap U \cap A \). Since \( y \in A \), we have \( r(y) = y \in U \) and hence \( r(y) \notin V \). This implies that \( r(W) \notin V \) because \( y \in W \). This is contrary to the weakly LC-continuity of \( r \). Consequently, \( A \) is a LC-closed set of \( X \).

Definition 7 the LC-frontier of a subset \( A \) of a space \( X \) denoted by \( LC-fr(A) \), is given by \( LC-fr(A) = [A]_{LC} \cap [X \setminus A]_{LC} \).

Theorem 2.5 The set of all points \( x \in X \) at which \( f : (X, \tau) \to (Y, \sigma) \) is not weakly LC-continuous is identical with the union of the LC-frontiers of the inverse images of open subsets of \( Y \) containing \( f(x) \).
Proof. Necessity. Suppose that \( f \) is not weakly \( LC \)-continuous at a point \( x \) of \( X \). Then, there exists an open set \( V \subset Y \) containing \( f(x) \) such that \( f(U) \) is not a subset of \( V \) for every \( U \in LC(X,x) \). Hence we have \( U \cap (X \setminus f^{-1}(V)) \neq \emptyset \) for every \( U \in LC(X,x) \). It follows that \( x \in [X \setminus f^{-1}(V)]_{LC} \). We also have \( x \in f^{-1}(V) \subset [f^{-1}(V)]_{LC} \). This means that \( x \in LC-fr(f^{-1}(V)) \).

Sufficiency. Suppose that \( x \in LC-fr(f^{-1}(V)) \) for some \( V \in O(Y, f(x)) \) Now, we assume that \( f \) is weakly \( LC \)-continuous at \( x \in X \). Then there exists \( U \in LC(X,x) \) such that \( f(U) \subset V \). Therefore, we have \( x \in U \subset f^{-1}(V) \). Thus \( x \notin [X \setminus f^{-1}(V)]_{LC} \). This is a contradiction. This means that \( f \) is not weakly \( LC \)-continuous at \( x \).

Definition 8 A filter base \( B \) is said to be \( LC \)-convergent to a point \( x \in X \) if for any locally closed set \( A \) containing \( x \), there exists \( B_1 \in B \) such that \( B_1 \subset A \).

Theorem 2.6 A function \( f : X \to Y \) is weakly \( LC \)-continuous if and only if for each point \( x \in X \) and each filter base \( B \) on \( X \) \( LC \)-converging to \( x \), the filter base \( f(B) \) is convergent to \( f(x) \).

Proof. Suppose that \( f \) is weakly \( LC \)-continuous. Let \( x \in X \) and \( B \) be any filter base \( LC \)-converging to \( x \). Since \( f \) is weakly \( LC \)-continuous, for each open set \( V \subset Y \) containing \( f(x) \), there exists a locally closed set \( U \in X \) containing \( x \) such that \( f(U) \subset V \). Since \( B \) is \( LC \)-converging to \( x \), then there exists \( B_1 \in B \) such that \( B_1 \subset U \). This implies that \( f(B_1) \subset V \). It follows that \( f(B_1) \) is convergent to \( f(x) \).

Conversely, let \( x \in X \) and \( V \) be any open set containing \( f(x) \). Suppose that \( B = LC(X,x) \). Then it follows that \( B \) is a filter base \( LC \)-converging to \( x \). Hence there exists \( U \in B \) such that \( f(U) \subset V \), as we wished to prove.

Definition 9 A space \( X \) is said to be \( LC \)-separate if for every pair of distinct points \( x \) and \( y \) in \( X \), there exist locally closed sets \( B_1 \) and \( B_2 \) containing \( x \) and \( y \), respectively, such that \( B_1 \cap B_2 = \emptyset \).

Let \( X = \{a,b\} \) with \( \tau = \{X, \emptyset, \{a\}\} \). (\( X, \tau \)) is \( LC \)-separate but not separate.
Theorem 2.7 If $f : X \to Y$ is a weakly $LC$-continuous injection and $Y$ is Hausdorff, then $X$ is $LC$-separate.

