ON FUZZY UPPER AND LOWER CONTRA-CONTINUOUS MULTIFUNCTIONS

M. Alimohammady, E. Ekici, S. Jafari and M. Roohi

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Abstract

This paper is devoted to the concepts of fuzzy upper and fuzzy lower contra-continuous multifunctions and also some characterizations of them are considered.

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Keywords: fuzzy topological space, fuzzy multifunctions, fuzzy lower contra-continuous multifunction, fuzzy upper contra-continuous multifunction.

1 Introduction

In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory can be found, specially in functional analysis and fixed point theory [5, 23, 24] etc. The initiation of fuzzy multifunctions is due to Papageorgiou [20]. He studied upper and lower semi-continuous multifunctions. Mukherjee and Malakar [15] have studied fuzzy multifunctions with q-coincidence. Recently many authors for example Albrycht and Maltoka, Nouh and El-Shafei [1, 17] and Beg [3, 4] have studied fuzzy multifunctions and have characterized, some properties of fuzzy multifunctions defined on a fuzzy topological space. Several authors have studied different types of fuzzy continuity for fuzzy multifunctions, for example see [2, 9, 20, 21] and also for more details on fuzzy multifunctions one can see [4]. On the other hand, Dontchev [8] introduced the notion of contra-continuous functions. It is shown in [8] that contra-continuous images of strongly S-closed spaces are compact. Joseph and Kwack [14] introduced another form of contra-continuity called \((\theta, s)\)-continuous functions in order to investigate S-closed spaces due to Thompson [25]. In recent years, several authors have studied some new forms of contra-continuity for functions and multifunctions, for example see [6, 11, 12, 13, 16]. In the present paper, we study the notions of fuzzy upper and fuzzy lower contra-continuous multifunctions. Also, some characterizations and properties of such notions are discussed.
2 Preliminaries

The class of all fuzzy sets on a universe $Y$ will be denoted by $I^Y$ and fuzzy sets on $Y$ will be denoted by $\mu$, $\eta$, etc. A family $\tau$ of fuzzy sets in $Y$ is called a fuzzy topology for $Y$ [7] if

1. $\emptyset, Y \in \tau$.
2. $\mu \land \eta \in \tau$ whenever $\mu, \eta \in \tau$.
3. If $\mu_i \in \tau$ for each $i \in I$, then $\bigvee \mu_i \in \tau$.

The pair $(Y, \sigma)$ is called a fuzzy topological space. Every member of $\sigma$ is called a fuzzy open set. A fuzzy set in $Y$ is called a fuzzy point if it takes the value 0 for all $y \in Y$ except one, say, $x \in Y$. If its value at $x$ is $\varepsilon$ $(0 < \varepsilon \leq 1)$, we denote this fuzzy point by $x_\varepsilon$, where the point $x$ is called its support [18, 19].

For any fuzzy point $x_\varepsilon$ and any fuzzy set $\mu$, $x_\varepsilon \in \mu$ if and only if $\varepsilon \leq \mu(x)$. A fuzzy point $x_\varepsilon$ is called quasi-coincident with a fuzzy set $\eta$, denoted by $x_\varepsilon \eta \eta$, if $\varepsilon + \eta(x) > 1$. A fuzzy set $\mu$ is called quasi-coincident with a fuzzy set $\eta$, denoted by $\mu \eta \eta$, if there exists a $x \in Y$ such that $\mu(x) + \eta(x) > 1$ [18, 19]. When they are not quasi-coincident, it will be denoted by $\mu \not\equiv \eta$.

Throughout this paper, $(X, \tau)$ or simply $X$ will stand for ordinary topological space and $(Y, \sigma)$ or simply $Y$ will be denoted a fuzzy topological space.

Let $X$ and $Y$ be a topological space in the classical sense and a fuzzy topological space, respectively. $F : X \to Y$ is called a fuzzy multifunction [20] if for each $x \in X$, $F(x)$ is a fuzzy set in $Y$. Throughout the paper, by $F : X \to Y$ we will mean that $F$ is a fuzzy multifunction from a classical topological space $X$ to a fuzzy topological space $Y$. For a fuzzy multifunction $F : X \to Y$, the upper inverse $F^+(\mu)$ and lower inverse $F^-(\mu)$ of a fuzzy set $\mu$ in $Y$ are defined as follows: $F^+(\mu) = \{x \in X : F(x) \leq \mu\}$ and $F^-(\mu) = \{x \in X : F(x) \geq \mu\}$. For any fuzzy set $\mu$ in $Y$, we have $F^-(1 - \mu) = X - F^+(\mu)$ [15]. We denote the interior and the closure of a subset $A$ of a topological space $X$ by $\text{Int}(A)$ and $\text{Cl}(A)$, respectively.

