ˇgα-closed sets in topological spaces

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Abstract

In this paper, we introduce the notion of ˇgα-closed sets in topological spaces and investigate some of their basic properties.

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1. Introduction and Preliminaries

Levine [6,7] introduced the concept of generalized closed sets and semi-closed sets in topological spaces. Maki et al. introduced generalized α-closed sets (briefly gα-closed sets) [9] and α-generalized closed sets (briefly αg-closed sets) [8]. The concept of ˇg-closed sets [16,17], *g-closed sets [14] and ˇggs-closed sets [15] are introduced by M.K.R.S. Veera Kumar. In this paper, we introduce a new class of sets, namely, ˇgα-closed sets and present some of its properties.

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A, respectively. P(X) denotes the power set of X.

We recall the following definitions which are useful in the sequel.

Definition 1.1. A subset A of a space (X, τ) is called

1. a pre-open set [10] if A ⊆ int(cl(A)) and a pre-closed set if cl(int(A)) ⊆ A,
2. a semi-open set [7] if $A \subseteq \text{cl}(\text{int}(A))$ and a semi-closed set [7] if $\text{int}(\text{cl}(A)) \subseteq A$,

3. an $\alpha$-open set [11] if $A \subseteq \text{int}(\text{cl}(A))$ and an $\alpha$-closed set [11] if $\text{cl}(\text{int}(A)) \subseteq A$,

4. a semi-preopen set [1] if $A \subseteq \text{cl}(\text{int}(A))$ and a semi-preclosed set [1] if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$ and

5. a regular open set if $A = \text{int}(\text{cl}(A))$ and a regular closed set if $\text{cl}(\text{int}(A)) = A$.

The pre-closure (resp. semi-closure, $\alpha$-closure, semi-preclosure) of a subset $A$ of a space $(X, \tau)$ is the intersection of all pre-closed (resp. semi-closed, $\alpha$-closed, semi-preclosed) sets that contain $A$ and is denoted by $\text{pcl}(A)$ (resp. $\text{scl}(A)$, $\alpha\text{cl}(A)$, $\text{spcl}(A)$).

**Definition 1.2.** A subset $A$ of a space $(X, \tau)$ is called a

1. a generalized closed (briefly $g$-closed) set [6] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$; the complement of a $g$-closed set is called a $g$-open set,

2. a semi-generalized closed (briefly sg-closed) set [3] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$,

3. a generalized semi-closed (briefly gs-closed) set [2] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$,

4. an $\alpha$-generalized closed (briefly $\alpha g$-closed) set [8] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$,

5. a generalized $\alpha$-closed (briefly $\alpha a$-closed) set [9] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$,

6. a $g_\alpha^*$-closed set [9] if $\alpha\text{cl}(A) \subseteq \text{int}(U)$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$,

7. a generalized semi-preclosed (briefly gsp-closed) set [4] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$,

8. a generalized preregular-closed (briefly $gpr$-closed) set [5] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$,
9. a $g^*$-closed set [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $(X, \tau)$,

10. a $\tilde{g}$-closed set [16,17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$; the complement of a $\tilde{g}$-closed set is called a $\tilde{g}$-open set,

11. a $^*g$-closed set [14] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^*g$-open in $(X, \tau)$; the complement of a $^*g$-closed set is called a $^*g$-open set,

12. a $^\sharp g$s-closed set [15] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^*g$s-open in $(X, \tau)$; the complement of a $^\sharp g$s-closed set is called a $^\sharp g$s-open set and

13. a $\tilde{g}$s-closed set [12] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^\sharp g$s-open in $(X, \tau)$.