Proof. Let $x$ and $y$ be distinct points of $X$. Then $f(x) \neq f(y)$. Since $Y$ is Hausdorff, there exist disjoint open sets $V$ and $W$ in $Y$ containing $f(x)$ and $f(y)$, respectively. Since $f$ is weakly $LC$-continuous, there exist locally closed sets $U_1$ and $U_2$ containing $x$ and $y$, respectively, such that $f(U_1) \subset V$ and $f(U_2) \subset W$. It follows that $U_1 \cap U_2 = \emptyset$. This shows clearly that $X$ is $LC$-separate.

Theorem 2.8 If $f, g : X \to Y$ are weakly $LC$-continuous functions and $Y$ is Hausdorff, then $A = \{x \in X : f(x) = g(x)\}$ is $LC$-closed in $X$.

Proof. Suppose that $x \notin A$. Then $f(x) \neq g(x)$. Since $Y$ is Hausdorff, there exist $V \in O(Y, f(x))$ and $W \in O(Y, g(x))$ such that $V \cap W = \emptyset$. Since $f$ and $g$ are weakly $LC$-continuous, there exist $U \in LC(X, x)$ and $G \in LC(X, x)$ such that $f(U) \subset V$ and $f(G) \subset W$. Set $D = U \cap G$, so $D \in LC(X, x)$. Hence we have $f(D) \cap g(D) \subset V \cap W = \emptyset$. This shows clearly that $x \notin [A]_{LC}$. It follows that $[A]_{LC} \subset A$, that is $A$ is $LC$-closed in $X$.

Definition 10 For a function $f : X \to Y$, the graph $G(f) = \{(x, f(x)) : x \in X\}$ is said to be $LC$-closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in LC(X, x)$ and $V \in O(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 2.9 A function $f : X \to Y$ has a $LC$-closed graph $G(f)$ if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in LC(X, x)$ and $V \in O(Y, y)$ such that $f(U) \cap V = \emptyset$.

Proof. It is an immediate consequence of Definition 10 and the fact that for any subsets $U \subset X$ and $V \subset Y$, $(U \times V) \cap G(f) = \emptyset$ if and only if $f(U) \cap V = \emptyset$.

Theorem 2.10 If $f : X \to Y$ is weakly $LC$-continuous and $Y$ is Hausdorff, then $G(f)$ is $LC$-closed in $X \times Y$. 


Proof. Let \((x, y) \in (X \times Y) \setminus G(f)\). It follows that \(f(x) \neq y\). Since \(Y\) is Hausdorff, there exist disjoint open sets \(V\) and \(W\) in \(Y\) containing \(f(x)\) and \(y\), respectively. Since \(f\) is weakly \(LC\)-continuous, there exists \(U \in LC(X, x)\) such that \(f(U) \subset V\). Therefore \(f(U) \cap W = \emptyset\) and \(G(f)\) is \(LC\)-closed in \(X \times Y\).

**Definition 11** Let \(A\) be a subset of \(X\), then we say that \(A\) is \(LC\)-compact relative to \(X\) if every cover of \(A\) by locally closed sets of \(X\) has a finite subcover. A space \(X\) is said to be \(LC\)-compact if \(X\) is \(LC\)-compact in \(X\).

**Theorem 2.11** If \(f : X \to Y\) is a weakly \(LC\)-continuous function and \(A\) is \(LC\)-compact relative to \(X\), then \(f(A)\) is compact relative to \(Y\).