3 Fuzzy upper and lower contra-continuous multifunctions

Definition 1 A fuzzy multifunction $F : X \to Y$ is called fuzzy lower contra-continuous multifunction if for each fuzzy closed set $\mu$ in $Y$ with $x \in F^-(\mu)$, there exists an open set $B$ in $X$ containing $x$ such that $B \subset F^-(\mu)$.

Definition 2 A fuzzy multifunction $F : X \to Y$ is called fuzzy upper contra-continuous multifunction if for each fuzzy closed set $\mu$ in $Y$ with $x \in F^+(\mu)$, there exists an open set $B$ in $X$ containing $x$ such that $B \subset F^+(\mu)$.

Theorem 3 The following are equivalent for a fuzzy multifunction $F : X \to Y$:

1. $F$ is fuzzy upper contra-continuous,
2. For each fuzzy closed set $\mu$ and $x \in X$ such that $F(x) \leq \mu$, there exists an open set $B$ containing $x$ such that if $y \in B$, then $F(y) \leq \mu$.
(3) $F^+(\mu)$ is open for any fuzzy closed set $\mu$ in $Y$,
(4) $F^-(\rho)$ is closed for any fuzzy open set $\rho$ in $Y$.

Proof. (1) $\iff$ (2): Obvious.
(1) $\Rightarrow$ (3): Let $\mu$ be any fuzzy closed set in $Y$ and $x \in F^+(\mu)$. By (1), there exists an open set $A_x$ containing $x$ such that $A_x \subseteq F^+(\mu)$. Thus, $x \in \text{Int}(F^+(\mu))$ and hence $F^+(\mu)$ is an open set in $X$.
(3) $\Rightarrow$ (4): Let $\rho$ be a fuzzy open set in $Y$. Then $Y \setminus \rho$ is a fuzzy closed set in $Y$. By (3), $F^+(Y \setminus \rho)$ is open. Since $F^+(Y \setminus \rho) = X \setminus F^-(\rho)$, then $F^-(\rho)$ is closed in $X$.
(4) $\Rightarrow$ (3): It is similar to that of (3) $\Rightarrow$ (4).
(3) $\Rightarrow$ (1): Let $\rho$ be any fuzzy closed set in $Y$ and $x \in F^+(\rho)$. By (3), $F^+(\rho)$ is an open set in $X$. Take $A = F^+(\rho)$. Then, $A \subseteq F^+(\rho)$. Thus, $F$ is fuzzy upper contra-continuous.

Definition 4 The set $\land \{\rho \in \tau : \mu \leq \rho\}$ is called the fuzzy kernel of a fuzzy set $\mu$ in a fuzzy topological space $(X, \tau)$ and is denoted by $\text{Ker}(\mu)$.

Lemma 5 For fuzzy set $\mu$ in a fuzzy topological space $(X, \tau)$, if $\mu \in \tau$, then $\mu = \text{Ker}(\mu)$.

Theorem 6 Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be a fuzzy multifunction. If $\text{Cl}(F^-(\mu)) \leq F^-(\text{Ker}(\mu))$ for every fuzzy set $\mu$ in $Y$, then $F$ is fuzzy upper contra-continuous.

Proof. Suppose that $\text{Cl}(F^-(\mu)) \leq F^-(\text{Ker}(\mu))$ for every fuzzy set $\mu$ in $Y$. Let $\rho \in \sigma$. By Lemma 5, $\text{Cl}(F^-(\rho)) \leq F^-(\text{Ker}(\rho)) = F^-(\rho)$. This implies that $\text{Cl}(F^-(\rho)) = F^-(\rho)$ and hence $F^-(\rho)$ is closed in $X$. Thus, by Theorem 3, $F$ is fuzzy upper contra-continuous.

Definition 7 A fuzzy multifunction $F : X \rightarrow Y$ is called
(1) fuzzy lower semi-continuous [15] if for each fuzzy open set $\mu$ in $Y$ with $x \in F^-(\mu)$, there exists an open subset $B$ of $X$ containing $x$ such that $B \subseteq F^-(\mu)$.
(2) fuzzy upper semi-continuous [15] if for each fuzzy open set $\mu$ in $Y$ with $x \in F^+(\mu)$, there exists an open subset $B$ of $X$ containing $x$ such that $B \subseteq F^+(\mu)$.