**Notation 1.3.** For a topological space $(X, \tau)$, $C(X, \tau)$ (resp. $\alpha C(X, \tau)$, $GC(X, \tau)$, $SGC(X, \tau)$, $GSC(X, \tau)$, $\alpha GC(X, \tau)$, $GaC(X, \tau)$, $G\alpha C(X, \tau)$, $GSPC(X, \tau)$, $GPRC(X, \tau)$, $G^*C(X, \tau)$, $^*GC(X, \tau)$, $^\sharp GSC(X, \tau)$, $\tilde{GSC}(X, \tau)$) denotes the class of all closed (resp. $\alpha$-closed, $g$-closed, $sg$-closed, $gs$-closed, $ga$-closed, $ga$-$^*$-closed, $gsp$-closed, $gpr$-closed, $g^*$-closed, $^*g$-closed, $^\sharp g$s-closed, $\tilde{g}s$-closed) subsets of $(X, \tau)$.

2. $\tilde{g}a$-closed sets

We introduce the following definition.

**Definition 2.1.** A subset $A$ of $(X, \tau)$ is called a $\tilde{g}a$-closed set if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^\sharp g$s-open in $(X, \tau)$.

**Theorem 2.2.** Every $\alpha$-closed set is a $\tilde{g}a$-closed set and thus every closed set is $\tilde{g}a$-closed.

**Proof.** Let $A$ be an $\alpha$-closed set in $(X, \tau)$, then $A = acl(A)$. Let $A \subseteq U$ such that $U$ is $^\sharp g$s-open in $(X, \tau)$. Since $A$ is $\alpha$-closed, $A = acl(A) \subseteq U$. This shows that $A$ is $\tilde{g}a$-closed. The second part of the theorem follows from the fact that every closed set is $\alpha$-closed.

The converse of Theorem 2.2 is not true as it can be seen by the following example.

**Example 2.3.** Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$. Here $\alpha C(X, \tau) = \{a, b\}$.
Example 2.9. Let \( \{X, \phi, \{c\}\} \) and \( \widetilde{G}_{\alpha}C(X, \tau) = \{X, \phi, \{c\}, \{b, c\}, \{a, c\}\} \) and let \( A = \{b, c\} \). Then \( A \) is not an \( \alpha \)-closed and thus it is not closed. However \( A \) is a \( \tilde{g}_{\alpha} \)-closed set.

Thus the class of \( \tilde{g}_{\alpha} \)-closed sets properly contains the classes of \( \alpha \)-closed sets and closed sets.

**Theorem 2.4.**

(a) Every \( \tilde{g}_{\alpha} \)-closed set is a \( gs \)-closed set and thus \( gsp \)-closed and \( gpr \)-closed.

(b) Every \( \tilde{g}_{\alpha} \)-closed set is a \( g_{\alpha} \)-closed set and thus \( \alpha g \)-closed.

(c) Every \( \tilde{g}_{\alpha} \)-closed set is a \( sg \)-closed set and thus semi-preclosed.

**Proof.** It follows from the definitions.

The following examples show that these implications are not reversible.

**Example 2.5.** Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \). Here \( GSC(X, \tau) = P(X) \), \( GSPC(X, \tau) = P(X) \), \( GPRC(X, \tau) = P(X) \) and \( \widetilde{G}_{\alpha}C(X, \tau) = \{X, \phi, \{a\}, \{b, c\}\} \) and let \( A = \{b\} \). Then \( A \) is \( gs \)-closed, \( gsp \)-closed and \( gpr \)-closed. However \( A \) is not a \( \tilde{g}_{\alpha} \)-closed set.

**Example 2.6.** Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{b\}, \{b, c\}\} \). Here \( G_{\alpha}C(X, \tau) = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \alpha GC(X, \tau) = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and let \( A = \{a, b\} \). Then \( A \) is \( g_{\alpha} \)-closed and \( \alpha g \)-closed. However \( A \) is not a \( \tilde{g}_{\alpha} \)-closed set.

**Example 2.7.** Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \). Here \( SGC(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\} \), \( SPC(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\} \) and \( \widetilde{G}_{\alpha}C(X, \tau) = \{X, \phi, \{c\}, \{b, c\}, \{a, c\}\} \) and let \( A = \{a\} \). Then \( A \) is \( sg \)-closed and semi-preclosed. However \( A \) is not a \( \tilde{g}_{\alpha} \)-closed set.

