Proof that there are no odd perfect numbers

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Abstract

If \( y \) is an odd perfect number, let \( p \) be one of the prime factors of \( y \), the exponent of \( p \) be an integer \( n \) (\( n \geq 1 \)), the prime factors other than \( p \) and different from each other be \( p_1, p_2, \ldots, p_r \) and the even exponent of \( p_k \) be \( q_k \).

\[
y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} \left(1 + p_k + p_k^2 + \cdots + p_k^{q_k}\right)/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}
\]

must be satisfied. Let \( m \) be a non negative integer and \( q \) be a positive integer,

\[
n = 4m + 1
\]

\[
p = 4q + 1
\]

Let \( a \) and \( b \) be odd integers, satisfying following expressions,

\[
a = \prod_{k=1}^{r} \left(1 + p_k + p_k^2 + \cdots + p_k^{q_k}\right)
\]

\[
b = \prod_{k=1}^{r} p_k^{q_k}
\]

\[
a(p^n + \cdots + 1) - 2bp^n = 0
\]

is established. This is a known content that has been proven by Euler. Let \( v \) be a rational number,

\[
v = a/b
\]

holds. By the consideration of this research paper, because we proved that when \( v \) is integer or not, there are no solutions that satisfies the equation \( a(p^n + \cdots + 1) - 2bp^n = 0 \) other than inappropriate solution \((a, b, p, n) = (1, 1, 1, 1)\), we have obtained a conclusion that there are no odd perfect numbers.

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1. Introduction
The perfect number is one in which the sum of the divisors other than itself is the same value as itself and the smallest perfect number is
\[1 + 2 + 3 = 6\]
It is 6. Whether odd perfect numbers exist or not is currently an unsolved problem in mathematics.

2. Proof
Let \(y\) be an odd perfect number, one of the prime factors of \(y\) be an odd prime \(p\) and an exponent of \(p\) be an integer \(n (n \geq 1)\). Let the prime factors other than \(p\) and different from each other be \(p_1, p_2, \cdots, p_r\), \(q_k\) be the index of \(p_k\) and an integer \(a\) be the product of series of prime numbers other than prime \(p\).
\[
a = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \ldots ①
\]
The number of terms \(N\) of variable \(a\) is
\[
N = \prod_{k=1}^{r} (q_k + 1) \ldots ②
\]
When \(y\) is a perfect number,
\[
y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)
\]
is established.
\[
a \sum_{k=0}^{n} p^k / 2 = y
\]
\[
a \sum_{k=0}^{n} p^k / (2p^n) = y/p^n \ldots ③
\]

2.1. When \(q_k\) has at least one odd integer
Let the number of terms where \(q_k\) is odd be a positive integer \(u\). Because \(y/p^n = \prod_{k=1}^{r} p_k^{q_k}\) is odd and the denominator on the left side of the expression ③ has a prime factor 2, from the expression ② variable \(a\) has prime factor 2 more than \(u\) and variable \(a\) is even. Therefore, \(n\) is even and \(u\) is 1 since \(\sum_{k=0}^{n} p^k\) must be an odd integer.
2.2. When all \( q_k \) are even integers

\( y/p^n \) is odd and the denominator on the left side of the expression ③ is even. Since \( N \) is odd when \( q_k \) are all even integers, variable \( a \) is odd. Therefore, \( \sum_{k=0}^{n} p^k \equiv 0 \ (mod \ 2) \) is established and \( n \) must be an odd integer since \( \sum_{k=0}^{n} p^k \) is necessary to include one prime factor 2.

From 2.1, 2.2, in order for odd perfect numbers to exist, only one exponent of the prime factor of \( y \) must be an odd integer. We consider the case of 2.2 below.

In order for \( y \) to be an odd perfect number, the following expression must be established.

\[
y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} \left(1 + p_k + p_k^2 + \cdots + p_k^{q_k}\right) / (2p^n) = \prod_{k=1}^{r} p_k^{q_k}
\]

However, \( q_1, q_2, \ldots, q_r \) are all even integers.

Here, let \( b \) be an odd integer.

\[
b = \prod_{k=1}^{r} p_k^{q_k} \ldots ④
\]

A following expression is established.

\[
y/p^n = a(1 + p + p^2 + \cdots + p^n) / (2p^n) = b
\]

\[
a(p^{n+1} - 1) / (2(p - 1)p^n) = b
\]

\[
(a - 2b)p^{n+1} + 2bp^n - a = 0 \ldots ⑤
\]

\[
(ap - 2bp + 2b)p^n = a
\]

Since \( ap - 2bp + 2b \) is odd, \( a/p^n \) is odd. Let \( a/p^n \) be an odd integer \( c \).

\[
ap - 2bp + 2b = c \ (c > 0) \ldots ⑥
\]

\[
(2b - a)p = 2b - c
\]

Since variable \( a \) is odd, \( 2b - a \) is odd and \( 2b - a \neq 0 \).

\[
p = (2b - c) / (2b - a)
\]
Since $n \geq 1$,
\[
a - c = cp^n - c \geq cp - c > 0
\]
a > c
is.

