Abstract. The identity $\sin^2(x) + \cos^2(x) = 1$ can be used additionally except for the Pythagorean theorem, in proof of Fermat’s last theorem (FLT).

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1. Introduction.

In a remarkable 1940 treatise entitled *The Pythagorean Proposition*, Elisha Scott Loomis (1852–1940) presented literally hundreds of distinct proofs of the Pythagorean theorem. Loomis provided both “algebraic proofs” that make use of similar triangles, as well as “geometric proofs” that make use of area reasoning. Notably, none of the proofs in Loomis’s book were of a style one would be tempted to call “trigonometric”. Indeed, toward the end of his book ([1, p.244]) Loomis asserted that all such proofs are circular: There are no trigonometric proofs [of the Pythagorean theorem, because all of the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean theorem; because of this theorem we say $\sin^2(x) + \cos^2(x) = 1$ etc. Along the same lines but more recently, in the discussion page behind Wikipedia’s Pythagorean theorem entry, one may read that a purported proof was once deleted from the entry because it “...depend[ed] on the veracity of the identity $\sin^2(x) + \cos^2(x) = 1$, which is the Pythagorean theorem . . .” [2].

Given this unique relation of the rectangle, it would be paradoxical to believe that Fermat’s last theorem, could only be based on this relation. But as we will show below, it produces a process that fully reveals the secret of proof.

2. Sine and cosine of acute angles

We begin by defining the sine and cosine functions for positive acute angles, independently of the Pythagorean theorem, as ratios of sides of similar right triangles. Given $x \in (0, \pi/2)$, let $R_\alpha$ be the set of all right triangles containing an angle of measure $x$, and let $T$ be one such triangle. Because the angle measures in $T$ add up to $\pi$ (see Euclid’s Elements, I.32), $T$ must have angle measures $\pi/2, \pi/2 - x$ and $x$. The side opposite to the right angle is the longest side (see Elements I.19), called the hypotenuse of the right triangle; we denote its length by $H$. Let $AC$ denote the length of the side of $\angle = C \angle$ adjacent to the angle of measure $x$, and $AB$ the length of the opposite side (the figure is given below (Fig.1)
The identity \( \sin^2(x) + \cos^2(x) = 1 \) applied to a right triangle with legs \( a, b \) and hypotenuse \( c \) gives

\[
\frac{c^2}{a^2} + \frac{b^2}{a^2} = 1 \quad \text{or} \quad c^2 + b^2 = a^2.
\]

Let us first prove what is going on with the integers solutions of the relation \( c^2 + b^2 = a^2 \), where \( c, b, a \in \mathbb{R}^+ \).

3. Finding integer solutions of the Pythagorean equation

But in Diophantine analysis we will have only all integers. We start with the relation \( c^2 + b^2 = a^2 \) or,

\[
\frac{c^2}{a^2} + \frac{b^2}{a^2} = 1 \quad (1).
\]

We can therefore here associate this relationship with the original trigonometric, i.e. will apply \( \sin^2(x) + \cos^2(x) = 1 \) were \( \sin^2(x) = \frac{c^2}{a^2} \) & \( \cos^2(x) = \frac{b^2}{a^2} \), where \( 0 \leq x < \frac{\pi}{2} \) &

\[ 0 \leq \sin(x) < 1, 0 \leq \cos(x) < 1. \]

The relation (2) it can be done if we divide by \( \cos(x) \), as follows

\[
\tan^2(x) + 1 = \frac{1}{\cos^2(x)}.
\]

The relation corresponds to this form will be ..
\[ \frac{c^2}{b^2} + 1 = \frac{a^2}{b^2} \] (3) i.e. if \( \frac{a^2}{b^2} = m^2 \geq 1 \) where \( m \in \mathbb{Z}^+ \), then \( \frac{c^2}{b^2} = (m-l)^2 \geq 1 \) (6). With relation 3 we satisfy the equation and we get..