Remark 8 The notions of fuzzy upper contra-continuous multifunctions and fuzzy upper semi-continuous multifunctions are independent as shown in the following examples.

Example 9 Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $Y = [0, 1]$, $\sigma = \{Y, \emptyset, \mu, \rho, \eta\}$, where $\mu(y) = 0.5$, $\rho(y) = 0.6$, $\eta(y) = 0.7$ for $y \in Y$. Define a fuzzy multifunction as follows: $F(a) = \mu$, $F(b) = \rho$, $F(c) = \eta$. Then the fuzzy multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is fuzzy upper contra-continuous but it is not fuzzy upper semi-continuous.
Example 10 Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b, c\}\}$ and $Y = [0, 1]$, $\sigma = \{Y, \emptyset, \mu, \rho, \eta\}$, where $\mu(y) = 0.3$, $\rho(y) = 0.2$, $\eta(y) = 0.6$, $\gamma(y) = 0.4$, $\zeta(y) = 0.5$ for $y \in Y$.
Define a fuzzy multifunction as follows: $F(a) = \gamma$, $F(b) = \zeta$, $F(c) = \eta$. Then the fuzzy multifunction $F : (X, \tau) \to (Y, \sigma)$ is fuzzy upper semi-continuous but it is not fuzzy upper contra-continuous.

Theorem 11 The following are equivalent for a fuzzy multifunction $F : X \to Y$:

1. $F$ is fuzzy lower contra-continuous,
2. For each fuzzy closed set $\mu$ and $x \in X$ such that $F(x)\mu$, there exists an open set $B$ containing $x$ such that if $y \in B$, then $F(y)\mu$,
3. $F^{-1}(\mu)$ is open for any fuzzy closed set $\mu$ in $Y$,
4. $F^+(\rho)$ is closed for any fuzzy open set $\rho$ in $Y$.

Proof. It is similar to that of Theorem 3.

Theorem 12 For a fuzzy multifunction $F : (X, \tau) \to (Y, \sigma)$, if $\text{Cl}(F^+(\rho)) \leq F^+(\text{Ker}(\rho))$ for every fuzzy set $\rho$ in $Y$, then $F$ is fuzzy lower contra-continuous.

Proof. Suppose that $\text{Cl}(F^+(\rho)) \leq F^+(\text{Ker}(\rho))$ for every fuzzy set $\rho$ in $Y$.
Let $\rho \in \sigma$. We have $\text{Cl}(F^+(\rho)) \leq F^+(\text{Ker}(\rho)) = F^+(\rho)$. Thus, $\text{Cl}(F^+(\rho)) \leq F^+(\rho)$ and hence $F^+(\rho)$ is closed in $X$. By Theorem 11, $F$ is fuzzy lower contra-continuous.

Theorem 13 If $F_i : X \to Y$ are fuzzy upper contra-continuous multifunctions for $i = 1, 2, ..., n$, then $\vee_{i=1}^n F_i$ is a fuzzy upper contra-continuous multifunction.

Proof. Let $\mu$ be a fuzzy closed set of $Y$. We will show that $(\vee_{i=1}^n F_i)^+(\mu) = \{x \in X : \vee_{i=1}^n F_i(x) \leq \mu\}$ is open in $X$. Let $x \in (\vee_{i=1}^n F_i)^+(\mu)$. Then $F_i(x) \leq \mu$ for $i = 1, 2, ..., n$. Since $F_i : X \to Y$ is fuzzy upper contra-continuous multifunction for $i = 1, 2, ..., n$, there exists an open set $U_x$ containing $x$ such that for all $z \in U_x$, $F_i(z) \leq \mu$. Let $U = \bigwedge_{i=1}^n U_x$. Then $U \subset (\vee_{i=1}^n F_i)^+(\mu)$. Thus, $(\vee_{i=1}^n F_i)^+(\mu)$ is open and hence $\vee_{i=1}^n F_i$ is a fuzzy upper contra-continuous multifunction.

Lemma 14 ([4]) Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy sets in a fuzzy topological space $X$. Then a fuzzy point $x$ is quasi-coincident with $\forall \mu_i$ if and only if there exists an $i_0 \in I$ such that $x q \mu_{i_0}$.

