ON A FINER TOPOLOGICAL SPACE THAN $\tau_\theta$
AND SOME MAPS

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Abstract

In 1943, Fomin [7] introduced the notion of $\theta$-continuity. In 1966, the notions of $\theta$-open subsets, $\theta$-closed subsets and $\theta$-closure were introduced by Veličko [18] for the purpose of studying the important class of $H$-closed spaces in terms of arbitrary filterbases. He also showed that the collection of $\theta$-open sets in a topological space $(X, \tau)$ forms a topology on $X$ denoted by $\tau_\theta$ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3] have also obtained several new and interesting results related to these sets.

In this paper, we will offer a finer topology on $X$ than $\tau_\theta$ by utilizing the new notions of $\omega_\theta$-open and $\omega_\theta$-closed sets. We will also discuss some of the fundamental properties of such sets and some related maps. Key words and phrases: Topological spaces, $\theta$-open sets, $\theta$-closed sets, $\omega_\theta$-open sets, $\omega_\theta$-closed sets, anti locally countable, $\omega_\theta$-continuity. 2000 Mathematics Subject Classification: 54B05, 54C08; Secondary: 54D05.

1 Introduction

In 1982, Hdeib [8] introduced the notion of $\omega$-closedness by which he introduced and investigated the notion of $\omega$-continuity. In 1943, Fomin [7] introduced the notion of $\theta$-continuity. In 1966, the notions of $\theta$-open subsets, $\theta$-closed subsets and $\theta$-closure were introduced by Veličko [18] for the purpose of studying the important class of $H$-closed spaces in terms of arbitrary filterbases. He also showed that the collection of $\theta$-open sets in a topological space $(X, \tau)$ forms a topology on $X$ denoted by $\tau_\theta$ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3]
have also obtained several new and interesting results related to these sets. In this paper, we will offer a finer topology on $X$ than $\tau_\theta$ by utilizing the new notions of $\omega_\theta$-open and $\omega_\theta$-closed sets. We will also discuss some of the fundamental properties of such sets and some related maps.

Throughout this paper, by a space we will always mean a topological space. For a subset $A$ of a space $X$, the closure and the interior of $A$ will be denoted by $cl(A)$ and $int(A)$, respectively. A subset $A$ of a space $X$ is said to be $\alpha$-open [14] (resp. preopen [13], regular open [17], regular closed [17]) if $A \subseteq int(cl(A))$ (resp. $A \subseteq int(cl(A))$, $A = int(cl(A))$, $A = cl(int(A))$)

A point $x \in X$ is said to be in the $\theta$-closure [18] of a subset $A$ of $X$, denoted by $\theta-cl(A)$, if $cl(U) \cap A \neq \emptyset$ for each open set $U$ of $X$ containing $x$. A subset $A$ of a space $X$ is called $\theta$-closed if $A = \theta-cl(A)$. The complement of a $\theta$-closed set is called $\theta$-open. The $\theta$-interior of a subset $A$ of $X$ is the union of all open sets of $X$ whose closures are contained in $A$ and is denoted by $\theta-int(A)$. Recall that a point $p$ is a condensation point of $A$ if every open set containing $p$ must contain uncountably many points of $A$. A subset $A$ of a space $X$ is $\omega$-closed [8] if it contains all of its condensation points. The complement of an $\omega$-closed subset is called $\omega$-open. It was shown that the collection of all $\omega$-open subsets forms a topology that is finer than the original topology on $X$. The union of all $\omega$-open sets of $X$ contained in a subset $A$ is called the $\omega$-interior of $A$ and is denoted by $\omega-int(A)$.

The family of all $\omega$-open (resp. $\theta$-open, $\alpha$-open) subsets of a space $(X, \tau)$ is denoted by $\omega O(X)$ (resp, $\tau_\theta = \theta O(X)$, $\alpha O(X)$).