**Proof.** Suppose that \(f : X \to Y\) is weakly \(LC\)-continuous and let \(A\) be \(LC\)-compact relative to \(X\). Let \(\{V_\alpha : \alpha \in \nabla\}\) be an open cover of \(f(A)\). For each point \(x \in A\), there exists \(\alpha(x) \in \nabla\) such that \(f(x) \in V_{\alpha(x)}\). Since \(f\) is weakly \(LC\)-continuous, there exists \(U_x \in LC(X, x)\) such that \(f(U_x) \subset V_{\alpha(x)}\). The family \(\{U_x : x \in A\}\) is a cover of \(A\) by locally closed sets of \(X\) and hence there exists a finite set \(A_0\) of \(A\) such that \(A \subset \cup_{x \in A_0} U_x\). Therefore, we obtain \(f(A) \subset \cup_{x \in A_0} V_{\alpha(x)}\). This shows that \(f(A)\) is compact in \(Y\).

**Definition 12** A space \(X\) is said to be \(LC\)-connected if \(X\) can not be expressed as the union of two nonempty \(LC\)-open sets.

Observe that the Sierpinski space is connected but it is not \(LC\)-connected.

**Theorem 2.12** If \(f : X \to Y\) is a weakly \(LC\)-continuous function and \(X\) is \(LC\)-connected, then \(Y\) is connected.

**Proof.** Suppose that \(Y\) is not connected. Then there exist nonempty open sets \(V\) and \(W\) such that \(V \cap W = \emptyset\) and \(V \cup W = Y\). It follows that \(f^{-1}(V) \cap f^{-1}(W) = \emptyset\) and \(f^{-1}(V) \cup f^{-1}(W) = X\). By weakly \(LC\)-continuity of \(f\), it follows from Theorem 2.1 that \(f^{-1}(V)\) and \(f^{-1}(W)\) are nonempty \(LC\)-open sets in \(X\). This shows that \(X\) is not \(LC\)-connected. But this is a contradiction. Hence \(Y\) is connected.
**Definition 13** The intersection of all locally closed sets containing a set $A$ is called the $LC^*$-closure of $A$ and is denoted by $[A]^*_{LC}$. This is, for any $A \subset X$, $[A]^*_{LC} = \cap \{F \in LC(X): A \subset F\}$.

**Remark 2.13** If $B$ is a locally closed set in a space $X$, then $[B]^*_{LC} = B$. The converse is false. If $X$ denote the real line with the cofinite topology and if $B = \{\frac{1}{n}: n \in N\}$. Then $[B]^*_{LC} = B$. But $B$ is not locally closed. However, the converse is true if the space $X$ is an Alexandorff space. A space is said to be Alexandorff if the intersection of any open sets of $X$ is open in $X$.

**Definition 14** Let $p$ be a point of $X$ and $N$ be a subset of $X$. $N$ is called a $LC$-neighborhood of $p$ in $X$ if there exists a locally open set $O$ of $X$ such that $p \in O \subset N$.

**Lemma 2.14** Let $A$ be a subset of $X$. Then, $p \in [A]^*_{LC}$ if and only if for any $LC$-neighborhood $N_p$ of $p$ in $X$, $A \cap N_p \neq \phi$.

**Proof.** Necessity. Suppose that $p \in [A]^*_{LC}$. If there exists a $LC$-neighborhood $N$ of the point $p$ in $X$ such that $N \cap A = \phi$, then by definition, there exists a locally open set $O_p$ such that $p \in O_p \subset N$. Therefore, we have $O_p \cap A = \phi$, so that $A \subset X \setminus O_p$. Since $X \setminus O_p$ is locally closed, then $[A]^*_{LC} \subset X \setminus O_p$. As $p \notin [A]^*_{LC}$ which is contrary to the hypothesis.

Sufficiency. If $p \notin [A]^*_{LC}$, then by definition of $[A]^*_{LC}$, there exists a locally closed set $F$ of $X$ such that $A \subset F$ and $p \notin F$. Therefore, we have $p \in X \setminus F$ such that $X \setminus F$ is a locally open set. Hence $X \setminus F$ is a $LC$-neighborhood of $p$ in $X$, but $(X \setminus F) \cap A = \phi$. This is contrary to the hypothesis.

