ON ENTIRE FUNCTIONS-MINORANTS FOR SUBHARMONIC FUNCTIONS OUTSIDE OF A SMALL EXCEPTIONAL SET

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\( \mathbb{N} := \{1, 2, \ldots \} \) is the set of all natural numbers, and \( \mathbb{N}_0 := \{0\} \cup \mathbb{N}. \) \#\( S \) is the cardinality of a set \( S. \) We consider the set \( \mathbb{R} \) of real numbers mainly as the real axis in the complex plane \( \mathbb{C}. \) By this means, \( \mathbb{R}^+ := \{x \in \mathbb{R}: x \geq 0\} \) is the positive semi-axis in \( \mathbb{C}. \) Besides, \( \mathbb{R}_+ := \mathbb{R}^+ \setminus \{0\}, \) \( \mathbb{R}_{+\infty} := \mathbb{R}^+ \cup \{+\infty\}, \mathbb{R}_{-\infty} := \mathbb{R}_{+\infty} \cup (-\mathbb{R}_{+\infty}). \)

Let \( S \subset \mathbb{C}. \) \( \mathcal{B}(S) \) is the class of all Borel subsets \( B \subset S, \) and \( \mathcal{B}_c(S) \subset \mathcal{B}(S) \) is the class of all compact Borel subset in \( S. \)

Let \( S \in \mathcal{B}(\mathbb{C}). \) \( \text{Meas}(S) \) is the class of all countably additive functions \( \nu \) on \( \mathcal{B}(S) \) with values in \( \mathbb{R}_{+\infty} \) such that \( \nu(K) \in \mathbb{R} \) for each \( K \in \mathcal{B}_c(S). \) Elements from \( \text{Meas}(S) \) are called charges, and \( \text{Meas}^+(S) \subset \text{Meas}(S) \) is the subclass of positive charges called measures.

The classes \( \text{sbh}(S), \text{har}(S) := \text{sbh}(S) \cap (-\text{sbh}(S)), \text{Hol}(S) \) consist of the restrictions to \( S \) of subharmonic, harmonic, holomorphic functions on open sets containing \( S \) resp., and \( \text{sbh}_+(S) := \{u \in \text{sbh}(S): u \neq -\infty\}, \text{Hol}_+(S) := \text{Hol}(S) \setminus \{0\}. \) The Riesz measure of \( u \in \text{sbh}(S) \) is the measure \( \frac{1}{2\pi} \Delta u \in \text{Meas}^+(S) \) where \( \Delta \) is the Laplace operator acting in the sense of the theory of distributions [1], [2].

Given \( z \in \mathbb{C} \) and \( r \in \mathbb{R}^+, \) \( D(z, r) := \{z' \in \mathbb{C}: |z' - z| < r\} \) is an open disk of radius \( r \) centered at \( z; D(r) := D(0, r); \mathbb{D} := D(1) \) is the unit disk. Besides, \( \overline{D}(z, r) := \{z' \in \mathbb{C}: |z' - z| \leq r\} \) is a closed disk; \( \overline{D}(r) := \overline{D}(0, r); \mathbb{D} := \overline{D}(1), \) and \( \partial \overline{D}(z, r) \) is a circle of radius \( r \) centered at \( z; \partial \overline{D}(r) := \partial \overline{D}(0, r); \partial \mathbb{D} := \partial \overline{D}(1) \) is the unit circle.

Given \( \nu \in \text{Meas}(\mathbb{C}) \), we denote by \( \nu^+, \nu^- := (-\nu)^+ \) and \( |\nu| := \nu^+ + \nu^- \) the upper, lower, and total variations of \( \nu, \) and define the counting function of \( \nu \) at \( z \in \mathbb{C} \) as \( \nu(z, r) := \nu(\overline{D}(z, r)), \) and the radial counting function of \( \nu \) as \( \nu^{\text{rad}}(r) := \nu(0, r), r \in \mathbb{R}^+. \)

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For a function $v: \overline{D}(z, r) \to \mathbb{R}_{\pm \infty}$, we define [1, Definition 2.6.7], [2]

\begin{align*}
C_v(z, r) &:= \frac{1}{2\pi} \int_{0}^{2\pi} v(z + re^{i\theta}) \, d\theta, \quad C_v(r) := C_v(0, r), \quad (1C) \\
B_v(z, r) &:= \frac{2}{r^2} \int_{0}^{r} C_v(z, t) \, dt, \quad B_v(r) := B_v(0, r), \quad (1B) \\
M_v(z, r) &:= \sup_{z' \in \partial \overline{D}(z, r)} v(z'), \quad M_v(r) := M_v(0, r), \quad (1M)
\end{align*}

where

\[ M_v(z, r) := \sup_{z' \in \overline{D}(z, r)} v(z') \]

if $v \in \text{sbh}(\overline{D}(z, r))$ [1, Definition 2.6.7], [2].