**Theorem 2.8.** Every \( \tilde{g}_{\alpha} \)-closed set is \( \tilde{g}_{\beta} \)-closed set.

**Proof.** It follows from the definitions.

The converse of Theorem 2.8 need not be true by the following example.

**Example 2.9.** Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}\} \). Here \( \widetilde{G}_{\gamma}SC(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\} \), \( \widetilde{G}_{\alpha}C(X, \tau) = \{X, \phi, \{b\}, \{c\}, \{b, c\}\} \). Let \( A = \{a\} \). Then \( A \) is \( \tilde{g}_{\beta} \)-closed but not a \( \tilde{g}_{\alpha} \)-closed set.
Theorem 2.10.

(a) $\tilde{g}$α-closedness is independent of $g$-closedness, $g^*$-closedness and $^*g$-closedness.

(b) $\tilde{g}$α-closedness is independent of $\tilde{g}$-closedness.

(c) $\tilde{g}$α-closedness is independent of $g\alpha^*$-closedness.

Proof. It follows from the following examples.

Example 2.11. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$. Here $G\alpha C(X, \tau) = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}$, $G^*C(X, \tau) = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}$, $^*G\alpha C(X, \tau) = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}$ and $\tilde{G}\alpha C(X, \tau) = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Then $\{a, b\}$ is $g$-closed, $g^*$-closed and $^*g$-closed, but not $\tilde{g}$α-closed, but not even a $g$-closed, $g^*$-closed and $^*g$-closed.

Example 2.12. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$. Here $G\alpha C(X, \tau) = \{X, \phi, \{b\}, \{b, c\}\}$ and $\tilde{G}\alpha C(X, \tau) = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Then $\{c\}$ is $\tilde{g}$α-closed, but not a $\tilde{g}$-closed set.

Example 2.13. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Here $G\alpha C(X, \tau) = P(X)$ and $\tilde{G}\alpha C(X, \tau) = \{X, \phi, \{a\}, \{b, c\}\}$. Then $\{b\}$ is $g\alpha^*$-closed, but not a $\tilde{g}$α-closed set.

Example 2.14. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Here $G\alpha C(X, \tau) = \{X, \phi, \{b\}, \{b, c\}\}$ and $\tilde{G}\alpha C(X, \tau) = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Then $\{b\}$ is $\tilde{g}$α-closed, but not a $g\alpha^*$-closed set.

Theorem 2.14. Let $A$ be a subset of $(X, \tau)$.

(a) If $A$ is $\tilde{g}$α-closed, then $\text{acl}(A) - A$ does not contain any non-empty $^*g$-closed set.

(b) If $A$ is $\tilde{g}$α-closed and $A \subseteq B \subseteq \text{acl}(A)$, then $B$ is $\tilde{g}$α-closed.

Proof.

(a) Suppose that $A$ is $\tilde{g}$α-closed and let $F$ be a non-empty $^*g$-closed set with $F \subseteq \text{acl}(A) - A$. Then $A \subseteq X - F$ and so $\text{acl}(A) \subseteq X - F$. Hence $F \subseteq X - \text{acl}(A)$, a contradiction.
(b) Let \( U \) be a \(^{\sharp}gs\)-open set of \((X, \tau)\) such that \( B \subseteq U \). Then \( A \subseteq U \). Since \( A \) is \( \tilde{\alpha}\)-closed, \( acl(A) \subseteq U \). Now \( acl(B) \subseteq acl(acl(A)) \subseteq U \). Therefore \( B \) is also a \( \tilde{\alpha}\)-closed set of \((X, \tau)\).

**Theorem 2.15.** Let \( A \) and \( B \) be subspaces of a topological space \((X, \tau)\). Then the union of two \( \tilde{\alpha}\)-closed set is \( \tilde{\alpha}\)-closed set in \((X, \tau)\).