A function $f : X \to Y$ is said to be $\omega$-continuous [9] (resp. $\theta$-continuous [7]) if $f^{-1}(V)$ is $\omega$-open (resp. $\theta$-open) in $X$ for every open subset $V$ of $Y$. A function $f : X \to Y$ is called weakly $\omega$-continuous [6] if for each $x \in X$ and each open subset $V$ in $Y$ containing $f(x)$, there exists an $\omega$-open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq cl(V)$.

2 A finer topology than $\tau_\theta$

Definition 1 A subset $A$ of a space $(X, \tau)$ is called $\omega_\theta$-open if for every $x \in A$, there exists an open subset $B \subseteq X$ containing $x$ such that $B \setminus \theta-int(A)$ is countable. The complement of an $\omega_\theta$-open subset is called $\omega_\theta$-closed.

The family of all $\omega_\theta$-open subsets of a space $(X, \tau)$ is denoted by $\omega_\theta O(X)$.

Theorem 2 $(X, \omega_\theta O(X))$ is a topological space for a topological space $(X, \tau)$.  

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Example 5 (1) Let \( \omega \) be a countable subset of a space \( X \) and \( x \in A \cap B \). There exists an open set \( U \supseteq X \) containing \( x \) such that \( U' \cap \text{cl}(A) \) and \( V' \cap \text{cl}(B) \) are countable. Then \( (U' \cap C) \cap \text{cl}(A) \cap \text{cl}(B) \) is countable. Hence, \( A \cap B \in \omega O(X) \). Let \( \{ A_i : i \in I \} \) be a family of \( \omega \)-open subsets of \( X \) and \( x \in \bigcup_{i \in I} A_i \). Then \( x \in A_j \) for some \( j \in I \). This implies that there exists an open subset \( B \) of \( X \) containing \( x \) such that \( B' \cap \text{cl}(A_j) \) is countable. Since \( B' \cap \text{cl}((\bigcup_{i \in I} A_i)) \supset \bigcup_{i \in I} B' \cap \text{cl}(A_i) \), then \( B' \cap \text{cl}(\bigcup_{i \in I} A_i) \) is countable. Hence, \( \bigcup_{i \in I} A_i \in \omega O(X) \).

Theorem 3 Let \( A \) be a subset of a space \( (X, \tau) \). Then \( A \) is \( \omega \)-open if and only if for every \( x \in A \), there exists an open subset \( U \) containing \( x \) and a countable subset \( V \) such that \( U \cap V \subseteq \text{cl}(A) \).

Proof. Let \( A \in \omega O(X) \) and \( x \in A \). Then there exists an open subset \( U \) containing \( x \) such that \( U' \subseteq \text{cl}(A) \) is countable. Take \( V = U' \cap \text{cl}(A) = U \cap (X' \cap \text{cl}(A)) \). Thus, \( U' \subseteq \text{cl}(A) \).

Conversely, let \( x \in A \). There exists an open subset \( U \) containing \( x \) and a countable subset \( V \) such that \( U' \subseteq \text{cl}(A) \). Hence, \( U' \subseteq \text{cl}(A) \) is countable.

Remark 4 The following diagram holds for a subset \( A \) of a space \( X \):

\[
\begin{array}{ccc}
\text{\( \omega \)-open} & \longrightarrow & \text{\( \omega \)-open} \\
\uparrow & & \uparrow \\
\text{\( \theta \)-open} & \longrightarrow & \text{open}
\end{array}
\]

The following examples show that these implications are not reversible.

Example 5 (2) Let \( R \) be the real line with the topology \( \tau = \{ \emptyset, R, R' \} \). Then the set \( R \setminus (0, 1) \) is open but it is not \( \omega \)-open.

Example 6 Let \( X = \{ a, b, c, d \} \) and \( \tau = \{ X, \emptyset, \{ a \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ a, b, c \}, \{ a, c, d \} \} \). Then the set \( A = \{ a, b, d \} \) is \( \omega \)-open but it is not open.