From the equation ⑥,
\[
2b(p - 1) - (ap - c) = 0
\]
\[
2b - c(p^{n+1} - 1)/(p - 1) = 0
\]
Since $(p^n + \cdots + 1)/2$ is odd, $n = 4m + 1$ must be hold with $m$ as an integer.
\[
2b(p - 1) = c(p^{n+1} - 1)
\]
\[
2b = c(p^n + \cdots + 1)
\]
\[
2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) \ldots ⑦
\]
Since $b$ is odd when $p + 1$ is not a multiple of 4, $p - 1$ must be a multiple of 4. A positive integer is taken as $q$.
\[
p = 4q + 1
\]
is established. The above conditions have been proved by Euler.

When $p > 1$,
\[
p^n - 1 < p^n
\]
\[
(p^n - 1)/(p - 1) < p^n/(p - 1)
\]
\[
p^{n-1} + \cdots + 1 < p^n/(p - 1)
\]
Since $p$ is an odd prime number satisfying $p = 4q + 1$ and $p \geq 5$,
\[
p^{n-1} + \cdots + 1 < p^n/4
\]
\[
2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)
\]
\[
2b - a < cp^n/4 = a/4
\]
\[
2b < 5a/4
\]
\[
a > 8b/5 \ldots ⑧
Let $a_k$ and $b_k$ be odd integers and if

$$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}, \quad b_k = p_k^{q_k}$$

$$a_k - b_k < b_k/(p_k - 1)$$

$$a_k < b_k p_k/(p_k - 1)$$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑧.

From the expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of $b$, let $d_k$ be the index.

$$(p + 1)/2 = \prod_{k=1}^{r} p_k^{d_k}$$

$p = 2 \prod_{k=1}^{r} p_k^{d_k} - 1 \cdots \text{⑪}$$

From $a = cp^n$ and the expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1 \cdots (A)$

When $r = 1$,

$a = (p_1^{q_1+1} - 1)/(p_1 - 1)$

$b = p_1^{q_1}$

The equation (A) does not hold since there are no odd perfect numbers when $r = 1$. 
Let $A_k$ and $B_k$ to be odd integers.

\[ A_k = (p_k^{q_k+1} - 1)/(p_k - 1) \]

\[ B_k = p_k^{q_k} \]

$a = \prod_{k=1}^{r} A_k$ and $b = \prod_{k=1}^{r} B_k$ hold. When the equation (A) holds,

\[ A_1 A_2 \ldots A_r/(B_1 B_2 \ldots B_r) = 2p^n/(p^n + \cdots + 1) \ldots (B) \]

is established.

Assume that $a/b$ on the left of the equation (B) is not an integer.

\[ 2B_1 B_2 \ldots B_r = c(p_x^{n_x} + \cdots + 1) \]

holds. $p_x^{n_x} + \cdots + 1$ is a multiple of the prime factors $p_1$ to $p_r$. If $p_x^{n_x} + \cdots + 1 = B_1 B_2 \ldots B_r$ holds, a contradiction arises since $c = 2$ and $c$ is even. If $p_x^{n_x} + \cdots + 1 = 2B_1 B_2 \ldots B_r$, $c = 1$ holds. From the above, when $a/b$ is not an integer and $c > 1$ holds, at least one of the prime factors $p_1$ to $p_r$ of the denominator on the left side remains in the denominator on the left side after transposition.

The product of the prime factors remaining in the denominator on the left side among the prime factors $p_1$ to $p_r$ of the denominator on the left side of the equation (B) is defined as an odd integer $P$. If the numerator of the left side is a multiple of $P$, it becomes inconsistent since the left side is an integer and the right side is not an integer. Thereby, when the left side is reduced, at least one of the prime factors $p_s$ of $P$ remains in the denominator. At this time, it becomes a contradiction since $p_s$ does not exist in the denominator of the right side. From the above, the equation (A) does not hold when $c > 1$.