**Proof.** Let \( A \) and \( B \) be \( \tilde{\alpha}\)-closed sets. Let \( A \cup B \subseteq U \) such that \( U \) is \( ^{\sharp}gs\)-open. Since \( A \) and \( B \) are \( \tilde{\alpha}\)-closed sets, \( acl(A) \subseteq U \) and \( acl(B) \subseteq U \). This implies that \( acl(A \cup B) = acl(A) \cup acl(B) \subseteq U \), (since \( \tau=\alpha\)-open set forms a topology [9]) and so \( acl(A \cup B) \subseteq U \). Therefore \( A \cup B \) is \( \tilde{\alpha}\)-closed.

We need the following notations:

For a subset \( E \) of a space \((X, \tau)\), we define the following subsets of \( E \).

\[ E_{\tau} = \{ x \in E/\{x\} \in \tau \}; \]

\[ E_{F} = \{ x \in E/\{x\} \text{ is closed in (X,}\tau) \}; \]

\[ E_{\tilde{gao}} = \{ x \in E/\{x\} \text{ is } \tilde{\alpha}\text{-open in (X,}\tau) \}; \]

\[ E_{\tilde{gsc}} = \{ x \in E/\{x\} \text{ is } ^{\sharp}gs\text{-closed in (X,}\tau) \}. \]

**Lemma 2.16.** For any space \((X, \tau)\), \( X = X_{\tilde{gsc}} \cup X_{\tilde{gao}} \) holds.

**Proof.** Let \( x \in X \). Suppose that \( \{x\} \) is not \( ^{\sharp}gs\)-closed set in \((X, \tau)\). Then \( X \) is a unique \( ^{\sharp}gs\)-open set containing \( X - \{x\} \). Thus \( X - \{x\} \) is \( \tilde{\alpha}\)-closed in \((X, \tau)\) and so \( \{x\} \) is \( \tilde{\alpha}\)-open. Therefore \( x \in X_{\tilde{gsc}} \cup X_{\tilde{gao}} \) holds.

We need more notations:

For a subset \( A \) of \((X, \tau)\), \( ker(A) = \cap\{U/U \in \tau \text{ and } A \subseteq U\} \);

\[ ^{\sharp}GSO-ker(A) = \cap\{U/U \in \, ^{\sharp}GSO(X, \tau) \text{ and } A \subseteq U\}. \]

**Theorem 2.17.** For a subset \( A \) of \((X, \tau)\), the following conditions are equivalent.

1. \( A \) is \( \tilde{\alpha}\)-closed in \((X, \tau)\).

2. \( acl(A) \subseteq ^{\sharp}GSO-ker(A) \) holds.

3. (i) \( acl(A) \cap X_{\tilde{gsc}} \subseteq A \) and (ii) \( acl(A) \cap X_{\tilde{gao}} \subseteq ^{\sharp}GSO-ker(A) \) holds.

**Proof.**

(1) \( \Rightarrow \) (2) Let \( x \notin ^{\sharp}GSO-ker(A) \). Then there exists a set \( U \in ^{\sharp}GSO(X, \tau) \) such
that $x /\notin U$ and $A \subseteq U$. Since $A$ is $\tilde{\alpha}a$-closed, $\alpha cl(A) \subseteq U$ and so $x /\notin \alpha cl(A)$. This shows that $\alpha cl(A) \subseteq \overset{\ddagger}{GSO}\ker(A)$.

$\overset{\circ}{(2)} \Rightarrow (1)$ Let $U \in \overset{\overset{\circ}{\circ}}{GSO}(X, \tau)$ such that $A \subseteq U$. Then we have that $\overset{\ddagger}{GSO}\ker(A) \subseteq U$ and so by $(2) \alpha cl(A) \subseteq U$. Therefore $A$ is $\tilde{\alpha}a$-closed.

