ON SOME NEW CLASSES OF SETS AND A NEW DECOMPOSITION OF CONTINUITY VIA GRILLS

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ABSTRACT. In this paper, we present and study some new classes of sets and give a new decomposition of continuity in terms of grills.

1. INTRODUCTION AND PRELIMINARIES

The idea of grill on a topological space was first introduced by Choquet [7]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds ([5], [6], [8]). In [2], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. We are utilizing the same procedure in this paper.

Throughout this paper, $X$ or $(X, \tau)$ represent a topological space with no separation axioms assumed unless explicitly stated. For a subset $A$ of a space $X$, the closure of $A$ and the interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. The power set of $X$ will be denoted by $\wp(X)$. A collection $G$ of a nonempty subsets of a space $X$ is called a grill [7] on $X$ if (i) $A \in G$ and $A \subseteq B \Rightarrow B \in G$, (ii) $A, B \subseteq X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$. For any point $x$ of a topological space $(X, \tau)$, $\tau(x)$ denote the collection of all open neighborhoods of $x$. Let $(X, \tau)$ be a topological space. A subset $A$ in $X$ is said to be a $t$-set ([3] and [4]) if $\text{Int}(\text{Cl}(A)) = \text{Int}(A)$. A subset $A$ in $X$ is said to be a $B$-set [4] if there is a $U \in \tau$ and a $t$-set $A$ in $(X, \tau)$ such that $H = U \cap A$, respectively. A subset $A$ in $X$ is said to be preopen [1] (resp. regular open) if $A \subseteq \text{Int}(\text{Cl}(A))$ (resp. $\text{Int}(\text{Cl}(A)) = A$).

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Definition 1.1 ([2]). Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). The mapping \(\Phi: \varnothing(X) \to \varnothing(X)\), denoted by \(\Phi_G(A, \tau)\) for \(A \in \varnothing(X)\) or simply \(\Phi(A)\) called the operator associated with the grill \(G\) and the topology \(\tau\) and is defined by \(\Phi_G(A) = \{x \in X \mid A \cap U \in G, \forall U \in \tau(x)\}\).

Proposition 1.1 ([2]). Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). Then for all \(A, B \subseteq X\):

i) \(\Phi(A \cup B) = \Phi(A) \cup \Phi(B)\);

ii) \(\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)\).

Let \(G\) be a grill on a space \(X\). Then a map \(\Psi: \varnothing(X) \to \varnothing(X)\) is defined by \(\Psi(A) = A \cup \Phi(A)\), for all \(A \in \varnothing(X)\). The map \(\Psi\) satisfies Kuratowski closure axioms. Corresponding to a grill \(G\) on a topological space \((X, \tau)\), there exists a unique topology \(\tau_G\) on \(X\) given by \(\tau_G = \{U \subseteq X \mid \Psi(X - U) = X - U\}\), where for any \(A \subseteq X\), \(\Psi(A) = A \cup \Phi(A) = \tau_G - Cl(A)\). For any grill \(G\) on a topological space \((X, \tau)\), \(\tau \subseteq \tau_G\) [2]. If \((X, \tau)\) is a topological space and \(G\) is a grill on \(X\), then we denote a grill topological space by \((X, \tau, G)\).

Let \((X, \tau)\) be a topological space and \(G\) be any grill on \(X\). Then \(A \cup B \subseteq X\) implies \(\Phi(A) \cup \Phi(B)\) [2].

Theorem 1.1 ([2]). i) If \(G_1\) and \(G_2\) are two grills on a space \(X\) with \(G_1 \subseteq G_2\), then \(\tau_{G_1} \subseteq \tau_{G_2}\).

ii) If \(G\) is a grill on a space \(X\) and \(B \notin G\), then \(B\) is closed in \((X, \tau, G)\).

iii) For any subset \(A\) of a space \(X\) and any grill \(G\) on \(X\), \(\Phi(A)\) is \(\tau_G\)-closed.

Theorem 1.2 ([2]). Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). If \(U \in \tau\), then \(U \cap \Phi(A) = U \cap \Phi(U \cap A)\) for any \(A \subseteq X\).

