There is a correspondence between the positive solutions of a diophantine equation and the divisors of natural numbers

Premise
All this research has been done using Mathematica®, a symbolic computation environment that uses a programming language called Wolfram Language.

I investigated the positive solutions of Diophantine equation \( x + y + x \cdot y + x^2 = n \) where \( n \) is a natural number for convenience starting from 2. In terms of Mathematica all this can be expressed as:

\[
\text{Reduce}\left[ x + y + x \cdot y + y^2 \overset{\text{\(\equiv\)}}{=} n \land x > 0 \land y > 0, \{x, y\}, \text{Integers} \right]
\]

Reduce is the function used to solve diophantine equations and has three arguments:
- the set of conditions that must be satisfied by the variables joined together by the symbol \( \land \) which stands for ‘and’
- the set of variable names (in this case \( x \) and \( y \))
- the domain of the values (in this case the domain of integers)

To complete the informations needed to understand the following we have another functions:

\[
\text{Divisors}[n]
\]

This function given a natural \( n \) returns the list expressed as a comma separated sequence of values between braces that starts from 1 and goes to \( n \). Example:

\[
\text{In[\cdot\cdot]} := \text{Divisors}[30] \\
\text{Out[\cdot\cdot]} := \{1, 2, 3, 5, 6, 10, 15, 30\}
\]

Now lets try to calculate the solutions of the above equation replacing \( n \) with 30:
What we get is a list of solutions, each enclosed in round brackets and separated by symbol || which stands for ‘or’ followed by the list of divisors of 30. The interesting thing is that if you calculate in each solution x+y you get the same values in the second half of divisors list except the last number.

Following this concept what is happen when we use for n a prime number? A prime number p has the divisors list like {1,p} and hence the second half except last corresponds to empty solutions list:

We have used above number 30 which has an even number of divisors. But what happens with a number with odd number of divisors:

We don’t start from the second half but from the number in the middle of the sequence.

Now we start with n=4 to check what we have verified. Obviously this is true for n<4 because we have four terms in the expression (x+y+x*y+y^2) and x>0 and y>0 so each term will be at last 1.
In[7]:= Reduce[x+y+x*y+y^2==6&&x>0&&y>0,{x,y},Integers]
Divisors[6]

Out[7]= x = 2 & y = 1
Out[7]= {1, 2, 3, 6}

In[8]:= Reduce[x+y+x*y+y^2==7&&x>0&&y>0,{x,y},Integers]
Divisors[7]

Out[8]= False
Out[8]= {1, 7}

In[9]:= Reduce[x+y+x*y+y^2==8&&x>0&&y>0,{x,y},Integers]
Divisors[8]

Out[9]= x = 3 & y = 1
Out[9]= {1, 2, 4, 8}

In[10]:= Reduce[x+y+x*y+y^2==9&&x>0&&y>0,{x,y},Integers]
Divisors[9]

Out[10]= x = 1 & y = 2
Out[10]= {1, 3, 9}

In[11]:= Reduce[x+y+x*y+y^2==10&&x>0&&y>0,{x,y},Integers]
Divisors[10]

Out[11]= x = 4 & y = 1
Out[11]= {1, 2, 5, 10}

In[12]:= Reduce[x+y+x*y+y^2==11&&x>0&&y>0,{x,y},Integers]
Divisors[11]

Out[12]= False
Out[12]= {1, 11}

In[13]:= Reduce[x+y+x*y+y^2==12&&x>0&&y>0,{x,y},Integers]
Divisors[12]

Out[13]= (x = 2 && y = 2) || (x = 5 && y = 1)
Out[13]= {1, 2, 3, 4, 6, 12}

In[14]:= Reduce[x+y+x*y+y^2==13&&x>0&&y>0,{x,y},Integers]
Divisors[13]

Out[14]= False
Out[14]= {1, 13}

In[15]:= Reduce[x+y+x*y+y^2==14&&x>0&&y>0,{x,y},Integers]
Divisors[14]

Out[15]= x = 6 & y = 1
Out[15]= {1, 2, 7, 14}
\( \text{Reduce}[x+y+x*y+y^2\equiv 15&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Divisors}[15]\)

\(\text{Out} = x = 3 \&\& y = 2\)

\(\text{Out} = \{3, 5, 15\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 16&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Divisors}[16]\)

\(\text{Out} = (x = 1 \&\& y = 3) \text{ || } (x = 7 \&\& y = 1)\)

\(\text{Out} = \{2, 4, 8, 16\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 17&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Out} = \text{False}\)

\(\text{Out} = \{1, 17\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 18&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Divisors}[18]\)

\(\text{Out} = (x = 4 \&\& y = 2) \text{ || } (x = 8 \&\& y = 1)\)

\(\text{Out} = \{2, 3, 6, 9, 18\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 19&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Out} = \text{False}\)

\(\text{Out} = \{1, 19\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 20&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Divisors}[20]\)

\(\text{Out} = (x = 2 \&\& y = 3) \text{ || } (x = 9 \&\& y = 1)\)

\(\text{Out} = \{2, 4, 5, 10, 20\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 21&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Divisors}[21]\)

\(\text{Out} = x = 5 \&\& y = 2\)

\(\text{Out} = \{1, 3, 7, 21\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 22&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Divisors}[22]\)

\(\text{Out} = x = 10 \&\& y = 1\)

\(\text{Out} = \{1, 2, 11, 22\}\)

\( \text{Reduce}[x+y+x*y+y^2\equiv 23&&x>0&&y>0, \{x,y\}, \text{Integers}] \)

\(\text{Out} = \text{False}\)

\(\text{Out} = \{1, 23\}\)
Main Results

We have found a way to identify prime numbers by an equation. A number p is prime iff there are no positive solutions to diophantine equation \( x+y+x\cdot y+x^2=p \)

For the non prime numbers all the proper divisors are identified; let \((x,y)\) a solution of diophantine equation \( x+y+x\cdot y+x^2=n \) then \( x+y \) is a divisor of \( n \).