Example 7 Let \( A \) be an \( \omega \)-closed subset of a space \( X \). Then \( \text{cl}(A) \subseteq K \cup V \) for a closed subset \( K \) and a countable subset \( V \).
**Proof.** Since \( A \) is \( \omega_\theta \)-closed, then \( X \setminus A \) is \( \omega_\theta \)-open. For every \( x \in X \setminus A \), there exists an open set \( U \) containing \( x \) and a countable set \( V \) such that \( U \setminus V \subset \theta\text{-int}(X \setminus A) = X \setminus \theta\text{-cl}(A) \). Hence, \( \theta\text{-cl}(A) \subset X \setminus (U \setminus V) = X \cap ((X \setminus U) \cup V) = (X \setminus U) \cup V \). Take \( K = X \setminus U \). Thus, \( K \) is closed and \( \theta\text{-cl}(A) \subset K \cup V \).

**Definition 8** The intersection of all \( \omega_\theta \)-closed sets of \( X \) containing a subset \( A \) is called the \( \omega_\theta \)-closure of \( A \) and is denoted by \( \omega_\theta\text{-cl}(A) \).

**Lemma 9** Let \( A \) be a subset of a space \( X \). Then

1. \( \omega_\theta\text{-cl}(A) \) is \( \omega_\theta \)-closed in \( X \).
2. \( \omega_\theta\text{-cl}(X \setminus A) = X \setminus \omega_\theta\text{-int}(A) \).
3. \( x \in \omega_\theta\text{-cl}(A) \) if and only if \( A \cap G \neq \emptyset \) for each \( \omega_\theta \)-open set \( G \) containing \( x \).
4. \( A \) is \( \omega_\theta \)-closed in \( X \) if and only if \( A = \omega_\theta\text{-cl}(A) \).

**Definition 10** A subset \( A \) of a topological space \( (X, \tau) \) is said to be an \((\omega_\theta, \omega)\)-set if \( \omega_\theta\text{-int}(A) = \omega\text{-int}(A) \).

**Definition 11** A subset \( A \) of a topological space \( (X, \tau) \) is said to be an \((\omega_\theta, \theta)\)-set if \( \omega_\theta\text{-int}(A) = \theta\text{-int}(A) \).

**Remark 12** Every \( \omega_\theta \)-open set is an \((\omega_\theta, \omega)\)-set and every \( \theta \)-open set is an \((\omega_\theta, \theta)\)-set but not conversely.

**Example 13** (1) Let \( \mathbb{R} \) be the real line with the topology \( \tau = \{\emptyset, \mathbb{R}, Q'\} \) where \( Q' \) is the set of irrational numbers. Then the natural number set \( N \) is an \((\omega_\theta, \omega)\)-set but it is not \( \omega_\theta \)-open.

(2) Let \( \mathbb{R} \) be the real line with the topology \( \tau = \{\emptyset, \mathbb{R}, (2, 3)\} \). Then the set \( A = (1, \frac{5}{2}) \) is an \((\omega_\theta, \theta)\)-set but it is not \( \theta \)-open.

**Theorem 14** Let \( A \) be a subset of a space \( X \). Then \( A \) is \( \omega_\theta \)-open if and only if \( A \) is \( \omega \)-open and an \((\omega_\theta, \omega)\)-set.

**Proof.** Since every \( \omega_\theta \)-open is \( \omega \)-open and an \((\omega_\theta, \omega)\)-set, it is obvious.

Conversely, let \( A \) be an \( \omega \)-open and \((\omega_\theta, \omega)\)-set. Then \( A = \omega\text{-int}(A) = \omega_\theta\text{-int}(A) \). Thus, \( A \) is \( \omega_\theta \)-open.
Theorem 15 Let $A$ be a subset of a space $X$. Then $A$ is $\theta$-open if and only if $A$ is $\omega_\theta$-open and an $(\omega_\theta, \theta)$-set.

Proof. Necessity. It follows from the fact that every $\theta$-open set is $\omega_\theta$-open and an $(\omega_\theta, \theta)$-set.

Sufficiency. Let $A$ be an $\omega_\theta$-open and $(\omega_\theta, \theta)$-set. Then $A = \omega_\theta$-$\text{int}(A) = \theta$-$\text{int}(A)$. Thus, $A$ is $\theta$-open. □

Recall that a space $X$ is called locally countable if each $x \in X$ has a countable neighborhood.