**Definition 15** A function $f : X \to Y$ is said to be $LC^*$-continuous if the inverse image of every closed in $Y$ is locally closed in $X$.

**Theorem 2.15** Let $f : X \to Y$ be a function.

(i) The following statements are equivalent:

(a) $f$ is $LC^*$-continuous.

(b) The inverse image of each open set of $Y$ is locally open in $X$. 

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(ii) If \( f \) is \( LC^a \)-continuous, then \( f([A]_{LC}^*) \subseteq Cl(f(A)) \) for every \( A \subseteq X \).

(iii) The following statements are equivalent:

(a) For each point \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists a locally open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \).

(b) \( f([A]_{LC}^*) \subseteq Cl(f(A)) \) for every \( A \subseteq X \).

(iv) For the following statements \((a) \Rightarrow (b) \Rightarrow (c)\), and they are equivalent if \( X \) is Alexandorff.

(a) \( f \) is \( LC^a \)-continuous.

(b) \( f([A]_{LC}^*) \subseteq Cl(f(A)) \) for every \( A \subseteq X \).

(c) \( [f^{-1}(B)]_{LC}^* \subseteq f^{-1}(Cl(B)) \) for every \( B \subseteq Y \).

Proof. (i) The equivalence is proved by definitions.

(ii) Since \( A \subseteq f^{-1}(Cl(f(A))) \), it is obtained that \( f([A]_{LC}^*) \subseteq Cl(f(A)) \) by using assumptions.

(iii) \((a) \Rightarrow (b)\): Let \( y \in f([A]_{LC}^*) \) and let \( V \) any open neighborhood of \( y \). Then, there exists a point \( x \in X \) and a locally open set \( U \) such that \( f(x) = y, x \in U, x \in [A]_{LC}^* \) and \( f(U) \subseteq V \). Since \( x \in [A]_{LC}^*, U \cap A \neq \emptyset \) holds and hence \( f(A) \cap V \neq \emptyset \). Therefore we have

\[
y = f(x) \in Cl(f(A)).
\]

(b) \( \Rightarrow (a)\): Let \( x \in X \) and \( V \) be any open set containing \( f(x) \). Let \( A = f^{-1}(Y \setminus V) \), then \( x \notin A \). Since \( f([A]_{LC}^*) \subseteq Cl(f(A)) \subseteq (Y \setminus V) \), it is shown that \( [A]_{LC}^* = A \). Then, since \( x \notin [A]_{LC}^* \), there exists a locally open set \( U \) containing \( x \) such that \( U \cap A = \emptyset \) and hence \( f(U) \subseteq f(X \setminus A) \subseteq V \).

(iv) \((a) \Rightarrow (b)\): Let \( A \) be any subset of \( X \). Let \( y \notin Cl(f(A)) \). Then there exist \( V \in O(Y, y) \) such that \( V \cap f(A) = \emptyset \); hence \( A \cap f^{-1}(V) = \emptyset \). By (i), \( f^{-1}(V) \in LO(X) \) and \( A \subseteq X \setminus f^{-1}(V) \in LC(X) \). Therefore, we have \([A]_{LC}^* \subseteq X \setminus f^{-1}(V) \) and hence \([A]_{LC}^* \cap f^{-1}(V) = \emptyset \).

We obtain \( f([A]_{LC}^*) \cap V = \emptyset \) and \( y \notin f([A]_{LC}^*) \). Hence \( f([A]_{LC}^*) \subseteq Cl(f(A)) \).

(b) \( \Rightarrow (c) \): Let \( B \) be any subset of \( Y \). By (b) \( f([f^{-1}(B)]_{LC}^*) \subseteq Cl(B) \) and \([f^{-1}(B)]_{LC}^* \subseteq f^{-1}(Cl(B)) \).

