

BALANCED MATRICES

T. AGAMA AND G. KIBITI

ABSTRACT. In this paper we introduce a particular class of matrices. We study the concept of a matrix to be balanced. We study some properties of this concept in the context of matrix operations. We examine the behaviour of various matrix statistics in this setting. The crux will be to understanding the determinants and the eigenvalues of balanced matrices. It turns out that there does exist a direct communication among the leading entry, the trace, determinants and, hence, the eigenvalues of these matrices of order 2×2 . These matrices have an interesting property that enables us to predict their quadratic forms, even without knowing their entries but given their spectrum.

1. Introduction and motivation

Matrix theory is one of the most established areas of linear algebra, deeply embedded in both theoretical research and practical applications. Its pervasiveness has made certain fundamental concepts, such as eigenvalues, eigenvectors, and quadratic forms, accessible not only to mathematicians but also to practitioners in other fields. In fact, the vastness of matrix theory is such that even non-mathematicians frequently encounter and understand its core elements, reflecting the extent to which this discipline has matured. Yet, matrix theory continues to offer fertile ground for discovery, particularly in the classification and analysis of new matrix types with distinct structural properties.

Consider a typical 2×2 matrix of the form

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A crucial step in understanding the behavior of this matrix is determining its spectrum, which consists of its eigenvalues. These eigenvalues can be found by solving the characteristic equation

$$|A - \lambda I| = 0,$$

where λ denotes any eigenvalue of A . The spectrum provides insight into the matrix's action on a vector space, encapsulating information about scaling factors in linear transformations. However, calculating the spectrum, especially for higher-dimensional matrices, can be computationally intensive, necessitating sophisticated techniques.

Date: October 4, 2024.

2000 Mathematics Subject Classification. Primary 54C40, 14E20; Secondary 46E25, 20C20.

Key words and phrases. balanced, fully-balanced, vertically balanced, horizontally balanced, discrepancy.

For symmetric matrices, the quadratic form provides an alternative way to explore the matrix's properties. In the case of matrix A , the quadratic form is given by

$$F(x, y) := ax^2 + bxy + dy^2,$$

which offers a geometric perspective on the matrix's influence on vectors in \mathbb{R}^2 . The quadratic form is particularly important in optimization and geometry, where it aids in studying curvature and other key geometric properties.

Despite the wealth of tools available for matrix analysis, the computation of the spectrum and quadratic form typically requires a detailed understanding of the matrix's entries. In what follows, we study a special class of matrices for which both the spectrum and the quadratic form can be effectively determined without the need to solve the characteristic equation or fully compute the matrix entries. In this class, the eigenvalues and quadratic forms are directly related to simple operations on the entries of the matrix, thereby simplifying the analysis.

For any matrix A in this class, we find that the sums of the row and column entries approximate the maximum eigenvalue in the spectrum:

$$\sum_{r=1}^2 a_{i,r} \approx \sum_{s=1}^2 a_{s,j} \approx \max(M),$$

and that the differences between the row and column entries approximate the minimum eigenvalue:

$$|a_{i,1} - a_{i,2}| \approx |a_{1,j} - a_{2,j}| \approx \min(M),$$

where M is the spectrum of A . These simple relations provide a direct method for estimating the spectrum of the matrix, bypassing the need for solving complex characteristic equations.

Moreover, for symmetric matrices in this class, the quadratic form can be reconstructed directly from the eigenvalues, without requiring explicit knowledge of the matrix. In particular, the quadratic form is approximated by one of the following expressions, depending on the eigenvalues λ_1 and λ_2 :

$$F(x, y) \approx \left(\frac{\lambda_2 - |\lambda_1|}{2} \right) (x + y)^2 + 2|\lambda_1|xy,$$

or

$$F(x, y) \approx \left(\frac{\lambda_2 + |\lambda_1|}{2} \right) (x + y)^2 - 2|\lambda_1|xy.$$

This framework not only simplifies the process of matrix analysis but also highlights a new class of matrices where key characteristics such as the spectrum and quadratic form can be efficiently deduced from elementary operations on the entries. This novel approach offers potential applications in areas that require rapid or simplified matrix diagnostics, particularly in high-dimensional settings where traditional methods may be computationally prohibitive.