Theorem 16 Let $(X, \tau)$ be a locally countable space and $A \subset X$.

(1) $\omega_\theta O(X)$ is the discrete topology.

(2) $A$ is $\omega_\theta$-open if and only if $A$ is $\omega$-open.

Proof. (1) : Let $A \subset X$ and $x \in A$. Then there exists a countable neighborhood $B$ of $x$ and there exists an open set $U$ containing $x$ such that $U \subset B$. We have $U \setminus \theta$-$\text{int}(A) \subset B \setminus \theta$-$\text{int}(A) \subset B$. Thus $U \setminus \theta$-$\text{int}(A)$ is countable and $A$ is $\omega_\theta$-open. Hence, $\omega_\theta O(X)$ is the discrete topology.

(2) : Necessity. It follows from the fact that every $\omega_\theta$-open set is $\omega$-open.

Sufficiency. Let $A$ be an $\omega$-open subset of $X$. Since $X$ is a locally countable space, then $A$ is $\omega_\theta$-open. □

Corollary 17 If $(X, \tau)$ is a countable space, then $\omega_\theta O(X)$ is the discrete topology.

A space $X$ is called anti locally countable if nonempty open subsets are uncountable. As an example, observe that in Example 5 (1), the topological space $(R, \tau)$ is anti locally countable.

Theorem 18 Let $(X, \tau)$ be a topological space and $A \subset X$. The following hold:

(1) If $X$ is an anti locally countable space, then $(X, \omega_\theta O(X))$ is anti locally countable.

(2) If $X$ is anti locally countable regular space and $A$ is $\theta$-open, then $\theta$-$\text{cl}(A) = \omega_\theta$-$\text{cl}(A)$.

Proof. (1) : Let $A \in \omega_\theta O(X)$ and $x \in A$. There exists an open subset $U \subset X$ containing $x$ and a countable set $V$ such that $U \setminus V \subset \theta$-$\text{int}(A)$. Thus, $\theta$-$\text{int}(A)$ is uncountable and $A$ is uncountable.
(2): It is obvious that $\omega_\theta$-cl$(A) \subset \theta$-cl$(A)$.

Let $x \in \theta$-cl$(A)$ and $B$ be an $\omega_\theta$-open subset containing $x$. There exists an open subset $V$ containing $x$ and a countable set $U$ such that $V \setminus U \subset \theta$-int$(B)$. Then $(V \setminus U) \cap A \subset \theta$-int$(B) \cap A$ and $(V \cap A) \setminus U \subset \theta$-int$(B) \cap A$. Since $X$ is regular, $x \in V$ and $x \in \theta$-cl$(A)$, then $V \cap A \neq \emptyset$. Since $X$ is regular and $V$ and $A$ are $\omega_\theta$-open, then $V \cap A$ is $\omega_\theta$-open. This implies that $V \cap A$ is uncountable and hence $(V \cap A) \setminus U$ is uncountable. Since $B \cap A$ contains the uncountable set $\theta$-int$(B) \cap A$, then $B \cap A$ is uncountable. Thus, $B \cap A \neq \emptyset$ and $x \in \omega_\theta$-cl$(A)$. □

**Corollary 19** Let $(X, \tau)$ be an anti locally countable regular space and $A \subset X$. The following hold:

1. If $A$ is $\theta$-closed, then $\theta$-int$(A) = \omega_\theta$-int$(A)$.
2. The family of $(\omega_\theta, \theta)$-sets contains all $\theta$-closed subsets of $X$.

**Theorem 20** If $X$ is a Lindelof space, then $A \setminus \theta$-int$(A)$ is countable for every closed subset $A \in \omega_\theta O(X)$.