When $c = 1$, $a_k$ is a power of $p$ since $a = p^n$. Let $s_k$ be an integer,

\[ a_k = p^{s_k} \]

is established. From this expression,

\[ p_1^{q_1+1} = p^{s_1}(p_1 - 1) + 1 \]

\[ p_2^{q_2+1} = p^{s_2}(p_2 - 1) + 1 \]

\[ \ldots \]

\[ p_r^{q_r+1} = p^{s_r}(p_r - 1) + 1 \]
When the both sides are multiplied,
\[ b \prod_{k=1}^{r} p_k = \prod_{k=1}^{r} (p^{s_k}(p_k - 1) + 1) \]
\[ (p^n + \cdots + 1) \prod_{k=1}^{r} p_k = 2 \prod_{k=1}^{r} (p^{s_k}(p_k - 1) + 1) \]
\[ \prod_{k=1}^{r} p_k \equiv 2 \pmod{p} \]

Let \( t \) be a positive odd integer,
\[ \prod_{k=1}^{r} p_k = tp + 2 \]

Let \( w \) be a positive odd integer,
\[ w = \prod_{k=1}^{r} p_k \]
\[ \prod_{k=1, d_k > 0}^{r} p_k = (tp + 2)/w \]
From the expression (9),
\[ (p + 1)/2 = (tp + 2)/w \times \prod_{k=1}^{r} p_k^{d_k-1} \quad \text{(C)} \]
\[ w(p + 1) = 2(tp + 2) \prod_{k=1}^{r} p_k^{d_k-1} \]
\[ 4 \prod_{k=1}^{r} p_k^{d_k-1} \equiv w \pmod{p} \]

Let \( x \) be an odd integer,
\[ 4 \prod_{k=1}^{r} p_k^{d_k-1} = xp + w \quad \text{(D)} \]
From the expression (C),
\[ (p + 1)/2 = (tp + 2)/w \times (xp + w)/4 \]
\[ txp = -(2x + (t - 2)w) \]
\[
\prod_{k=1}^{r} p_k = t \left( 2 \prod_{k=1}^{r} p_k^{d_k} - 1 \right) + 2
\]

\(-t + 2 \equiv 0 \pmod{p_k, d_k > 0}\)

t > 3 \text{ holds since } t \text{ is a positive odd integer.}

If \(2x + (t - 2)w < 0\) holds, \(x > 0\) must hold. However, it becomes a contradiction since \(2x + (t - 2)w > 0\) holds when \(x > 0\). Therefore, \(x < 0\) and \(2x + (t - 2)w > 0\)

\(2 > -(t - 2)w/x\)

must be satisfied.

From the expression (D),

\[
4 \prod_{k=1}^{r} p_k^{d_k-1} = xp + \prod_{k=1, d_k=0}^{r} p_k
\]

\[
4 \prod_{k=1}^{r} p_k^{d_k} = xp \prod_{k=1, d_k>0}^{r} p_k + \prod_{k=1}^{r} p_k
\]

\[
2(p + 1) = xp \prod_{k=1, d_k>0}^{r} p_k + tp + 2
\]

\[
xp \prod_{k=1, d_k>0}^{r} p_k = -(t - 2)p
\]

\[
\prod_{k=1, d_k>0}^{r} p_k = -(t - 2)/x
\]

By the inequality described above,

\[
w \prod_{k=1, d_k>0}^{r} p_k < 2
\]

is established. However, a contradiction arises since this expression obviously does not hold. From the above, all points \((a, b, p, n)\) where \(a/b\) is not an integer cannot be solutions of the equation (A).
(a, b, p, n) = (1, 1, 1, 1) is a solution of the equation (A).
y = bp^n = 1
Therefore, this is not an appropriate solution.

If $a/b$ is an integer, when the equation (A) is satisfied at points other than the point
(1, 1, 1, 1), let $v$ be an integer,

$v = a/b = 2p^n/(p^n + \cdots + 1)$

$2p^n = v(p^n + \cdots + 1)$

Let $z$ be an integer and if $v = zp^n$ holds,

$2 = z(p^n + \cdots + 1)$

When $p \equiv 1 \pmod{4}$, $p \geq 5$ and $n \equiv 1 \pmod{4}$, $n \geq 1$,

$p^n + \cdots + 1 \geq 6$

At this time, it is not appropriate since $z$ does not become an integer. Therefore, if $v$
is an integer, except for (1, 1, 1, 1) there are no solutions satisfying the equation (A).
From the above, there are no odd perfect numbers.
3. Complement

From the equation (5),
\[ 2b^n(p - 1) = a(p^{n+1} - 1) \]
\[ 2 = a(p^{n+1} - 1)/(bp^n(p - 1)) \]
\[ 2 = (p_1 q_{1+1} - 1)(p_2 q_{2+1} - 1) \cdots (p_r q_{r+1} - 1)(p^{n+1} - 1) \]
\[ / (p_1 q_1 p_2 q_2 \cdots p_r q_r p^n(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)(p - 1)) \]
\[ 2(p_{q_{1+1}} - p_1 q_{1})(p_{q_{2+1}} - p_2 q_{2}) \cdots (p_{q_{r+1}} - p_r q_{r})(p^{n+1} - p^n) \]
\[ = (p_1 q_{1+1} - 1)(p_2 q_{2+1} - 1) \cdots (p_r q_{r+1} - 1)(p^{n+1} - 1) \]