\[(m-l)^2 + 1 = m^2 \] then apply \( m = \frac{l^2 + 1}{2l} \) & \( m - l = \frac{1 - l^2}{2l} \) (7) where \( m, l \in \mathbb{Q}^+ \). But because we don't want fractional numbers i.e. \( a, b, c \in \mathbb{Q}^+ - \mathbb{Z}^+ \) but only \( a, b, c \in \mathbb{Z}^+ \). To purpose that's what we choose \( l = \frac{f}{n} \) where \( (f,n)=1 \), where \( f, n \in \mathbb{Z}^+ \) relatively primes. Now the relation (4) becomes

\[ m = \frac{f^2 + n^2}{2f \cdot n} \] & \[ m - l = \frac{n^2 - f^2}{2f \cdot n} \] (8). From the relations (5&6) if \( b = 2f \cdot n \cdot v \), where \( v \in \mathbb{Z}^+ \), arise..

\[ a = v \cdot (f^2 + n^2) \] & \[ c = v \cdot (f^2 - n^2) \]. The final relation therefore for its integers solutions of relation \( c^2 + b^2 = a^2 \) are...

\[ a = v \cdot (f^2 + n^2) \] & \[ c = v \cdot (f^2 - n^2) \] & \( b = 2f \cdot n \cdot v \), \( \{f, n, v\} \in \mathbb{Z}^+ \)

4. The last theorem of Fermat and how is proved trigonometrically

If we have three positive integers \( a, b, \) and \( c \) satisfy the equation \( c^n + b^n = a^n \) for any integer value of \( n \) greater than 2. The cases \( n = 1 \) and \( n = 2 \) have been known since antiquity to have an infinite number of solutions [3]. The proposition was first conjectured by Pierre de Fermat in 1637 in the margin of a copy of Arithmetica; Fermat added that he had a proof that was too large to fit in the margin. However, there were first doubts about it since the publication was done by his son without his consent, after Fermat's death. After 358 years of effort by mathematicians, the first successful proof was released in 1994 by Andrew Wiles, and formally published in 1995; it was described as a "stunning advance" in the citation for Wiles's Abel Prize award in 2016.

This unsolved problem stimulated the development of algebraic number theory in the 19th century and the proof of the modularity theorem in the 20th century. It is among the most notable theorems in the history of mathematics and prior to its proof was in the Guinness Book of World Records as the "most difficult mathematical problem" in part because the theorem has the largest number of unsuccessful proofs.
The conjecture..

As we have mentioned, we can equate the equation \( c^n + b^n = a^n \) (1) with the trigonometric equation form

\[
\sin^n(x) + \cos^n(x) = 1 \tag{2}
\]

if apply

\[
\cos(x) = \frac{c^n}{a^n} \quad \text{and} \quad \sin(x) = \frac{b^n}{a^n} = \cos^n(x) \tag{3}
\]

The proof passes though 2 parts to prove that, it does not apply for power for even numbers greater than 2, i.e. \( n > 2 \) and odd numbers greater of the unit i.e. \( n > 1 \).

Part 1.

The equivalent Diophantine trigonometric equation \( \sin^{2k+1}(x) + \cos^{2k+1}(x) = 1 \tag{i1} \) has no solutions for \( k \geq 1 \).

Proof...

Let’s assume that \( x \) is a solution of equation (i1). We can easily (because \( 0 \leq \cos(x) \leq 1 \) & \( 0 \leq \sin(x) \leq 1 \) find that.. 

\[
\cos^{2k+1}(x) \leq \cos^2(x) \quad \text{and} \quad \sin^{2k+1}(x) \leq \sin^2(x) \tag{i2}
\]

if in at least one of the relations (i2), the inequality applies then if we add in parts we will have..

\[
\sin^{2k+1}(x) + \cos^{2k+1}(x) < 1 \tag{i3}
\]

Therefore the trigonometric solution of (i1) will result from the group..

\[
S = \left\{ \begin{array}{ll}
\cos^{2k+1}(x) = \cos^2(x) \\
\sin^{2k+1}(x) = \sin^2(x)
\end{array} \right. \Rightarrow \left\{ \begin{array}{ll}
\cos^2(x)(\cos^{2k-1}(x) - 1) = 0 \\
\sin^2(x)(\sin^{2k-1}(x) - 1) = 0
\end{array} \right. \Rightarrow \left\{ \begin{array}{ll}
\cos(x) = 0 \vee \cos(x) = 1 \\
\sin(x) = 0 \vee \sin(x) = 1
\end{array} \right. \Rightarrow
\]

The system \( <S> \) ends up the solutions are .. (\( t \in Z, x = 2\pi t \)) | (\( t \in Z, (x = \frac{\pi}{2} + 2\pi t ) \)).