**Proof.** Let $A \in \omega_\theta O(X)$ be a closed set. For every $x \in A$, there exists an open set $U_x$ containing $x$ such that $U_x \setminus \theta$-int$(A)$ is countable. Thus, \{U$_x$ : $x \in A$\} is an open cover for $A$. Since $A$ is Lindelof, it has a countable subcover \{U$_n$ : $n \in N$\}. Since $A \setminus \theta$-int$(A) = \cup_{n \in N}(U_n \setminus \theta$-int$(A))$, then $A \setminus \theta$-int$(A)$ is countable. □

**Theorem 21** If $A$ is $\omega_\theta$-open subset of $(X, \tau)$, then $\omega_\theta O(X)|_A \subset \omega_\theta O(A)$.

**Proof.** Let $G \in \omega_\theta O(X)|_A$. We have $G = V \cap A$ for some $\omega_\theta$-open subset $V$. Then for every $x \in G$, there exist $U, W \in \tau$ containing $x$ and countable sets $K$ and $L$ such that $U \setminus K \subset \theta$-int$(V)$ and $W \setminus L \subset \theta$-int$(A)$. We have $x \in A \cap (U \cap W) \in \tau|_A$. Thus, $K \cup L$ is countable and $A \cap (U \cap W) \setminus (K \cup L) \subset (U \cap W) \setminus (X \setminus K) \cap (X \setminus L) = (U \setminus K) \cap (W \setminus L) \subset \theta$-int$(V) \cap \theta$-int$(A) \cap A = \theta$-int$(V \cap A) \cap A = \theta$-int$(G) \cap A \subset \theta$-int$_A(G)$. Hence, $G \in \omega_\theta O(A)$. □

### 3 Continuities via $\omega_\theta$-open sets

**Definition 22** A function $f : X \to Y$ is said to be $\omega_\theta$-continuous if for every $x \in X$ and every open subset $V$ in $Y$ containing $f(x)$, there exists an $\omega_\theta$-open subset $U$ in $X$ containing $x$ such that $f(U) \subset V$. 


Theorem 23 For a function $f : X \to Y$, the following are equivalent:

1. $f$ is $\omega_\theta$-continuous.
2. $f^{-1}(A)$ is $\omega_\theta$-open in $X$ for every open subset $A$ of $Y$.
3. $f^{-1}(K)$ is $\omega_\theta$-closed in $X$ for every closed subset $K$ of $Y$.

Proof. (1) $\Rightarrow$ (2) : Let $A$ be an open subset of $Y$ and $x \in f^{-1}(A)$. By (1), there exists an $\omega_\theta$-open set $B$ in $X$ containing $x$ such that $B \subset f^{-1}(A)$. Hence, $f^{-1}(A)$ is $\omega_\theta$-open.

(2) $\Rightarrow$ (1) : Let $A$ be an open subset in $Y$ containing $f(x)$. By (2), $f^{-1}(A)$ is $\omega_\theta$-open. Take $B = f^{-1}(A)$. Hence, $f$ is $\omega_\theta$-continuous.

(2) $\Leftrightarrow$ (3) : Let $K$ be a closed subset of $Y$. By (2), $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$ is $\omega_\theta$-open. Hence, $f^{-1}(K)$ is $\omega_\theta$-closed.

Theorem 24 The following are equivalent for a function $f : X \to Y$:

1. $f$ is $\omega_\theta$-continuous.
2. $f : (X, \omega_\theta O(X)) \to (Y, \sigma)$ is continuous.

Definition 25 A function $f : X \to Y$ is called weakly $\omega_\theta$-continuous at $x \in X$ if for every open subset $V$ in $Y$ containing $f(x)$, there exists an $\omega_\theta$-open subset $U$ in $X$ containing $x$ such that $f(U) \subset \text{cl}(V)$. If $f$ is weakly $\omega_\theta$-continuous at every $x \in X$, it is said to be weakly $\omega_\theta$-continuous.

Remark 26 The following diagram holds for a function $f : X \to Y$:

$$
\begin{array}{ccc}
\text{weakly } \omega_\theta\text{-continuous} & \longrightarrow & \text{weakly } \omega\text{-continuous} \\
\uparrow & & \uparrow \\
\omega_\theta\text{-continuous} & \longrightarrow & \omega\text{-continuous} \\
\uparrow & & \uparrow \\
\theta\text{-continuous} & \longrightarrow & \text{continuous}
\end{array}
$$

The following examples show that these implications are not reversible.