Consider a function $d: \mathbb{C} \to \mathbb{R}^+$. Given $S \subset \mathbb{C}$ and $r: \mathbb{C} \to \mathbb{R}$, we define

\[ S^{\bullet d} := \bigcup_{z \in S} D(z, d(z)) \subset \mathbb{C}, \]

\[ r^{\bullet d}: z \mapsto \sup \left\{ r(z'): z' \in D(z, d(z)) \right\} \in \mathbb{R}^+_{\pm \infty}, \quad z \in \mathbb{C}, \]

and denote the indicator function of set $S$ by

\[ 1_S: z \mapsto \begin{cases} 
1 & \text{if } z \in S \\
0 & \text{if } z \notin S
\end{cases}, \quad z \in \mathbb{C}. \]

\textbf{Theorem 1} (cf. [3, Normal Points Lemma], [4, § 4. Normal points, Lemma]). Let $r: \mathbb{C} \to \mathbb{R}^+$ be a Borel function such that

\[ d := 2 \sup \{ r(z): z \in \mathbb{C} \} < +\infty, \quad (2) \]

and $\mu \in \text{Meas}^+(\mathbb{C})$ be a measure with

\[ E := \left\{ z \in \mathbb{C}: \int_{0}^{r(z)} \frac{\mu(z, t)}{t} \, dt > 1 \right\} \subset \mathbb{C}. \quad (3) \]

Then there is a no-more-than countable set of disks $D(z_k, t_k)$, $k = 1, 2, \ldots$, such that

\[ z_k \in E, \quad t_k \leq r(z_k), \quad E \subset \bigcup_{k} D(z_k, t_k), \]

\[ \sup_{z \in \mathbb{C}} \# \{ k: z \in D(z_k, t_k) \} \leq 2020, \quad (4) \]
i.e., the multiplicity of this covering \( \{D(z_k, t_k)\}_{k=1,2,...} \) of set \( E \) not larger than 2020, and for every \( \mu \)-measurable subset \( S \subset E \),

\[
\sum_{S \cap D(z_k, t_k) \neq \emptyset} t_k \leq 2020 \int_{S^{s.d}} r^{*r} \ d\mu \leq 2020 \int_{S^{s.d}} r^{*d} \ d\mu. \tag{5}
\]

**Proof.** By definition (3), there is a number

\[ t_z \in (0, r(z)) \quad \text{such that} \quad 0 < t_z < r(z) \mu(z, t_z) \quad \text{for each} \quad z \in E. \tag{6} \]

Thus, the system \( \mathcal{D} = \{D(z, t_z)\}_{z \in E} \) of these disks has properties

\[ E \subset \bigcup_{z \in E} D(z, t_z), \quad 0 < t_z \leq r(z) \leq R. \tag{7} \]

By the Besicovitch covering theorem \([5, 2.8.14]-[10, I.1, \text{Remarks}]\) in the Landkof version \([11, \text{Lemma 3.2}]\), one can select some no-more-than counting subsystem in \( \mathcal{D} \) of disks \( D(z_k, t_k) \in \mathcal{D}, \ k = 1, 2, \ldots, t_k := t_{z_k}, \) such that properties (4) are fulfilled.

Consider a \( \mu \)-measurable subset \( S \subset E \). In view of (6) it is easy to see that

\[
\bigcup \left\{ D(z_k, t_k) : S \cap D(z_k, t_k) \neq \emptyset \right\} \subset \bigcup_{z \in S} D(z, d) = S^{s.d}. \tag{8}
\]

Hence, in view of (6) and (4), we obtain

\[
\sum_{S \cap D(z_k, t_k) \neq \emptyset} t_k := \sum_{S \cap D(z_k, t_k) \neq \emptyset} t_{z_k} \leq \sum_{S \cap D(z_k, t_k) \neq \emptyset} r(z_k) \mu(z, t_k)
\]

\[
= \sum_{S \cap D(z_k, t_k) \neq \emptyset} \int_{D(z_k, t_k)} r(z_k) \ d\mu(z) \leq \sum_{S \cap D(z_k, t_k) \neq \emptyset} \int_{D(z_k, t_k)} r^{*r} \ d\mu
\]

\[
= \sum_{S \cap D(z_k, t_k) \neq \emptyset} \int_{S^{s.d}} 1_{D(z_k, t_k)} r^{*r} \ d\mu
\]

\[
= \int_{S^{s.d}} \left( \sum_{S \cap D(z_k, t_k) \neq \emptyset} 1_{D(z_k, t_k)} \right) r^{*r} \ d\mu \leq 2020 \int_{S^{s.d}} r^{*d} \ d\mu \tag{4}
\]

that gives (5). \( \square \)

**Theorem 2** ([12, Corollary 2]). Let \( w \in \text{sbh}_s(\mathbb{C}), \ P \in \mathbb{R}^+, \) and

\[
p: \ z \mapsto \frac{1}{(1 + |z|)^P}, \quad z \in \mathbb{C}. \tag{9}
\]