2. Some new classes of sets

Definition 2.1. Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). A subset \(A\) in \(X\) is said to be:

i) \(\Phi\)-open if \(A \subseteq Int(\Phi(A))\);

ii) \(g\)-set if \(Int(\Phi(A)) = Int(A)\);

iii) \(g\Phi\)-set if \(Int(\Phi(A)) = Int(A)\).

Remark 2.1. It should be noted that:

i) Open set and \(\Phi\)-open set are independent from each other.

ii) Every \(g\Phi\)-set is a \(g\)-set, but it is not conversely.

Example 2.1. Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}\). If \(G = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}\), then \(G\) is a grill on \(X\) such that \(\tau - \{\emptyset\} \subseteq G\).
Take $A = \{a,b,d\} \in \tau$, but it is not $\Phi$-open, since $\Phi(\{a,b,d\}) = \{a\}$. And take $B = \{a,b\} \notin \tau$, but it is a $\Phi$-open since $\Phi(\{a,b\}) = X$. Furthermore, $A = \{a,b,d\}$ is a $g$-set, but it is not a $g\Phi$-set.

**Proposition 2.1.** A $\tau_G$-closed set is equivalent to a $g$-set.

**Proof.** Let $A$ be a subset in $(X, \tau, G)$. Then $\text{Int}(\Psi(\Phi(A))) = \text{Int}(\Phi(A) \cup \Phi(\Phi(A))) = \text{Int}(\Phi(A))$, i.e. $\Phi(A)$ is a $g$-set.

**Definition 2.2.** A subset $A$ of $(X, \tau, G)$ is said to be $G$-regular if $\text{Int}(\text{Int}(A)) = A$.

**Proposition 2.2.** Every $G$-regular open set is a $g$-set.

**Proof.** Obvious.

**Example 2.2 ([2]).** Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$. If $G = \{\{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$, then $G$ is a grill on $X$ such that $\tau - \{\emptyset\} \subset G$. Take $A = \{a,c\}$, then $A$ is a $g$-set but it is not a $G$-regular set.

**Proposition 2.3.** A $t$-set is a $g$-set.

**Proof.** Let $A$ be a $t$-set. Then

\[
\text{Int}(A) \supset \text{Int}(\text{Int}(A)) = \text{Int}(A \cup \Phi(A)) \supset \text{Int}(A \cup \text{Cl}(A)) = \text{Int}(\text{Cl}(A)) = \text{Int}(A).
\]

Therefore, $A$ is a $g$-set.

**Remark 2.2.** The converse of Proposition 2.3 is false. By the same conditions as in Example 2.2, take $A = \{a,c\}$. Then $A$ is a $g$-set and also a $g\Phi$-set, but it is not a $t$-set.

**Proposition 2.4.** If $A, B$ are two $g$-sets, then $A \cap B$ is a $g$-set.

**Proof.** $\text{Int}(A \cap B) \supset \text{Int}(\text{Int}(A \cap B)) = \text{Int}(\text{Int}(A \cap B) \cap \text{Int}(A \cap B)) = \text{Int}(\text{Int}(A \cap B) \cap \text{Int}(\text{Int}(A \cap B))) = \text{Int}(A \cap B) = \text{Int}(A \cap B)$. Then $A \cap B$ is a $g$-set.

**Definition 2.3.** Let $(X, \tau)$ be a topological space and $G$ be a grill on $X$. A subset $A$ in $X$ is said to be $G$-preopen set if $A \subset \text{Int}(\Psi(A))$.

**Example 2.3.** In Example 2.2, take $A = \{a,c\}$. Then $A$ is preopen, but it is not $G$-preopen.

**Proposition 2.5.** A $G$-preopen set $A$ is a preopen set.

**Proof.** Let $A$ be a $G$-preopen. Then

\[
A \supset \text{Int}(\text{Int}(A)) = \text{Int}(A \cup \Phi(A)) \supset \text{Int}(A \cup \text{Cl}(A)) = \text{Int}(\text{Cl}(A)).
\]

Therefore, $A$ is a preopen set.
Remark 2.3. By Example 2.9 in [2], since if \( G = \emptyset(X) - \{0\} \) in \((X, \tau)\), then \( \tau_G = \tau \), G-preopen and preopen sets are equivalent.