Example 27 Let $R$ be the real line with the topology $\tau = \{\emptyset, R, (2,3)\}$. Let $Y = \{a, b, c\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(x) = \begin{cases} a & \text{if } x \in (0,1) \\ b & \text{if } x \notin (0,1) \end{cases}$. Then $f$ is weakly $\omega_\theta$-continuous but it is not $\omega_\theta$-continuous.
Example 28 Let $R$ be the real line with the topology $\tau = \{\emptyset, R, Q'\}$ where $Q'$ is the set of irrational numbers. Let $Y = \{a, b, c, d\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, c, d\}\}$. Define a function $f : (R, \tau) \to (Y, \sigma)$ as follows: $f(x) = \begin{cases} a & \text{if } x \in Q' \cup \{1\} \\ b & \text{if } x \notin Q' \cup \{1\}. \end{cases}$ Then $f$ is $\omega$-continuous but it is not weakly $\omega\theta$-continuous.

Example 29 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = a$, $f(c) = c$, $f(d) = a$. Then $f$ is $\omega\theta$-continuous but it is not $\theta$-continuous.

For the other implications, the contra examples are as shown in [6, 9].

Definition 30 A function $f : X \to Y$ is said to be $(\omega\theta, \omega)$-continuous if $f^{-1}(A)$ is an $(\omega\theta, \omega)$-set for every open subset $A$ of $Y$.

Definition 31 A function $f : X \to Y$ is said to be $(\omega\theta, \theta)$-continuous if $f^{-1}(A)$ is an $(\omega\theta, \theta)$-set for every open subset $A$ of $Y$.

Remark 32 Every $\theta$-continuous function is $(\omega\theta, \theta)$-continuous and every $\omega\theta$-continuous function is $(\omega\theta, \omega)$-continuous but not conversely.

Example 33 Let $R$ be the real line with the topology $\tau = \{\emptyset, R, Q'\}$ where $Q'$ is the set of irrational numbers. Define a function $f : (R, \tau) \to (R, \tau)$ as follows: $f(x) = \begin{cases} \pi & \text{if } x \in N \\ 1 & \text{if } x \notin N. \end{cases}$ Then $f$ is $(\omega\theta, \omega)$-continuous but it is not $\omega\theta$-continuous.

Example 34 Let $R$ be the real line with the topology $\tau = \{\emptyset, R, (2, 3)\}$. Let $A = (1, \frac{3}{2})$ and $\sigma = \{R, \emptyset, A, R\setminus A\}$. Define a function $f : (R, \tau) \to (R, \sigma)$ as follows: $f(x) = \begin{cases} \frac{5}{4} & \text{if } x \in (1, 2) \\ 4 & \text{if } x \notin (1, 2). \end{cases}$ Then $f$ is $(\omega\theta, \theta)$-continuous but it is not $\theta$-continuous.

Definition 35 A function $f : X \to Y$ is coweakly $\omega\theta$-continuous if for every open subset $A$ in $Y$, $f^{-1}(fr(A))$ is $\omega\theta$-closed in $X$, where $fr(A) = cl(A)\setminus int(A)$. 
**Theorem 36** Let \( f : X \to Y \) be a function. The following are equivalent:

1. \( f \) is \( \omega_\theta \)-continuous,
2. \( f \) is \( \omega \)-continuous and \((\omega_\theta, \omega)\)-continuous,
3. \( f \) is weakly \( \omega_\theta \)-continuous and coweakly \( \omega_\theta \)-continuous.