2. Balanced matrices

Definition 2.1. Let $\mathbb{M}_{n \times m}(\mathbb{R})$ be the space of $n \times m$ matrices with real entries. Then $A = (a_{ij}) \in \mathbb{M}_{n \times m}(\mathbb{R})$, a non-zero matrix, is said to be horizontally balanced if

$$\sum_{j=1}^m a_{r,j}^2 \approx \sum_{j=1}^m a_{s,j}^2,$$

for $1 \leq s < r \leq n$. Similarly, It is said to be vertically balanced if

$$\sum_{i=1}^n a_{i,r}^2 \approx \sum_{i=1}^n a_{i,s}^2,$$

for $1 \leq s < r \leq n$. Any matrix A is said to be fully balanced if it is both vertically and horizontally balanced.

Example 2.2. Perhaps a good straight-forward example of a fully balanced matrix is the identity matrix, since it abides by the above criterion. Another obvious example of a fully-balanced matrix is given by

$$\lambda \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

for $\lambda \in \mathbb{R}$. Hence for $A \in \mathbb{M}_3(\mathbb{R})$, the unity matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

the definition 2.1 about fully balanced-matrices holds, for we have

$$(2.1) \quad 1^2 + 1^2 + 1^2 = 1^2 + 1^2 + 1^2 = 3 \quad \text{Horizontally}$$

$$(2.2) \quad 1^2 + 1^2 + 1^2 = 1^2 + 1^2 + 1^2 = 3 \quad \text{Vertically.}$$

Through out this paper we choose for simplicity to specialize our study to fully-balanced square matrices. Letting the paper to be taken this way brings more questions around.

3. Elementary properties of fully balanced matrices

In this section we examine some properties of fully balanced matrices. We investigate how these properties are preserved under various matrix operations. We prove the theorem for the sums and products of 2×2 matrices. Later, we will prove a result that will enable us to extend these properties to higher order matrices.

Theorem 3.1. *Let $A, B \in \mathbb{M}_n(\mathbb{R})$ be fully-balanced matrices and let $\lambda \in \mathbb{R}$. Then the following remain valid:*

- (i) *The transpose A^T is also fully balanced.*
- (ii) *The multiple λA is also fully balanced.*

- (iii) *The sum of any 2×2 fully-balanced matrix with positive entries is still fully-balanced. In other words, the notion of balanced matrices is preserved under matrix addition.*
- (iii) *The product of any 2×2 fully-balanced matrices with positive entries is still fully balanced.*
- (iv) *The inverse of any 2×2 non-singular fully-balanced matrix is still fully balanced. That is, if A is a non-singular fully-balanced 2×2 matrix, then so is A^{-1} .*

Proof. (i) Let $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ be fully-balanced. Then by definition 2.1, it is both vertically and horizontally balanced. Since the transpose of a vertically balanced matrix becomes a horizontally-balanced matrix and vice-versa, it follows that the transpose A^T must be fully balanced.

(ii) The fact that λA is also fully balanced is obvious.

- (iii) Consider the 2×2 matrices

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Then by definition 2.1 the following holds

$$(3.1) \quad a_1^2 + b_1^2 \approx c_1^2 + d_1^2, \quad a_2^2 + b_2^2 \approx c_2^2 + d_2^2$$

and

$$(3.2) \quad b_1^2 + d_1^2 \approx a_1^2 + c_1^2, \quad b_2^2 + d_2^2 \approx a_2^2 + c_2^2.$$

Using the relation $a_1^2 + b_1^2 \approx c_1^2 + d_1^2$ and $a_1^2 + c_1^2 \approx b_1^2 + d_1^2$, we observe that $c_1^2 \approx b_1^2$. Since the entries are positive, we must have $c_1 \approx b_1$. Using the equation further shows that $a_1 \approx d_1$, $a_2 \approx d_2$ and $b_2 \approx c_2$. Their sum is given by