Theorem 15 If $F_i : X \to Y$ are fuzzy lower contra-continuous multifunctions for $i = 1, 2, ..., n$, then $\vee_{i=1}^n F_i$ is a fuzzy lower contra-continuous multifunction.

Proof. Let $\mu$ be a fuzzy closed set of $Y$. We will show that $(\vee_{i=1}^n F_i)^-(\mu) = \{x \in X : (\vee_{i=1}^n F_i)(x)\mu\}$ is open in $X$. Let $x \in (\vee_{i=1}^n F_i)^-(\mu)$. Then $(\vee_{i=1}^n F_i)(x)\mu$ and hence $F_{i_0}(x)\mu$ for an $i_0$. Since $F_{i_0} : X \to Y$ is fuzzy lower contra-continuous multifunction, then there exists an open set $U_x$ containing $x$ such
that for all \( z \in U \), \( F_{i_0}(z) \mu U \). Then \( (\bigvee_{i=1}^{n} F_i)(z) \mu U \) and hence \( U \subset (\bigvee_{i=1}^{n} F_i)^{(\mu)} \). Thus, \( (\bigvee_{i=1}^{n} F_i)^{(\mu)} \) is open and hence \( \bigvee_{i=1}^{n} F_i \) is a fuzzy lower contra-continuous multifunction. ■

**Theorem 16** Let \( F : X \rightarrow Y \) be a fuzzy multifunction and \( \{ U_i : i \in I \} \) be an open cover for \( X \). Then the following are equivalent:

1. \( F_i = F |_{U_i} \) is a fuzzy lower contra-continuous multifunction for all \( i \in I \),
2. \( F \) is fuzzy lower contra-continuous.

**Proof.** (1) \( \Rightarrow \) (2): Let \( x \in X \) and \( \mu \) be a fuzzy closed set in \( Y \) with \( x \in F^{-}(\mu) \). Since \( \{ U_i : i \in I \} \) is an open cover for \( X \), then \( x \in U_{i_0} \) for an \( i_0 \in I \). We have \( F(x) = F_{i_0}(x) \) and hence \( x \in F_{i_0}^{-}(\mu) \). Since \( F |_{U_{i_0}} \) is fuzzy lower contra-continuous, then there exists an open set \( B = G \cap U_{i_0} \) in \( U_{i_0} \) such that \( x \in B \) and \( F^{-}(\mu) \cap U_{i_0} = F |_{U_{i_0}}^{-}(\mu) \cap B = G \cap U_{i_0} \), where \( G \) is open in \( X \). We have \( x \in B = G \cap U_{i_0} \subset F |_{U_{i_0}}^{-}(\mu) = F^{-}(\mu) \cap U_{i_0} \subset F^{-}(\mu) \). Hence, \( F \) is fuzzy lower contra-continuous.

(2) \( \Rightarrow \) (1): Let \( x \in X \) and \( x \in U_i \). Let \( \mu \) be a fuzzy closed set in \( Y \) with \( F_i(x) \mu U \). Since \( F \) is lower contra-continuous and \( F(x) = F_i(x) \), then there exists an open set \( U \) containing \( x \) such that \( U \subset F^{-}(\mu) \). Take \( B = U_i \cap U \). Then \( B \) is open in \( U_i \) containing \( x \). We have \( B \subset F_i^{-}(\mu) \). Thus, \( F_i \) is a fuzzy lower contra-continuous.

**Theorem 17** Let \( F : X \rightarrow Y \) be a fuzzy multifunction and \( \{ U_i : i \in I \} \) be an open cover for \( X \). Then the following are equivalent:

1. \( F_i = F |_{U_i} \) is a fuzzy upper contra-continuous multifunction for all \( i \in I \),
2. \( F \) is fuzzy upper contra-continuous.

**Proof.** It is similar to that of Theorem 16. ■

Recall that for a multifunction \( F_1 : X \rightarrow Y \) and a fuzzy multifunction \( F_2 : Y \rightarrow Z \), the fuzzy multifunction \( F_2 \circ F_1 : X \rightarrow Z \) is defined by \( (F_2 \circ F_1)(x) = F_2(F_1(x)) \) for \( x \in X \).