**Proof.** (1) \( \Leftrightarrow \) (2): It is an immediate consequence of Theorem 14.

(1) \( \Rightarrow \) (3): Let \( f \) be weakly \( \omega_\theta \)-continuous and coweakly \( \omega_\theta \)-continuous. Let \( x \in X \) and \( V \) be an open subset of \( Y \) such that \( f(x) \in V \). Since \( f \) is weakly \( \omega_\theta \)-continuous, then there exists an \( \omega_\theta \)-open subset \( U \) of \( X \) containing \( x \) such that \( f(U) \subset cl(V) \). We have \( fr(V) = cl(V) \setminus V \) and \( f(x) \notin fr(V) \). Since \( f \) is coweakly \( \omega_\theta \)-continuous, then \( x \in U \setminus f^{-1}(fr(V)) \) is \( \omega_\theta \)-open in \( X \). For every \( y \in f(U \setminus f^{-1}(fr(V))) \), \( y = f(x_1) \) for a point \( x_1 \in U \setminus f^{-1}(fr(V)) \). We have \( f(x_1) = y \in f(U) \subset cl(V) \) and \( y \notin fr(V) \). Hence, \( f(x_1) = y \notin fr(V) \) and \( f(x_1) \in V \). Thus, \( f(U \setminus f^{-1}(fr(V))) \subset V \) and \( f \) is \( \omega_\theta \)-continuous. \( \blacksquare \)

**Theorem 37** The following are equivalent for a function \( f : X \to Y \):

1. \( f \) is \( \theta \)-continuous,
2. \( f \) is \( \omega_\theta \)-continuous and \((\omega_\theta, \theta)\)-continuous.

**Proof.** It is an immediate consequence of Theorem 15. \( \blacksquare \)

**Theorem 38** Let \( f : X \to Y \) be a function. The following are equivalent:

1. \( f \) is weakly \( \omega_\theta \)-continuous,
2. \( \omega_\theta \)-cl\( (f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K)) \) for every subset \( K \) of \( Y \),
3. \( \omega_\theta \)-cl\( (f^{-1}(int(A))) \subset f^{-1}(A) \) for every regular closed set \( A \) of \( Y \),
4. \( \omega_\theta \)-cl\( (f^{-1}(A)) \subset f^{-1}(cl(A)) \) for every open set \( A \) of \( Y \),
5. \( f^{-1}(A) \subset \omega_\theta \)-int\( (f^{-1}(cl(A))) \) for every open set \( A \) of \( Y \),
6. \( \omega_\theta \)-cl\( (f^{-1}(A)) \subset f^{-1}(cl(A)) \) for each preopen set \( A \) of \( Y \),
7. \( f^{-1}(A) \subset \omega_\theta \)-int\( (f^{-1}(cl(A))) \) for each preopen set \( A \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( K \subset Y \) and \( x \in X \setminus f^{-1}(cl(K)) \). Then \( f(x) \in Y \setminus cl(K) \). This implies that there exists an open set \( A \) containing \( f(x) \) such that \( A \cap K = \emptyset \). We have, \( cl(A) \cap int(cl(K)) = \emptyset \). Since \( f \) is weakly \( \omega_\theta \)-continuous, then there exists an \( \omega_\theta \)-open set \( B \) containing \( x \) such that \( f(B) \subset cl(A) \). We have \( B \cap f^{-1}(int(cl(K))) = \emptyset \). Thus, \( x \in X \setminus \omega_\theta \)-cl\( (f^{-1}(int(cl(K)))) \) and \( \omega_\theta \)-cl\( (f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K)) \).

(2) \( \Rightarrow \) (3): Let \( A \) be any regular closed set in \( Y \). Thus, \( \omega_\theta \)-cl\( (f^{-1}(int(A))) \) = \( \omega_\theta \)-cl\( (f^{-1}(int(cl(int(A)))) \subset f^{-1}(cl(int(A))) = f^{-1}(A) \).
Thus \( x \in \text{cl}(A) \) and hence \( f(B) \subset \text{cl}(A) \). We have \( f(B) \cap A = \emptyset \) and hence \( B \cap f^{-1}(A) = \emptyset \). Thus, \( x \in X \setminus \omega\text{-cl}(f^{-1}(A)) \) and \( \omega\text{-cl}(f^{-1}(A)) \subset f^{-1}(\text{cl}(A)) \).