$$A + B = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}.$$

We claim that the matrix $A + B$ is also fully balanced. For we observe that

$$(3.3) \quad \begin{aligned} (a_1 + a_2)^2 + (b_1 + b_2)^2 &= a_1^2 + a_2^2 + 2|a_1||a_2| + b_1^2 + b_2^2 + 2|b_1||b_2| \\ &= (a_1^2 + b_1^2 + 2|b_1||b_2|) + (a_2^2 + b_2^2 + 2|a_1||a_2|) \\ &\approx (c_1^2 + d_1^2 + 2|b_1||b_2|) + (c_2^2 + d_2^2 + 2|a_1||a_2|) \\ &\approx (c_1^2 + c_2^2 + 2|c_1||c_2|) + (d_1^2 + d_2^2 + 2|d_1||d_2|) \\ &\approx (c_1 + c_2)^2 + (d_1 + d_2)^2 \end{aligned}$$

by leveraging the relations in 3.1 and 3.2. Thus the matrix $A + B$ is horizontally balanced. Similarly we observe that

$$\begin{aligned} (b_1 + b_2)^2 + (d_1 + d_2)^2 &= b_1^2 + b_2^2 + 2|b_1||b_2| + d_1^2 + d_2^2 + 2|d_1||d_2| \\ &= (b_1^2 + d_1^2 + 2|d_1||d_2|) + (b_2^2 + d_2^2 + 2|b_1||b_2|) \\ &\approx (a_1^2 + a_2^2 + 2|a_1||a_2|) + (c_1^2 + c_2^2 + 2|c_1||c_2|) \\ &\approx (a_1 + a_2)^2 + (c_1 + c_2)^2 \end{aligned}$$

where we have used the relation 3.1 and 3.2. Thus the matrix $A + B$ is also vertically balanced. Therefore it must be fully balanced.

(iv) We now show that their product is also fully balanced. Their product is given by

$$\begin{aligned} AB &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} (a_1a_2 + b_1c_2)^2 + (a_1b_2 + b_1d_2)^2 &= (a_1^2a_2^2 + a_1^2b_2^2) + (b_1^2c_2^2 + b_1^2d_2^2) + 2a_1a_2b_1c_2 + 2a_1b_2b_1d_2 \\ &\approx a_1^2(c_2^2 + d_2^2) + b_1^2(c_2^2 + d_2^2) + 2d_1a_2c_1c_2 + 2b_2c_1d_1d_2 \\ &\approx d_1^2(c_2^2 + d_2^2) + c_1^2(a_2^2 + b_2^2) + 2d_1a_2c_1c_2 + 2b_2c_1d_1d_2 \\ &\approx (c_1a_2 + d_1c_2)^2 + (c_1b_2 + d_1d_2)^2 \end{aligned}$$

and the product is horizontally balanced. A similar argument will show that, the product is also vertically balanced. Therefore, the product is fully balanced.

(iv) The fact that A^{-1} is fully balanced, given that A is fully balanced is obvious. \square

4. Trace, determinants and eigenvalues associated with balanced matrices

In this section we examine various statistics associated with balanced matrices. We study the behaviour of their trace, their determinants, their eigenvalues, their eigenvectors and their corresponding interplay in this setting.

Proposition 4.1. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a fully-balanced square matrix with positive real entries. If $a < \epsilon$ then $Tr(A) \leq N_\epsilon$ for any $\epsilon > 0$ and where N_ϵ is a constant depending on ϵ .

Proof. By invoking Theorem 3.1, the result follows immediately. \square

Remark 4.1. Theorem 3.1 relates the leading entry of a 2×2 fully-balanced matrix to their trace. Indeed if the leading entry is small enough then the trace must not be too big. Similarly if the leading entry is somewhat large then their trace must be large. This property is archetypal of balanced matrices.

Proposition 4.1 does highlights the importance of balanced matrices. It tells us for the most part that the leading entry or more generally the diagonal entry of any 2×2 fully-balanced matrices has a profound connection with their eigenvalues, and hence influences their eigen-vectors. Indeed by using the well-known elementary relation

$$\lambda_1 + \lambda_2 = Tr(A),$$

where λ_1, λ_2 are the eigen-values of the fully-balanced matrix A , then by leveraging Proposition 4.1, we observe that if the leading entry is small enough then each of the eigen-values must not be too big provided the spectrum is real and has only

positive eigen values. Similarly if the leading entry is somewhat large then at least one of the eigen-values must be large under the requirement of the structure of the spectrum. A similar description could be carried out to relate the leading entry of balanced matrices to their determinants, using the well-known elementary relation (See [2])

$$\det(A) = \lambda_1 \lambda_2,$$

where each λ_i for $1 \leq i \leq 2$ is an eigenvalue of A . This is a description characteristic of very rare class of matrices of which balanced matrices is a sub-class.