**Definition 18** Let \( X \) and \( Y \) be topological spaces. A multifunction \( F : X \rightarrow Y \) is called

1. lower semi-continuous [21] if for each open subset \( A \subset Y \) with \( x \in F^{-}(A) \), there exists an open set \( B \) in \( X \) containing \( x \) such that \( B \subset F^{-}(A) \).
2. upper semi-continuous [21] if for each open subset \( A \subset Y \) with \( x \in F^{+}(A) \), there exists an open set \( B \) in \( X \) containing \( x \) such that \( B \subset F^{+}(A) \).

**Theorem 19** If \( F_1 : X \rightarrow Y \) is an upper semi-continuous multifunction and \( F_2 : Y \rightarrow Z \) is a fuzzy upper contra-continuous multifunction, then \( F_2 \circ F_1 \) is fuzzy upper contra-continuous.
Proof. Let \( x \in X \) and \( \mu \) be a fuzzy closed set in \( Z \). We have \((F_2 \circ F_1)^+(\mu) = F_1^+(F_2^+(\mu))\). Since \( F_2 \) is fuzzy upper contra-continuous, then \( F_2^+(\mu) \) is open in \( Y \). Since \( F_1 \) is upper semi-continuous, then \( F_1^+(F_2^+(\mu)) = (F_2 \circ F_1)^+(\mu) \) is open in \( X \). Thus, \( F_2 \circ F_1 \) is fuzzy upper contra-continuous. \( \blacksquare \)

Definition 20 A fuzzy set \( \mu \) in a fuzzy topological space \( X \) is called a fuzzy \( cl \)-neighbourhood of a fuzzy point \( x \) in \( X \) if there exists a fuzzy closed set \( \rho \) in \( X \) such that \( x \in \rho \leq \mu \).

Theorem 21 If \( F : X \rightarrow Y \) is a fuzzy upper contra-continuous multifunction, then for each point \( x \) of \( X \) and each fuzzy \( cl \)-neighbourhood \( \mu \) of \( F(x) \), \( F^+(\mu) \) is a neighbourhood of \( x \).

Proof. Let \( x \in X \) and \( \mu \) be a fuzzy \( cl \)-neighbourhood of \( F(x) \). There exists a fuzzy closed set \( \rho \) in \( Y \) such that \( F(x) \leq \rho \leq \mu \). We have \( x \in F^+(\rho) \leq F^+(\mu) \). Since \( F^+(\rho) \) is an open set, \( F^+(\mu) \) is a neighbourhood of \( x \). \( \blacksquare \)

Remark 22 ([26]) A subset \( A \) of a topological space \((X, \tau)\) can be considered as a fuzzy set with characteristic function defined by

\[
A(x) = \begin{cases} 
1 & , x \in A \\
0 & , x \notin A
\end{cases}
\]

Let \((Y, \sigma)\) be a fuzzy topological space. The fuzzy sets of the form \( A \times \rho \) with \( A \in \tau \) and \( \rho \in \sigma \) form a basis for the product fuzzy topology \( \tau \times \sigma \) on \( X \times Y \), where for any \((x, y) \in X \times Y \),

\[
(A \times \rho)(x, y) = \min\{A(x), \rho(y)\}
\]

Definition 23 ([15]) For a fuzzy multifunction \( F : X \rightarrow Y \), the fuzzy graph multifunction \( G_F : X \rightarrow X \times Y \) of \( F \) is defined by \( G_F(x) = x_1 \times F(x) \) for every \( x \in X \).

Theorem 24 If the fuzzy graph multifunction \( G_F \) of a fuzzy multifunction \( F : X \rightarrow Y \) is fuzzy lower contra-continuous, then \( F \) is fuzzy lower contra-continuous.

Proof. Suppose that \( G_F \) is fuzzy lower contra-continuous and \( x \in X \). Let \( \mu \) be a fuzzy closed set in \( Y \) such that \( F(x) \mu \). Then there exists \( y \in Y \) such that \((F(x))(y) + \mu(y) > 1\). Then \((G_F(x))(x, y) + (X \times \mu)(x, y) = (F(x))(y) + \mu(y) > 1\). Hence, \( G_F(x)q(X \times \mu) \). Since \( G_F \) is fuzzy lower contra-continuous, there exists an open set \( B \) in \( X \) such that \( x \in B \) and \( G_F(b)q(X \times \mu) \) for all \( b \in B \).