**Proposition 2.6.** If \( A \) is a G-preopen, then \( Cl(Int(\Psi(A))) = Cl(A) \)

*Proof.* \( Cl(A) \subset Cl(Int(\Psi(A))) \subset Cl(\Psi(A)) = Cl(A \cup \Phi(A)) = Cl(A) \cup Cl(\Phi(A)) = Cl(A) \cup \Phi(A) \subset Cl(A) \).

**Proposition 2.7.** Every \( \Phi \)-open set \( A \) is G-preopen.

*Proof.* Let \( A \) be a \( \Phi \)-open. Then \( A \subset Int(\Phi(A)) \subset Int(A \cup \Phi(A)) = Int(\Psi(A)) \). Therefore \( A \) is G-preopen.

**Proposition 2.8.** Let \((X, \tau, G)\) be a grill topological space with I arbitrary index set. Then:

i) If \( \{A_i \mid i \in I\} \) are G-preopen sets, then \( \cup \{A_i \mid i \in I\} \) is a G-preopen set.

ii) If \( A \) is a G-preopen set and \( U \in \tau \), then \( (A \cap U) \) is a G-preopen set.

*Proof.* i) Let \( \{A_i \mid i \in I\} \) be G-preopen sets, then \( A_i \subset Int(\Psi(A_i)) \) for every \( i \in I \). Thus

\[
\bigcup A_i \subset \bigcup(\Psi(A_i)) \subset \bigcup(\Psi(A_i)) = Int(A_i \cup \Phi(A_i)) =
\]

\[
Int(A_i \cup \Phi(A_i)) = Int(\Psi(A_i) \cup \Phi(A_i)) = Int(\Psi(A_i)).
\]

ii) Let \( A \) be a G-preopen set and \( U \in \tau \). By Theorem 1.2,

\[
U \cap A \subset U \cap Int(\Psi(A)) = U \cap Int(A \cup \Phi(A)) = Int(U \cap (A \cup \Phi(A))) = Int(U \cap A \cup (U \cap \Phi(A))) = Int(U \cap A \cup (U \cap \Phi(U \cap A))) \subset Int((U \cap A) \cup \Phi(U \cap A)) = Int(\Psi(U \cap A)).
\]

**Definition 2.4.** Let \((X, \tau, G)\) be a topological space and \( G \) a grill on \( X \). A subset \( A \) in \( X \) is said to be G-set (resp. \( \Phi \)-set) if there is a \( U \in \tau \) and a g-set (resp. \( \Phi \)-set) \( A \) in \((X, \tau, G)\) such that \( H = U \cap A \), respectively.

**Proposition 2.9.** i) A g-set \( A \) is a G-set.

ii) A g\( \Phi \)-set \( A \) is a G\( \Phi \)-set.

*Proof.* Obvious.

**Proposition 2.10.** An open set \( U \) is a G-set (resp. \( \Phi \)-set).

*Proof.* \( U = U \cap X \), \( Int(\Psi(X)) = Int(X) \).

**Proposition 2.11.** A \( \tau_G \)-closed set \( C \) is a G-set

*Proof.* It follows from Proposition 2.1 and Proposition 2.9.
Proposition 2.12. i) A $B$-set is a $G$-set.

ii) A $G$-set is a $G\Phi$-set.

Proof. i) Let $H$ be a $B$-set. Then $H = U \cap A$, where $U \in \tau$ and $A$ is a $t$-set. $H = U \cap \text{Int}(A) = U \cap \text{Int}(\text{Cl}(A)) = U \cap \text{Int}(A \cup \text{Cl}(A)) \supset U \cap \text{Int}(A \cup \Phi(A)) = U \cap \text{Int}(\Psi(A)) \supset U \cap \text{Int}(A) = H$. Therefore $H$ is a $G$-set.

ii) Similar to i).

The converse of Proposition 2.12 is false as it is shown by the following example.

Example 2.4. In Example 2.2 $A = \{a, c\}$ is a $G$-set and also a $G\Phi$-set, but it is not $B$-set. In Example 2.1, $A = \{a, b, d\}$ is a $G$-set, but it is not $G\Phi$-set.

Proposition 2.13. A subset $S$ in a space $(X, \tau, G)$ is open if and only if it is a $G$-preopen and a $G$-set.

Proof. Necessity. It follows from Proposition 2.10 and the obvious fact that every open set is G-preopen.