(6) \( \Rightarrow \) (7) : Let \( A \) be any preopen set of \( Y \). Since \( Y \setminus \text{cl}(A) \) is open in \( Y \), then \( X \setminus \omega\text{-int}(f^{-1}(\text{cl}(A))) = \omega\text{-cl}(f^{-1}(Y \setminus \text{cl}(A))) \subset f^{-1}(\text{cl}(Y \setminus \text{cl}(A))) \subset X \setminus f^{-1}(A) \). Hence, \( f^{-1}(A) \subset \omega\text{-int}(f^{-1}(\text{cl}(A))) \).

(7) \( \Rightarrow \) (1) : Let \( x \in X \) and \( A \) any open set of \( Y \) containing \( f(x) \). Then \( x \in f^{-1}(A) \subset \omega\text{-int}(f^{-1}(\text{cl}(A))) \). Take \( B = \omega\text{-int}(f^{-1}(\text{cl}(A))) \). Then \( f(B) \subset \text{cl}(A) \). Thus, \( f \) is weakly \( \omega\text{-continuous} \) at \( x \in X \).

**Theorem 39** The following properties are equivalent for a function \( f : X \to Y \):

1. \( f : X \to Y \) is weakly \( \omega \)-continuous at \( x \in X \).
2. \( x \in \omega\text{-int}(f^{-1}(\text{cl}(A))) \) for each neighborhood \( A \) of \( f(x) \).

**Proof.** (1) \( \Rightarrow \) (2) : Let \( A \) be any neighborhood of \( f(x) \). There exists an \( \omega\text{-open} \) set \( B \) containing \( x \) such that \( f(B) \subset \text{cl}(A) \). Since \( B \subset f^{-1}(\text{cl}(A)) \) and \( B \) is \( \omega \)-open, then \( x \in B \subset \omega\text{-int}(f^{-1}(\text{cl}(A))) \subset \omega\text{-int}(f^{-1}(\text{cl}(A))) \).

(2) \( \Rightarrow \) (1) : Let \( x \in \omega\text{-int}(f^{-1}(\text{cl}(A))) \) for each neighborhood \( A \) of \( f(x) \). Take \( U = \omega\text{-int}(f^{-1}(\text{cl}(A))) \). Thus, \( f(U) \subset \text{cl}(A) \) and \( U \) is \( \omega \)-open. Hence, \( f \) is weakly \( \omega \)-continuous at \( x \in X \).

**Theorem 40** Let \( f : X \to Y \) be a function. The following are equivalent:

1. \( f \) is weakly \( \omega \)-continuous.
2. \( f(\omega\text{-cl}(K)) \subset \theta\text{-cl}(f(K)) \) for each subset \( K \) of \( X \),
3. \( \omega\text{-cl}(f^{-1}(A)) \subset f^{-1}(\theta\text{-cl}(A)) \) for each subset \( A \) of \( Y \),
4. \( \omega\text{-cl}(f^{-1}(\text{int}(\theta\text{-cl}(A)))) \subset f^{-1}(\theta\text{-cl}(A)) \) for every subset \( A \) of \( Y \).
Theorem 43 If $f : X \to Y$ is a weakly $\omega$-continuous surjection and $X$ is $\omega$-connected, then $Y$ is connected.
Proof. Suppose that $Y$ is not connected. There exist nonempty open sets $U$ and $V$ of $Y$ such that $Y = U \cup V$ and $U \cap V = \emptyset$. This implies that $U$ and $V$ are clopen in $Y$. By Theorem 38, $f^{-1}(U) \subset \omega_{\theta}\text{-int}(f^{-1}(cl(U))) = \omega_{\theta}\text{-int}(f^{-1}(U))$. Hence $f^{-1}(U)$ is $\omega_{\theta}$-open in $X$. Similarly, $f^{-1}(V)$ is $\omega_{\theta}$-open in $X$. Hence, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, $X = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Thus, $X$ is not $\omega_{\theta}$-connected.

References


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