Balanced matrices are very important theoretically and could have real use application in areas of applied mathematics. The simple and the most basic example of a fully-balanced matrix, as we have seen, is the identity matrix. The determinant of this matrix is always 1. This gives us a clue of the distribution of balanced matrices.

Remark 4.2. Henceforth, when we say a balanced matrix it will imply a fully-balanced matrix. Otherwise, we will specify the context of balanced matrix.

Eigenvalues and eigenvectors are extremely important statistics in the study of matrices. Knowing these two for any matrix can be useful in practice. The quest to find an eigenvalue-value and, hence, eigenvector features very often in other various applied areas such as physics. The next result helps us to predict upto a smaller error eigenvalues and hence eigenvectors of balanced matrices, without having to undergo the traditional procedure. This result relates the sums and differences of the entries of balanced matrices to the least and worst eigenvalue for 2×2 balanced matrices. It will be great to extend this result to matrices of higher orders. But for the time being we content ourselves with the following:

Theorem 4.3. *Let $A \in \mathbb{B}_2(\mathbb{R}^+)$, the spaces of 2×2 balanced matrices with each $a_{ij} \geq 1$. If $M = \{|\lambda_1|, |\lambda_2|\}$ is the set of eigen-values of A , then*

$$\sum_{r=1}^2 a_{i,r} \approx \sum_{s=1}^2 a_{s,j} \approx \max(M)$$

for $1 \leq s, r \leq 2$ and

$$|a_{i,1} - a_{i,2}| \approx |a_{1,j} - a_{2,j}| \approx \min(M)$$

where $1 \leq i, j \leq 2$.

Proof. Consider the fully-balanced matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then by recalling the well-known elementary relation (See [2])

$$\det(B) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{Tr}(B)$$

for any matrix B , we can write

$$\begin{aligned} \det(A) &= \lambda_1 \lambda_2 \\ &= ad - bc \\ (4.1) \quad &\approx a^2 - b^2. \end{aligned}$$

Since $Tr(A) = \lambda_1 + \lambda_2$, then it follows that $|a - b| \approx |\lambda_1|$ and $a + b \approx |\lambda_2|$ or vice-versa. Similarly we can write

$$\begin{aligned} \det(A) &= \lambda_1 \lambda_2 \\ &= ad - bc \\ (4.2) \quad &\approx d^2 - c^2. \end{aligned}$$

Again using the relation $Tr(A) = \lambda_1 + \lambda_2$, then it follows that $|c - d| \approx |\lambda_1|$ and $c + d \approx |\lambda_2|$ or vice-versa. Again it follows that

$$\begin{aligned} \det(A) &= \lambda_1 \lambda_2 \\ &= ad - bc \\ (4.3) \quad &\approx a^2 - c^2. \end{aligned}$$

Using the relation $Tr(A) = \lambda_1 + \lambda_2$, then it follows that $|a - c| \approx |\lambda_1|$ and $a + c \approx |\lambda_2|$ or vice-versa. Also, we have

$$\begin{aligned} \det(A) &= \lambda_1 \lambda_2 \\ &= ad - bc \\ (4.4) \quad &\approx d^2 - b^2 \end{aligned}$$

and it follows that $|b - d| \approx |\lambda_1|$ and $b + d \approx \lambda_2$ or vice-versa, by using the relation $Tr(A) = \lambda_1 + \lambda_2$. Without loss of generality, we let $\lambda_2 = \max(M)$ and $\min(M) = \lambda_1$. Then it follows that $b + d \approx a + c \approx a + b \approx c + d \approx \lambda_2 = \max(M)$ and $|b - d| \approx |a - c| \approx |c - d| \approx |a - b| \approx \lambda_1 = \min(M)$. For suppose $b + d \approx |b - d|$, then it follows that either $d \approx 0$ or $b \approx 0$, which contradicts the minimality of each of the a_{ij} 's. Similarly, let us suppose that $b + d \approx |a - c|$. Then it follows that