Let there exists a \( b_0 \in B \) such that \( F(b_0) \mu \). Then for all \( y \in Y \), \((F(b_0))(y) + \mu(y) \leq 1\). For any \((a, c) \in X \times Y \), we have \((G_F(b_0))(a, c) \leq (F(b_0))(c) \) and \((X \times \mu)(a, c) \leq \mu(c) \). Since for all \( y \in Y \), \((F(b_0))(y) + \mu(y) \leq 1\), then \((G_F(b_0))(a, c) + \)
Thus, if the upper inverse image of fuzzy closed sets containing $F$ is not fuzzy upper contra-continuous is identical with the union of the frontier equivalent:

Theorem 28

Let all points of $(G \mu)$ be fuzzy closed in $Y$ with $x \in F^-(\rho)$, the net $(x_i)_{i \in I}$ is eventually in $F^-(\rho)$.

Proof. (1) $\Rightarrow$ (2) : Let $x_i$ be a net converging to $x$ in $X$ and $\rho$ be any fuzzy closed set in $Y$ with $x \in F^-(\rho)$. Since $F$ is fuzzy lower contra-continuous, then there exists an open set $A \subset X$ containing $x$ such that $A \subset F^-(\rho)$. Since $x_i \rightarrow x$, then there exists an index $i_0 \in I$ such that $x_i \in A$ for every $i \geq i_0$. Hence, $(x_i)_{i \in I}$ is eventually in $F^-(\rho)$.

(2) $\Rightarrow$ (1) : Suppose that $F$ is not fuzzy lower contra-continuous. There exists a point $x$ and a fuzzy closed set $\mu$ containing $x$ with $x \in F^-(\mu)$ such that $B \notin F^-(\mu)$ for each open set $B \subset X$ containing $x$. Let $x_i \in B$ and $x_i \notin F^-(\mu)$ for each open set $B \subset X$ containing $x$. Then the neighborhood net $(x_i)$ converges to $x$ but $(x_i)_{i \in I}$ is not eventually in $F^-(\mu)$. This is a contradiction. ■

Theorem 27

Let $F : X \rightarrow Y$ be a fuzzy multifunction. Then the following are equivalent:

(1) $F$ is fuzzy upper contra-continuous,

(2) For each $x \in X$ and each net $(x_i)_{i \in I}$ converging to $x$ in $X$ and each fuzzy closed set $\rho$ in $Y$ with $x \in F^+(\rho)$, the net $(x_i)_{i \in I}$ is eventually in $F^+(\rho)$.

Proof. The proof is similar to that of Theorem 26. ■

Recall that the frontier of a subset $A$ of a topological space $X$, denoted by $Fr(A)$, is defined by $Fr(A) = Cl(A) \cap Cl(X \setminus A) = Cl(A) \setminus Int(A)$.

Theorem 28

The set all points of $X$ at which a fuzzy multifunction $F : X \rightarrow Y$ is not fuzzy upper contra-continuous is identical with the union of the frontier of the upper inverse image of fuzzy closed sets containing $F(x)$. 
Proof. Suppose $F$ is not fuzzy upper contra-continuous at $x \in X$. Then there exists a fuzzy closed set $\eta$ in $Y$ containing $F(x)$ such that $A \cap (X \setminus F^+(\eta)) \neq \emptyset$ for every open set $A$ containing $x$. We have $x \in Cl(X \setminus F^+(\eta)) = X \setminus Int(F^+(\eta))$ and $x \in F^+(\eta)$. Thus, $x \in Fr(F^+(\eta))$.

Conversely, let $\eta$ be a fuzzy closed set in $Y$ containing $F(x)$ with $x \in Fr(F^+(\eta))$. Suppose that $F$ is fuzzy upper contra-continuous at $x$. There exists an open set $A$ containing $x$ such that $A \subset F^+(\eta)$. We have $x \in Int(F^+(\eta))$. This is a contradiction. Thus, $F$ is not fuzzy upper contra-continuous at $x$. $lacksquare$

Theorem 29 The set all points of $X$ at which a fuzzy multifunction $F : X \to Y$ is not fuzzy lower contra-continuous is identical with the union of the frontier of the lower inverse image of fuzzy closed sets which are quasi-coincident with $F(x)$.

Proof. It is similar to that of Theorem 28. $lacksquare$

Theorem 30 If $F : X \to Y$ is a fuzzy upper contra-continuous point closed multifunction and $F(x) \land F(y) = \emptyset$ for each distinct pair $x, y \in X$, then $X$ is a $T_2$ space.

