There is a correspondence between the positive solutions of a diophantine equation and the divisors of natural numbers

Premise
All this research has been done using Mathematica®, a symbolic computation environment that uses a programming language called Wolfram Language.

I investigated the positive solutions of Diophantine equation \(x + y + xy + x^2 = n\) where \(n\) is a natural number for convenience starting from 2. In terms of Mathematica all this can be expressed as:

\[
\text{Reduce} \left[ x + y + xy + y^2 = n \land x > 0 \land y > 0, \{x, y\}, \text{Integers} \right]
\]

Reduce is the function used to solve diophantine equations and has three arguments:
- the set of conditions that must be satisfied by the variables joined together by the symbol \(\land\) which stands for ‘and’
- the set of variable names (in this case \(x\) and \(y\))
- the domain of the values (in this case the domain of integers)

To complete the informations needed to understand the following we have two other functions:

\text{Divisors}[n]

This function given a natural \(n\) returns the list expressed as a comma separated sequence of values between braces that starts from 1 and goes to \(n\). Example:

\[
\text{In}[1]:= \text{Divisors}[30]
\]
\[
\text{Out}[1]= \{1, 2, 3, 5, 6, 10, 15, 30\}
\]

\text{Ceiling}[r]

This function given a real number \(r\) returns the smallest integer greater than or equal to \(r\). Example:
In[1]:= Ceiling[3.141592]

Out[1]= 4

Now lets try to calculate the solutions of the above equation replacing n with 30:

In[2]:= Reduce[x + y + x*y + y^2 == 30 && x > 0 && y > 0, {x, y}, Integers]


Out[2]= (x == 2 && y == 4) || (x == 8 && y == 2) || (x == 14 && y == 1)

Out[2]= {1, 2, 3, 5, 6, 10, 15, 30}

What we get is a list of solutions, each enclosed in round brackets and separated by symbol || which stands for ‘or’ followed by the list of divisors of 30. The interesting thing is that if you calculate in each solution x+y you get the same values in the second half of divisors list except the last number.

Following this concept what is happen when we use for n a prime number? A prime number p has the divisors list like {1,p} and hence the second half except last corresponds to empty solutions list:

In[3]:= Reduce[x + y + x*y + y^2 == 31 && x > 0 && y > 0, {x, y}, Integers]


Out[3]= {1, 31}

Output False means that there are no solutions.

We have used above number 30 which has an even number of divisors. But what happens with a number with odd number of divisors:

In[4]:= Reduce[x + y + x*y + y^2 == 36 && x > 0 && y > 0, {x, y}, Integers]


Out[4]= (x == 1 && y == 5) || (x == 6 && y == 3) || (x == 10 && y == 2) || (x == 17 && y == 1)

Out[4]= {1, 2, 3, 4, 6, 9, 12, 18, 36}

We don’t start from the second half but from the number in the middle of the sequence.

Now we start with n=4 to check what we have verified. Obviously this is true for n<4 because we have four terms in the expression (x+y+x*y+y^2) and x>0 and y>0 so each term will be at last 1.

In[5]:= Reduce[x + y + x*y + y^2 == 4 && x > 0 && y > 0, {x, y}, Integers]


Out[5]= X = 1 && y = 1

Out[5]= {1, 2, 4}
In: Reduce\[x + y + x*y + y^2 \equiv 5 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[5]

Out: False

Out: \{1, 5\}

In: Reduce\[x + y + x*y + y^2 \equiv 6 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[6]

Out: \(x = 2 \&\& y = 1\)

Out: \{1, 2, 3, 6\}

In: Reduce\[x + y + x*y + y^2 \equiv 7 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[7]

Out: False

Out: \{1, 7\}

In: Reduce\[x + y + x*y + y^2 \equiv 8 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[8]

Out: \(x = 3 \&\& y = 1\)

Out: \{1, 2, 4, 8\}

In: Reduce\[x + y + x*y + y^2 \equiv 9 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[9]

Out: \(x = 1 \&\& y = 2\)

Out: \{1, 3, 9\}

In: Reduce\[x + y + x*y + y^2 \equiv 10 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[10]

Out: \(x = 4 \&\& y = 1\)

Out: \{1, 2, 5, 10\}

In: Reduce\[x + y + x*y + y^2 \equiv 11 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[11]

Out: False

Out: \{1, 11\}

In: Reduce\[x + y + x*y + y^2 \equiv 12 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[12]

Out: \((x = 2 \&\& y = 2) \| (x = 5 \&\& y = 1)\)

Out: \{1, 2, 3, 4, 6, 12\}

In: Reduce\[x + y + x*y + y^2 \equiv 13 \&\& x > 0 \&\& y > 0, \{x, y\}, \text{Integers}\] Divisors[13]

Out: False

Out: \{1, 13\}
In[1]:=

Reduce\left[x+y+xy+y^2=14\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[14]}

Out[1]= x = 6 \land y = 1

Out[1]= \{1, 2, 7, 14\}

In[1]:=

Reduce\left[x+y+xy+y^2=15\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[15]}

Out[1]= x = 3 \land y = 2

Out[1]= \{1, 3, 5, 15\}

In[1]:=

Reduce\left[x+y+xy+y^2=16\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[16]}

Out[1]= \text{False}

Out[1]= \{1, 16\}

In[1]:=

Reduce\left[x+y+xy+y^2=17\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[17]}

Out[1]= \text{False}

Out[1]= \{1, 17\}

In[1]:=

Reduce\left[x+y+xy+y^2=18\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[18]}

Out[1]= \text{False}

Out[1]= \{1, 18\}

In[1]:=

Reduce\left[x+y+xy+y^2=19\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[19]}

Out[1]= \text{False}

Out[1]= \{1, 19\}

In[1]:=

Reduce\left[x+y+xy+y^2=20\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[20]}

Out[1]= \text{False}

Out[1]= \{1, 20\}

In[1]:=

Reduce\left[x+y+xy+y^2=21\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[21]}

Out[1]= x = 5 \land y = 2

Out[1]= \{1, 5, 21\}

In[1]:=

Reduce\left[x+y+xy+y^2=22\land x>0\land y>0,\{x,y\}\right],\text{Integers}

\text{Divisors[22]}

Out[1]= x = 10 \land y = 1

Out[1]= \{1, 10, 22\}
Main Result

We have found a way to identify prime numbers by an equation. A number $p$ is prime iff there are no positive solutions to diophantine equation $x+y+x*y+x^2=p$.