$$\begin{aligned} b + d &\approx a - c \\ &\approx d - c \end{aligned}$$

and we have that $b \approx -c$ if and only if $c \approx 0$, which violates the minimality of a_{ij} for $1 \leq i, j \leq 2$. Also in the case where $b + d = -(a - c)$, then it follows that $d \approx 0$, which is a contradiction. Again if $b + d \approx |c - d|$, then we see that

$$\begin{aligned} b + d &\approx c - d \\ &\approx b - d \end{aligned}$$

and it follows that $d \approx 0$. On the other hand, we will have that $c \approx 0$, both of which contradicts the **minimality** of a_{ij} . Thus by leveraging the fact that A is fully-balanced, in a similar manner for other cases the result follows immediately. \square

Corollary 4.1. Let $A_1, A_2 \in \mathcal{B}_2(\mathbb{R}^+)$ with $a_{ij} \geq 1$ and let $E_{\max}(A_1)$ denotes the maximum eigen-value of A_1 . Then

$$E_{\max}(A_1 + A_2) \approx E_{\max}(A_1) + E_{\max}(A_2).$$

Proof. Consider the 2×2 fully-balanced matrices given by

$$A_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \text{ and } A_2 := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then by Theorem 3.1, their sum

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

is also fully-balanced. By Theorem 4.3, $E_{\max}(A_1 + A_2) \approx b_1 + b_2 + d_1 + d_2$, and the result follows immediately. \square

Remark 4.4. Before we state the next result, we review the following terminologies concerning matrices in general.

Definition 4.5. By a block $n \times m$ matrix, we mean any matrix of the form

$$A = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{m1} & C_{m2} & \cdots & C_{mn} \end{pmatrix},$$

and where each C_{ij} is a sub-matrix of A for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 4.6. Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ given by

$$A := \begin{pmatrix} a_{1.1} & a_{1.2} \cdots & a_{1.n} \\ a_{2.1} & a_{2.2} \cdots & a_{2.n} \\ \vdots & & \\ a_{m.1} & a_{m.2} & \cdots a_{m.n} \end{pmatrix}.$$

Then we say a matrix B is an interior of A if it is a sub-matrix of A .

Conjecture 4.1. Let $A \in \mathcal{B}_n(\mathbb{R})$, the space of square balanced-matrices. Then there exist some interior of A that is also balanced.

Remark 4.7. By thinking of a matrix as a system, Conjecture 4.1 roughly speaking conveys the notion that, if a bigger system is balanced, then there must be a sub-system that is also balanced.

5. Discrepancies of fully-balanced matrices

In this section, in the spirit of proving some weaker versions of Conjecture 4.1 we introduce the notion of discrepancy of fully-balanced matrices. It turns out that Conjecture 4.1 is somewhat easier to attack in this setting.

Definition 5.1. Let $A \in \mathbb{M}_{n \times m}(\mathbb{R})$. Then by the discrepancy of the matrix A along rows, we mean the value

$$\sum_{j=1}^m a_{ij}.$$

Similarly, by the discrepancy along columns, we mean the value

$$\sum_{i=1}^n a_{ij}.$$

Definition 5.2. Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ and let

$$M_i = \frac{1}{m} \sum_{j=1}^m a_{ij}.$$

Then we say the discrepancy is fair along rows if for each $1 \leq i \leq m$ then $|M_i - a_{ij}| < \epsilon$ for all $1 \leq j \leq m$, where $\epsilon > 0$ is small.

The discrepancy is unfair along rows if for some a_{ij} ($j = 1, 2 \dots m$), there exist some n_0 such that

$$|M_i - a_{ij}| > N$$

for all $N \geq n_0$.

Remark 5.3. Next we prove some few propositions concerning fully-balanced matrices, in the context of discrepancy.