There is an entire function \( f \in \text{Hol}_s(\mathbb{C}) \) such that

\[
|\ln|f(z)|| \leq B_w(z, p(z)) \leq C_w(z, p(z)) \quad \text{for each} \quad z \in \mathbb{C}. \tag{10}
\]
A function \( f: [a, +\infty) \to \mathbb{R}_{+\infty} \) is a function of finite type (with respect to an order \( p \in \mathbb{R}^+ \)) iff (see [13, 2.1, (2.1t)])

\[
\text{type}_p[f] := \text{type}_p^\infty[f] := \limsup_{r \to +\infty} \frac{f^+(r)}{r^p} < +\infty, \quad f^+ := \sup\{f, 0\}.
\]

A function \( v \in \text{sbh}(\mathbb{C}) \) of finite type (with respect to an order \( p \in \mathbb{R}^+ \)) iff \( \text{type}_p[v] := \text{type}_p[M_v] < +\infty \) [13, Remark 2.1].

The upper density \( \text{type}_1[v] \) of a charge \( \nu \in \text{Meas} \) is defined as \( \text{type}_1[\nu] := \text{type}_1[|\nu|^{\text{rad}}] \).

The order of a function \( f: [a, +\infty) \to \mathbb{R}_{+\infty} \) (near \( +\infty \)) is a value

\[
\text{ord}_\infty[f] := \inf\{p \in \mathbb{R}^+: \text{type}_p[f] < +\infty\}
\]

\[
= \limsup_{r \to +\infty} \frac{\ln(1 + f^+(r))}{\ln r} \in \mathbb{R}_{+\infty}^+. \tag{11}
\]

A charge \( \nu \in \text{Meas}(\mathbb{C}) \) of finite order iff \( \text{ord}_\infty[\nu] := \text{ord}_\infty[|\nu|^{\text{rad}}] < +\infty \).

A function \( v \in \text{sbh}(\mathbb{C}) \) of finite order iff \( \text{ord}_\infty[v] := \text{ord}_\infty[M_v] < +\infty \).

An trivial corollary of the Poisson–Jensen formula is

**Theorem 3.** Let \( w \in \text{sbh}_+(\mathbb{C}) \) with Riesz measure \( \mu = \frac{1}{2\pi} \Delta w \in \text{Meas}^+(\mathbb{C}) \). Then we have \( \text{ord}_\infty[\mu] = \text{ord}_\infty[C_w] = \text{ord}_\infty[B_v] \), and

\[
[type_p[\mu] < +\infty] \iff [type_p[C_w] < +\infty] \iff [type_p[B_v] < +\infty]
\]

for each \( p \in \mathbb{R}^+ \).

**Theorem 4.** Let \( w \in \text{sbh}_+(\mathbb{C}) \) be a function with \( \text{ord}_\infty[C_w] < +\infty \). Then for any \( P \in \mathbb{R}^+ \), there are \( h \in \text{Hol}_+(\mathbb{C}) \) with \( \text{ord}_\infty[|\ln|h|] \leq \text{ord}_\infty[w] \) and \( \text{type}_q[|\ln|h|] \leq \text{type}_q[w] \) for each \( q \in \mathbb{R}^+ \), and a no-more-than countable set of disks \( D(z_k, r_k), k = 1, 2, \ldots, \) such that

\[
\left\{ z \in \mathbb{C}: \ln|h(z)| > w(z) \right\} \subset \bigcup_k D(z_k, r_k), \tag{12I}
\]

\[
\sup_{k} t_k \leq 1, \quad \sum_{\|z_k\| \geq R} t_k = O\left(\frac{1}{R^P}\right), \quad R \to +\infty. \tag{12E}
\]

**Proof.** By Theorem 3, \( \text{ord}_\infty[\mu] = \text{ord}_\infty[C_w] < +\infty \) for \( \mu := \frac{1}{2\pi} \Delta w \).

Consider \( P \in 1 + \text{ord}_\infty[\mu] + \mathbb{R}^+ \) and an entire function \( f \) from Theorem 2 with (9)–(10). Then for \( h := e^{-1}f \in \text{Hol}_+(\mathbb{C}) \) we obtain

\[
\ln|h(z)| \leq C_w(z, p(z)) - 1 = w(z) + \int_0^{p(z)} \frac{\mu(z, t)}{t} \, dt - 1 \quad \text{for each} \ z \in \mathbb{C}.
\]
Hence, by Theorem 1 with \( r := p \) and \( S := E \setminus D(R) \), we have (12I) with properties (4)–(5)\( \Rightarrow \) (12E). The relations between the orders and types of \( w \) and \( \ln |h| \) are an obvious consequence of (12).

\[ \square \]

**REFERENCES**