We consider when \( r = 2 \).

\[ (p_1 q_{1+1} - 1)(p_2 q_{2+1} - 1)(p^{n+1} - 1) = 2(p_1 q_{1+1} - 1)(p_2 q_{2+1} - 2)(p^{n+1} - p^n) \]

Let \( s, t, u \) be integers,
\[ s = p_1 q_{1+1} - 1 \]
\[ t = p_2 q_{2+1} - 1 \]
\[ u = p^{n+1} - 1 \]
are.

\[ stu = 2(p_1 q_{1+1} - 1 - (p_1 q_1 - 1))(p_2 q_{2+1} - 1 - (p_2 q_2 - 1))(p^{n+1} - 1 - (p^n - 1)) \]
\[ stu = 2(s - (s + 1)/p_1 + 1)(t - (t + 1)/p_2 + 1)(u - (u + 1)/p + 1) \]
\[ pp_1 p_2 stu = 2((s + 1)p_1 - (s + 1))(t + 1)p_2 + (t + 1)((u + 1)p + (u + 1)) \]
\[ pp_1 p_2 stu = 2(s + 1)(p_1 - 1)(t + 1)(p_2 - 1)(u + 1)(p - 1) \]
\[ stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \]

Since \( stu/((s + 1)(t + 1)(u + 1)) \) is a monotonically increasing function for variables \( s, t \) and \( u \), if
\[ s \geq 3^{2+1} - 1 = 26, \ p_1 = 3, \ q_1 = 2 \]
\[ t \geq 7^{2+1} - 1 = 342, \ p_2 = 7, \ q_2 = 2 \]
\[ u \geq 5^2 - 1 = 24, \ p = 5, \ n = 1 \]
holds,
\[ stu/((s + 1)(t + 1)(u + 1)) \geq 26 \times 342 \times 24/(27 \times 343 \times 25) = 7904/8575 \]
\[ 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35 \]

Since \( stu/((s + 1)(t + 1)(u + 1)) \) is limited to 1 when \( s, t \) and \( u \) are infinite,
\[ stu/((s + 1)(t + 1)(u + 1)) < 1 \]
If \( f(p_1,p_2,p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \) holds, it is sufficient to consider a combination where \( f(p_1,p_2,p) < 1 \).

\[
\begin{align*}
  f(3,7,5) &= 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32/35 \\
  f(3,11,5) &= 2 \times 2 \times 10 \times 4 / (3 \times 11 \times 5) = 32/33 \\
  f(3,13,5) &= 2 \times 2 \times 12 \times 4 / (3 \times 13 \times 5) = 64/65 \\
  f(3,17,5) &= 2 \times 2 \times 16 \times 4 / (3 \times 17 \times 5) = 256/255 \\
  f(3,7,13) &= 2 \times 2 \times 6 \times 12 / (3 \times 7 \times 13) = 96/91 \\
  f(3,5,17) &= 2 \times 2 \times 4 \times 16 / (3 \times 5 \times 17) = 256/255
\end{align*}
\]

From the above, when \( r = 2 \), combinations \((p_1,p_2,p) = (3,7,5),(3,11,5),(3,13,5)\) can be considered.

Let \( q_k \) be 2 and \( n = 1 \), if \( g(p_1,p_2,p) = (p_1^3 - 1)(p_2^3 - 1)(p^2 - 1)/(p_1^3p_2^3p^2) \),
\[
\begin{align*}
  g(3,7,5) &= 26 \times 342 \times 24 / (3^3 7^3 5^2) = 7904/8575 > 32/35 \\
  g(3,11,5) &= 26 \times 1330 \times 24 / (3^3 11^3 5^2) = 55328/59895 \\
  g(3,13,5) &= 26 \times 2196 \times 24 / (3^3 13^3 5^2) = 3904/4225
\end{align*}
\]

Since the function \( g \) is the minimum in the case of \( q_k = 2 \) and \( n = 1 \), there is no solution \( q_k \) and \( n \) when \( g > f \), so the case of \((p_1,p_2,p) = (3,7,5)\) becomes unsuitable.

\[
\begin{align*}
  \text{stu}/((s + 1)(t + 1)(u + 1)) &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \\
  (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1)/(p_1^{q_1+1}p_2^{q_2+1}p^{n+1}) \\
  &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p)
\end{align*}
\]

If \( F(p_1,p_2,p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \),
\[
F(p_1^{q_1+1},p_2^{q_2+1},p^{n+1}) = 2F(p_1,p_2,p)
\]
4. Acknowledgement

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5. References

Hiroyuki Kojima "The world is made of prime numbers" Kadokawa Shoten, 2017
Fumio Sairaiji-Kenichi Shimizu "A story that prime is playing" Kodansha, 2015