This is the only solutions of the system and will be the immediate results will therefore be..

Great results

1. \( \sin(x) = 1, \cos(x) = 0 \Rightarrow b = 0 \) and \( c = a \)
2. \( \sin(x) = 0, \cos(x) = 1 \Rightarrow a = b \) and \( c = 0 \).
Part 2.

The equivalent Diophantine trigonometric equation \( \sin^{2k}(x) + \cos^{2k}(x) = 1 \) (i4) has no solutions for \( k > 1 \).

Proof..

For the same reasons as before assume that \( x \) is a solution of equation (i4). Because Apply the restrictions \( 0 \leq \cos(x) \leq 1 \) & \( 0 \leq \sin(x) \leq 1 \) find that..

\[
\cos^{2k}(x) \leq \cos^2(x) & \sin^{2k}(x) \leq \sin^2(x) \quad (i5)
\]

The trigonometric solution of (i4) as clustered system will take the form ..

\[
S' = \begin{cases} 
\cos^{2k}(x) = \cos^2(x) \\
\sin^{2k}(x) = \sin^2(x)
\end{cases} \Rightarrow \begin{cases} 
\cos^2(x)(\cos^{2k-2}(x) - 1) = 0 \\
\sin^2(x)(\sin^{2k-2}(x) - 1) = 0
\end{cases} \Rightarrow \begin{cases} 
\cos(x) = 0 \lor \cos(x) = \pm 1 \\
\sin(x) = 0 \lor \sin^2(x) = 1
\end{cases} \Rightarrow
\]

\[
\begin{cases} 
\cos(x) = 0 \lor \cos(x) = \pm 1 \\
\sin(x) = 0 \lor \cos(x) = 0
\end{cases}
\]

The system \( <S'> \) ends up the solutions are ..

1. \((t \in \mathbb{Z}, x = 2\pi t)\)\((t \in \mathbb{Z}, x = -\frac{\pi}{2} + 2\pi t, x = \frac{\pi}{2} + 2\pi t)\).

2. \((t \in \mathbb{Z}, x = 2\pi t + \pi)\)\((t \in \mathbb{Z}, x = -\frac{\pi}{2} + 2\pi t, x = \frac{\pi}{2} + 2\pi t)\).

The only system solutions will be...

**Great results**

1. \(\sin(x)=1 , \cos(x)=0\) \(\Rightarrow b=0 \) and \( c=a \)

2. \(\sin(x)=-1 , \cos(x)=0\) \(\Rightarrow b=0 \) and \( c=-a \)

3. \(\sin(x)=0 , \cos(x)=1\) \(\Rightarrow c=0 \) and \( b=a \)

4. \(\sin(x)=0 , \cos(x)=-1\) \(\Rightarrow c=0 \) and \( b=-a \)

So the relations that satisfy Fermat's equation (1) for the case than \( n = 2k + 1 \) & \( n=2k \), \( n > 2 \) indicate that there can be no positive integer solution for variables \( a, b, c \). This is because we are not allowed to have a variable zero in the equation of Fermat \( c^n + b^n = a^n \). Generally.
Epilogue..
As we have seen it is clear that the same method is used for the solution in the case \( n = 2k + 1 \) & \( n = 2k \), that is, we use the same logic of the Pythagorean Triad. The proof is based on the Pythagorean triples, i.e. \( n = 2 \) of the FERMAT equation and is the result of the absolute correspondence with the trigonometric equation of the rectangular triangle, proof that is correct and fully compatible.

References....
[2].https://www.academia.edu/39691479/P%CE%A5%CE%A4%CE%97%CE%91GOREAN_TRIPLES_ First_and_second_degree_ -Last_Theorem_Fermat _under_conditions.