Theorem 5.4. *Let $A \in \mathcal{B}_{2 \times 2}(\mathbb{R}^+)$, the space of fully-balanced 2×2 matrices. Then A has a fair discrepancy along rows if and only if it has a fair discrepancy along columns.*

Proof. Let $A \in \mathcal{B}_{2 \times 2}(\mathbb{R}^+)$ and suppose A has a fair discrepancy along rows. Then it follows that for each $1 \leq i \leq 2$

$$|M_i - a_{ij}| < \epsilon$$

for small arbitrary $\epsilon > 0$ and for all $1 \leq j \leq 2$. This implies that $|M_1 - a_{1j}| < \epsilon$ and hence $a_{11} \approx a_{12}$ and $|M_2 - a_{2j}| < \epsilon$ for $\epsilon > 0$ and hence $a_{21} \approx a_{22}$. Since A is fully-balanced, it follows that $a_{11} \approx a_{22}$ and $a_{21} \approx a_{12}$. It follows that A must have a fair discrepancy along columns. The converse, on the other hand, follows similar approach. \square

Proposition 5.1. *Let $A \in \mathcal{B}_{2 \times 2}(\mathbb{R}^+)$. If A has a fair discrepancy on exactly one row, then it must have fair discrepancy on rows.*

Proof. Specify $A \in \mathcal{B}_{2 \times 2}(\mathbb{R}^+)$ given by

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose A has a fair discrepancy along exactly one row. Without loss of generality, let us assume the fair discrepancy occurs on the first row, then by Theorem 5.4, it must be that $a \approx b$. Since A is fully-balanced, Theorem 3.1 tells us that $a \approx b \approx c \approx d$. This implies that A has a fair discrepancy on rows, and the proof of the proposition is complete. \square

Conjecture 5.1. Let $\epsilon > 0$ and let $A \in \mathbb{M}_{n \times m}(\mathbb{R})$ be a fully balanced matrix. The average discrepancy along rows is given by

$$M = \frac{1}{m} \sum_{j=1}^m a_{ij}.$$

If

$$|M_i - a_{ij}| < \epsilon$$

for a fixed $1 \leq i \leq n$, then $|M_i - a_{ij}| < \epsilon$ for all $1 \leq i \leq n$.

Remark 5.5. Conjecture 5.1 tells us that if the discrepancy of a fully-balanced matrix along a given row is fair, then it must be fair on all other rows. In other words, a fair discrepancy on a given row is propagated to all other rows.

In general the determinant of matrices is not an approximate homomorphism. That is, the determinant of the sums of matrices may not have the same distribution as the sum of each determinant. These two statistics may be close to each other and could very well be far from each other. Here is where the concept of balanced matrices plays an important role. Given k distinct matrices, we say the determinant is an approximate homomorphism if the relation holds:

$$\det\left(\sum_{k=1}^n A_k\right) \approx \sum_{k=1}^n \det(A_k).$$

The next result clarifies and gives a more formal context to the ensuing discussion.

Theorem 5.6. *Let $A, B \in \mathcal{B}_2(\mathbb{R}^+)$, where $\mathcal{B}_2(\mathbb{R}^+)$ is the space of 2×2 balanced-matrices with $a_{ij} \geq 1$ and $b_{ij} \geq 1$. Let $\mathcal{M} = \{|\lambda_1|, |\lambda_2|\}$ be the spectrum of A . If $\min(\mathcal{M}) \approx 0$ and B has a fair discrepancy along rows or columns, then*

$$\det(A + B) \approx \det(A) + \det(B).$$

Proof. Consider the 2×2 fully-balanced matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Then, by Theorem 3.1 we have the fully-balanced matrix

$$A + B := \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix}.$$

It follows that

$$\begin{aligned} \det(A + B) &= (a_1 + b_1)(a_4 + b_4) - (a_2 + b_2)(a_3 + b_3) \\ &= (a_1 a_4 - a_2 a_3) + (b_1 b_4 - b_2 b_3) + (a_1 b_4 + b_1 a_4 - a_2 b_3 - b_2 a_3) \\ &\approx \det(A) + \det(B) + 2(b_1 a_1 - a_2 b_2) \\ &\approx \det(A) + \det(B) + 2b_1(a_1 - a_2) \end{aligned}$$

where we have utilized the fact that A and B are fully-balanced matrices, and that B has a fair discrepancy along rows or columns. By using the fact that $\min(\mathcal{M}) \approx 0$, then the result follows from Theorem 4.3. \square

Remark 5.7. Theorem 5.6 tells us that the determinant can be made an approximate homomorphism on any two fully-balanced matrices of not-too-small entries, by making the least element in the spectrum of one matrix negligible and avoiding outliers in the entries and rows of the second.