Sufficiency. Since $S$ is a $G$-set, then $S = U \cap A$ where $U$ is an open set and $\text{Int}(\Psi(A)) = \text{Int}(A)$. Since $S$ is also $G$-preopen, we have

$$S \subset \text{Int}(\Psi(S)) = \text{Int}(\Psi(U \cap A)) = \text{Int}(\Psi(U) \cap \Psi(A)) \subset \text{Int}(\Psi(U)) \cap \text{Int}(\Psi(A)) = \text{Int}(U \cap \Phi(U)) \cap \text{Int}(\Psi(A)) \subset \text{Int}(\text{Cl}(U)) \cap \text{Int}(\Psi(A)) = \text{Int}(\text{Cl}(U)) \cap \text{Int}(A).$$

Hence

$$S = U \cap A = (U \cap A) \cap U \subset (\text{Int}(\text{Cl}(U)) \cap \text{Int}(A)) \cap U$$

$$= (\text{Int}(\text{Cl}(U)) \cap U) \cap \text{Int}(A) = U \cap \text{Int}(A).$$

Therefore, $S = U \cap A \subset U \cap \text{Int}(A)$ and $S = U \cap \text{Int}(A)$. Thus $S$ is an open set.

Corollary 2.1. If $S$ is both $G\Phi$-set and $\Phi$-open set in $(X, \tau, G)$, then $S$ is open.

Definition 2.5. Let $(X, \tau, G)$ be a grill space and $A \subset X$. A set $A$ is said to be $G$-dense in $X$, if $\Psi(A) = X$.

Proposition 2.14. A subset $A$ of a grill $G$ in a space $(X, \tau, G)$ is $G$-dense if and only if for every open set $U$ containing $x \in X$, $A \cap U \in G$.

Proof. Necessity. Let $A$ be a $G$-dense set. Then, for every open set $U$ containing $x$ in a space $X$, $x \in \Psi(A) = A \cup \Phi(A)$. Hence if $x \in A$, then $A \cap U \in G$ and if $x \in \Phi(A)$, then $A \cap U \in G$. 

Sufficiency. Let every \( x \in X \). Moreover, let every open subset \( U \) of \( X \) containing \( x \) such that \( A \cap U \in G \). Then if \( x \in A \) or \( x \in \Phi(A) \), we have \( A \cap U \in G \). It follows that \( x \in \Psi(A) \) and thus \( X \subset \Psi(A) \). Therefore \( \Psi(A) = X \).

\[ \square \]

**Proposition 2.15.** If \( U \) is an open set and \( A \) is a \( G \)-dense set in \((X, \tau, G)\), then \( \Psi(U) = \Psi(U \cap A) \).

**Proof.** Since \( A \cap U \subset U \), we have \( \Psi(U \cap A) \subset \Psi(U) \). Conversely, if \( x \in \Psi(U) \), \( x \in U \) and \( x \in \Phi(U) \). Then for every open set \( V \) containing \( x \), \( U \cap V \in G \). Put \( W = U \cap V \in \tau(x) \). Since \( \Psi(A) = X \), \( W \cap A \in G \), i.e. \( W = (U \cap A) \cap V \in G \). Therefore, \( x \in \Psi(U \cap A) \) and \( \Psi(U) = \Psi(U \cap A) \).

\[ \square \]

**Proposition 2.16.** For any subset \( A \) of a space \((X, \tau, G)\), the following are equivalent:

1. \( A \) is \( G \)-preopen;
2. there is a \( G \)-regular open set \( U \) of \( X \) such that \( A \subset U \) and \( \Psi(A) = \Psi(U) \);
3. \( A \) is the intersection of \( G \)-regular open set and a \( G \)-dense set;
4. \( A \) is the intersection of an open set and a \( G \)-dense set.

**Proof.** (1) \( \Rightarrow \) (2): Let \( A \) be \( G \)-preopen in \((X, \tau, G)\), i.e. \( A \subset Int(\Psi(A)) \). Let \( U = Int(\Psi(A)) \). Then \( U \) is \( G \)-regular open such that \( A \subset U \) and \( \Psi(A) \subset \Psi(U) = \Psi(\Phi(A)) \subset \Psi(\Psi(A)) = \Psi(A) \). Hence \( \Psi(A) = \Psi(U) \).