Let \( X \) be Alexandorff and we prove that \((c) \Rightarrow (a)\). Let \( F \) be any closed set of \( Y \). By (c), \([f^{-1}(B)]_{LC}^* \subseteq f^{-1}(Cl(F)) = f^{-1}(F) \) and hence \([f^{-1}(B)]_{LC}^* \subseteq f^{-1}(F) \). Since \( X \)
is Alexandorff, \([f^{-1}(B)]_{LC} \in LC(X)\) and \(f^{-1}(F)\) is locally closed. Therefore, \(f\) is \(LC^*\)-continuous.

**Theorem 2.16** If \(f : X \to Y\) be a function, and let \(g : X \to X \times Y\) be the graph function of \(f\), defined by \(g(x) = \{(x, f(x))\}\) for every \(x \in X\). If \(g\) is \(LC^*\)-continuous, then \(f\) is \(LC^*\)-continuous.

**Proof.** Let \(U\) be an open set in \(Y\). Then \(X \times U\) is an open set in \(X \times Y\). Since \(g\) is \(LC^*\)-continuous, it follows of Theorem 2.13(i) that \(f^{-1}(U) = g^{-1}(X \times U)\) is a locally open set in \(X\). Thus \(f\) is \(LC^*\)-continuous.

**Theorem 2.17** Let \(\{X_i : i \in I\}\) be any family of topological spaces. If \(f : X \to \prod X_i\) is a \(LC^*\)-continuous function, then \(Pr_i \circ f : X \to X_i\) is \(LC^*\)-continuous for each \(i \in I\), where \(Pr_i\) is the projection of \(\prod X_j\) onto \(X_i\).

**Proof.** We shall consider a fixed \(i \in I\). Suppose \(U_i\) is an arbitrary open set in \(X_i\). Then \(Pr_i^{-1}(U_i)\) is open in \(\prod X_i\). Since \(f\) is \(LC^*\)-continuous, \(f^{-1}(Pr_i^{-1}(U_i)) = (Pr_i \circ f)^{-1}(U_i)\) is locally open in \(X\). Therefore \(Pr_i \circ f\) is \(LC^*\)-continuous.

**Definition 16** A space \(X\) is said to be:

(i) \(L\)-connected if \(X\) can not be expressed as the union of two disjoint nonempty locally open sets.

(ii) \(L\)-normal if each pair of non-empty disjoint closed sets can be separated by disjoint locally open sets.

**Theorem 2.18** If \(f : X \to Y\) is a \(LC^*\)-continuous surjection and \(X\) is \(L\)-connected, then \(Y\) is connected.

**Proof.** Suppose that \(Y\) is not connected. Then there exist nonempty open sets \(V\) and \(W\) such that \(V \cap W = \emptyset\) and \(V \cup W = Y\). It follows that \(f^{-1}(V) \cap f^{-1}(W) = \emptyset\) and \(f^{-1}(V) \cup f^{-1}(W) = X\). By \(LC^*\)-continuity of \(f\), it follows that \(f^{-1}(V)\) and \(f^{-1}(W)\) are nonempty locally open sets in \(X\). This shows that \(X\) is not \(L\)-connected. But this is a contradiction. Hence \(Y\) is connected.
**Theorem 2.19** If \( f : X \to Y \) is a \( LC^* \)-continuous, closed injection and \( Y \) is normal, then \( X \) is \( L \)-normal.

**Proof.** Let \( F_1 \) and \( F_2 \) be disjoint closed subsets of \( X \). Since \( f \) is closed and injective, \( f(F_1) \) and \( f(F_2) \) are disjoint closed subsets of \( Y \). Since \( Y \) is normal, \( f(F_1) \) and \( f(F_2) \) are separated by disjoint open sets \( V_1 \) and \( V_2 \) respectively. Hence \( F_i \subset f^{-1}(V_i) \), \( f^{-1}(V_i) \in LO(X) \) for \( i = 1, 2 \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \) and thus \( X \) is \( L \)-normal.

**References**


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