Conjecture 5.2. *Let $A, B \in \mathcal{B}_n(\mathbb{R}^+)$, where $\mathcal{B}_n(\mathbb{R}^+)$ is the space of $n \times n$ balanced-matrices with $a_{ij} \geq 1$ and $b_{ij} \geq 1$. Let $\mathcal{M} = \{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\}$ be the spectrum of A . If $\min(\mathcal{M}) \approx 0$ and B has a fair discrepancy along rows or columns, then*

$$\det(A + B) \approx \det(A) + \det(B).$$

6. Quadratic forms associated with balanced matrices

In this section we examine various forms associated with balanced matrices. For the time being we study the quadratic forms associated with fully-balanced 2×2 matrices. We review therefore the following definitions.

Definition 6.1. Let

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any symmetric matrix. Then by the quadratic form of A , we mean any expression of the form

$$F(x, y) := ax^2 + 2bxy + dy^2.$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a fully-balanced symmetric matrix. Then the associated quadratic form can be written as

$$\begin{aligned} F(x, y) &:= ax^2 + 2bxy + dy^2 \\ &\approx a(x^2 + y^2) + 2bxy \\ &\approx a(x + y)^2 + 2(b - a)xy. \end{aligned}$$

By using Theorem 4.3, we can write

$$\begin{aligned} F(x, y) &\approx a(x + y)^2 + 2(b - a)xy \\ &\approx \left(\frac{\lambda_2 - |\lambda_1|}{2} \right) (x + y)^2 + 2|\lambda_1|xy \end{aligned}$$

if $b > a$. Similarly if $b < a$, then the quadratic form looks a lot like

$$\begin{aligned} F(x, y) &= ax^2 + 2bxy + dy^2 \\ &\approx \left(\frac{\lambda_2 + |\lambda_1|}{2} \right) (x + y)^2 - 2|\lambda_1|xy, \end{aligned}$$

where λ_2 and λ_1 are the worst and the least eigenvalues of A respectively. More formally we launch a proposition concerning the quadratic forms associated with 2×2 fully-balanced symmetric matrices.

Proposition 6.1. *Let A be a fully-balanced 2×2 symmetric matrix, such that $a_{ij} \geq 1$ for $1 \leq i, j \leq 2$. Let $\mathcal{N} := \{|\lambda_1|, |\lambda_2|\}$ be the spectrum of A , and where $\max(\mathcal{N}) = |\lambda_2|$ and $\min(\mathcal{N}) = |\lambda_1|$. Then one of the following is an approximation of the quadratic form of A*

$$F(x, y) \approx \left(\frac{\lambda_2 - |\lambda_1|}{2} \right) (x + y)^2 + 2|\lambda_1|xy$$

or

$$F(x, y) \approx \left(\frac{\lambda_2 + |\lambda_1|}{2} \right) (x + y)^2 - 2|\lambda_1|xy.$$

Proof. The result follows from the ensuing discussion concerning quadratic forms of fully-balanced matrices. \square

Remark 6.2. Proposition 6.1 tells us that we do not necessarily need the entries of a 2×2 symmetric fully-balanced matrices to compute the values of their quadratic forms. Given the eigenvalues of A , we can with some precision predict the quadratic form of any fully-balanced 2×2 symmetric matrices, without knowing the entries.

7. Further remarks

In this paper we have introduced the concept of balanced matrices, where we studied various matrix statistics underlying this concept. Much emphasis was placed on 2×2 fully-balanced matrices. This is just the beginning of a series of papers regarding this concept. There is much optimistic work in progress to extend these results for lower order square matrices to matrices of higher orders. Another quest, in the not too distant future, will be to find if there really exist some bit of interaction between this class of matrices and matrices in general. This could provide a new window through which to study matrix theory.

1.

REFERENCES

1. Meyer, Carl D, *Matrix analysis and applied linear algebra* vol. 71, Siam, 2000.
2. Roman, Steven and Axler, S and Gehring, FW, *Matrix analysis and applied linear algebra*, vol. 3, Springer, 2005.

DEPARTMENT OF MATHEMATICS, AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCE, GHANA
E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

DEPARTMENT OF MATHEMATICS, AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCE, GHANA
E-mail address: Gael@aims.edu.gh