Proof. Let $x$ and $y$ be any two distinct points in $X$. We have $F(x) \land F(y) = \emptyset$. Since $F$ is fuzzy upper contra-continuous and point closed, $F^+(F(x))$ and $F^+(F(y))$ are disjoint fuzzy open sets containing $x$ and $y$, respectively. Hence, $X$ is $T_2$. $lacksquare$

Definition 31 A fuzzy topological space $X$ is called fuzzy strongly $S$-closed [2] if every fuzzy closed cover of $X$ has a finite subcover.

Theorem 32 Let $F : X \to Y$ be a fuzzy upper contra-continuous surjective multifunction. Suppose that $F(x)$ is fuzzy strongly $S$-closed for each $x \in X$. If $X$ is compact, then $Y$ is fuzzy strongly $S$-closed.

Proof. Let $\{\mu_k\}_{k \in I}$ be a fuzzy closed cover of $Y$. Since $F(x)$ is fuzzy strongly $S$-closed for each $x \in X$, there exists a finite subset $I_x$ of $I$ such that $F(x) \leq \bigvee_{k \in I_x} \mu_k$. Take $\mu_x = \bigvee_{k \in I_x} \mu_k$. Since $F$ is fuzzy upper contra-continuous, there exists a fuzzy open set $U_x$ of $X$ containing $x$ such that $F(U_x) \leq \mu_x$. Then $\{U_x\}_{x \in X}$ is an open cover of $X$. Since $X$ is compact, there exist $x_1, x_2, x_3, \ldots, x_n$ in $X$ such that $X = \bigvee_{i=1}^n U_{x_i}$. We have $Y = F(X) = F(\bigcup_{i=1}^n U_{x_i}) \leq \bigvee_{i=1}^n F(U_{x_i}) \leq \bigvee_{i=1}^n \mu_{x_i} = \bigvee_{i=1}^n \bigvee_{k \in I_{x_i}} \mu_k$. Thus, $Y$ is fuzzy strongly $S$-closed. $lacksquare$

Definition 33 A fuzzy topological space $X$ is said to be disconnected [26] if $X = \mu \land \eta$, where $\mu$ and $\eta$ are nonempty fuzzy open sets in $X$ such that $\mu \land \eta = \emptyset$. 

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Theorem 34 Let $F : X \rightarrow Y$ be a fuzzy upper contra-continuous punctually fuzzy connected surjective multifunction. If $X$ is connected, then $Y$ is a fuzzy connected space.

Proof. Suppose that $Y$ is not fuzzy connected. Let $Y = \mu \cup \eta$ be a partition of $Y$. Then, $\mu$ and $\eta$ are fuzzy open and closed in $Y$. Since $F(x)$ is fuzzy connected for each $x \in X$, $F(x) \leq \mu$ or $F(x) \leq \eta$. This implies that $x \in F^+(\mu) \cup F^+(\eta)$. We have $F^+(\mu) \cup F^+(\eta) = X$ and $F^+(\mu) \cap F^+(\eta) = \emptyset$. Since $F$ is fuzzy upper contra-continuous, $F^+(\mu)$ and $F^+(\eta)$ are open in $X$. Thus, $X = F^+(\mu) \cup F^+(\eta)$ is a partition of $X$. This is a contradiction.

Theorem 35 Let $F : X \rightarrow Y$ be a fuzzy lower contra-continuous punctually fuzzy connected surjective multifunction. If $X$ is connected, then $Y$ is a fuzzy connected space.

Proof. Suppose that $Y$ is not fuzzy connected. Let $Y = \mu \cup \eta$ be a partition of $Y$. Then $\mu$ and $\eta$ are fuzzy open and closed in $Y$. Since $F$ is fuzzy lower contra-continuous multifunction, $F^+(\mu)$ and $F^+(\eta)$ are closed. Since $X = F^+(\mu) \cup F^+(\eta)$ and $F^+(\mu) \cap F^+(\eta) = \emptyset$, then $X$ is not connected. This is a contradiction.

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References


M. ALIMOHAMMADY
Department of Mathematics, University of Mazandaran,
Babolsar, Iran
E-mail: amohsen@umz.ac.ir

E. EKICI
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus,
17020 Canakkale/TURKEY
E-mail: eekici@comu.edu.tr

S. JAFARI
College of Vestjaelland South,
Herrestraede 11,
4200 Slagelse, Denmark
E-mail: jafari@stofanet.dk

M. ROOHI
Islamic Azad University-Ghaemshahr branch, Iran
E-mail: mehdi.roohi@gmail.com