(2) \( \Rightarrow \) (3): Suppose (2) holds. Let \( D = A \cup (X - U) \). Then \( D \) is a \( G \)-dense set. In fact \( \Psi(D) = \Psi(A \cup (X - U)) = \Psi(A) \cup \Psi(X - U) = \Psi(U) \cup \Psi(X - U) = \Psi(U \cup (X - U)) = \Psi(X) = X \). Therefore, \( A = D \cap G \), \( D \) is a \( G \)-dense set and \( U \) is a \( G \)-regular open set.

(3) \( \Rightarrow \) (4): Every \( G \)-regular open set is open.

(4) \( \Rightarrow \) (1): Suppose \( A = U \cap D \) with \( U \) and \( D \) \( G \)-dense. Then \( \Psi(A) = \Psi(U) \) since \( A = U \cap D \), \( \Psi(A) = \Psi(U \cap D) = \Psi(U) \). Hence \( A \subset U \subset \Psi(U) = \Psi(A) \), that is, \( A \subset Int(\Psi(A)) \).

\[ \square \]

**Proposition 2.17.** If \( A \) is both regular open and \( G \)-preopen set in \((X, \tau, G)\), then it is \( G \)-regular open.

**Proof.** \( A \subset Int(\Psi(A)) = Int(A \cup \Phi(A)) \subset Int(Cl(A)) = A \).

\[ \square \]

**Remark 2.4.** It should be noted that open sets and \( g \)-sets are independent and regular open sets and \( G \)-regular open sets are also independent. Every \( G \)-regular open set is open. Regular openness implies openness and \( G \)-regular open sets imply \( g \)-sets.
3. Decomposition of continuity

**Definition 3.1.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( B \)-continuous [4] if for each open set \( V \) in \( Y \), \( f^{-1}(V) \) is a \( B \)-set in \( X \).

**Definition 3.2.** A function \( f : (X, \tau, G) \to (Y, \sigma) \) is said to be \( G \)-continuous (resp. \( G \Phi \)-continuous, \( \Phi \)-continuous, \( G \)-precontinuous) if for each open set \( V \) in \( Y \), \( f^{-1}(V) \) is a \( G \)-set (resp. \( G \Phi \)-set, \( \Phi \)-open, \( G \)-preopen) in \( (X, \tau, G) \), respectively.

**Proposition 3.1.** i) A \( B \)-continuous function is \( G \)-continuous.

ii) A \( G \)-continuous function is \( G \Phi \)-continuous.

**Example 3.1.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \). If \( G = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \), then \( G \) is a grill on \( X \) such that \( \tau - \{\emptyset\} \subseteq G \) [2]. Let \( Y = \{a, b\} \) with topology \( \sigma = \{\emptyset, Y, \{a\}\} \). Define a function \( f(a) = f(c) = a \) and \( f(b) = b \). Then \( f \) is \( G \)-continuous, but it is neither \( B \)-continuous nor \( G \)-precontinuous.

**Remark 3.1.** \( G \)-precontinuous and \( G \)-continuous are independent from each other as in the following example:

**Example 3.2.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \). If \( G = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \), then \( G \) is a grill on \( X \) such that \( \tau - \{\emptyset\} \subseteq G \) [2]. Let \( Y = \{a, b\} \) with topology \( \sigma = \{\emptyset, Y, \{a\}\} \). Define a function \( f(a) = f(b) = a \) and \( f(c) = b \). Then \( f \) is \( G \)-precontinuous, but it is not \( G \)-continuous. In Example 3.1, \( f \) is \( G \)-continuous, but it is not \( G \)-precontinuous.

We have the following decomposition of continuity inspired by Proposition 2.13.

**Proposition 3.2.** A function \( f : (X, \tau, G) \to (Y, \sigma) \) is continuous if and only if it is both \( G \)-precontinuous and \( G \)-continuous.

**Proof.** It follows from Proposition 2.13.

**Proposition 3.3.** If a function \( f : (X, \tau, G) \to (Y, \sigma) \) is both \( \Phi \)-continuous and \( G \Phi \)-continuous, then \( f \) is continuous.

**Proof.** It follows from Corollary 2.1.

REFERENCES


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