$\overset{\circ}{(2)} \Rightarrow (3)$ (i) First we claim that $\overset{\ddagger}{GSO}\ker(A) \cap X_{\overset{\circ}{gsc}} \subseteq A$. Indeed, let $x \in \overset{\ddagger}{GSO}\ker(A) \cap X_{\overset{\circ}{gsc}}$ and assume that $x /\notin A$. Since the set $X - \{x\} \in \overset{\circ}{\circ}GSO(X, \tau)$ and $A \subseteq X - \{x\}$, $\overset{\ddagger}{GSO}\ker(A) \subseteq X - \{x\}$. Then we have that $x \in X - \{x\}$ and so this is a contradiction. Thus we show that $\overset{\ddagger}{GSO}\ker(A) \cap X_{\overset{\circ}{gsc}} \subseteq A$. By using $(2)$, $\alpha cl(A) \cap X_{\overset{\circ}{gsc}} \subseteq \overset{\ddagger}{GSO}\ker(A) \cap X_{\overset{\circ}{gsc}} \subseteq A$.

(ii) It is obtained by lemma $2.16$ and $(3)$, 

$$\alpha cl(A) = \alpha cl(A) \cap X = \alpha cl(A) \cap (X_{\overset{\circ}{gsc}} \cup X_{\overset{\circ}{gao}}) = (\alpha cl(A) \cap X_{\overset{\circ}{gsc}}) \cup (\alpha cl(A) \cap X_{\overset{\circ}{gao}}) = A \cup \overset{\ddagger}{GSO}\ker(A) = \overset{\ddagger}{GSO}\ker(A) \text{ holds.}$$

**Theorem 2.18.** Let $(X, \tau)$ be a space and $A$ and $B$ are subsets.

(i) If $A$ is $\overset{\ddagger}{\overset{\circ}{gsc}}$-open and $\tilde{\alpha}a$-closed, then $A$ is $\alpha$-closed in $(X, \tau)$.

(ii) Suppose that $(X, \tau)$ is an $\alpha$-space. A $\tilde{\alpha}a$-closed set $A$ is $\alpha$-closed in $(X, \tau)$ if and only if $\alpha cl(A) - A$ is $\alpha$-closed in $(X, \tau)$.

(iii) For each $x \in X$, $\{x\}$ is $\overset{\ddagger}{\overset{\circ}{gsc}}$-closed or $X - \{x\}$ is $\tilde{\alpha}a$-closed in $(X, \tau)$.

(iv) Every subset is $\tilde{\alpha}a$-closed in $(X, \tau)$ if and only if $\overset{\ddagger}{\overset{\circ}{gsc}}$-open set is $\alpha$-closed.

**Proof.**

(ii) (Necessity) If $A$ is $\alpha$-closed, then $\alpha cl(A) - A = \phi$.

(Sufficiency) Suppose that $A$ is $\tilde{\alpha}a$-closed and $\alpha cl(A) - A$ is $\alpha$-closed. It follows from assumptions that $\tau = \tau^\alpha$. Then, $\alpha cl(A) - A$ is $\overset{\ddagger}{\overset{\circ}{gsc}}$-closed in $(X, \tau)$ and by Theorem $2.14.$, $\alpha cl(A) - A = \phi$. Therefore $A$ is $\alpha$-closed in $(X, \tau)$.

(iii) If $\{x\}$ is not $\overset{\ddagger}{\overset{\circ}{gsc}}$-closed, then $X - \{x\}$ is not $\overset{\ddagger}{\overset{\circ}{gsc}}$-open. Therefore $X - \{x\}$ is $\tilde{\alpha}a$-closed in $(X, \tau)$.

(iv) (Necessity) Let $U$ be a $\overset{\ddagger}{\overset{\circ}{gsc}}$-open set. Then we have that $\alpha cl(U) \subseteq U$ and hence $U$ is $\alpha$-closed.
(Sufficiency) Let $A$ be a subset and $U$ is a $g_{\text{open}}$ set such that $A \subseteq U$. Then $\text{ocl}(A) \subseteq \text{ocl}(U) = U$ and hence $A$ is $\tilde{g}_{\alpha}$-closed.

**Remark 2.19.** The following diagram shows the relationships established between $\tilde{g}_{\alpha}$-closed sets and some other sets. $A \rightarrow B$ represents $A$ implies $B$ but not conversely.

\[
\begin{array}{cccc}
\alpha - \text{closed} & gs - \text{closed} & g\alpha - \text{closed} \\
\downarrow & \uparrow & \uparrow \\
\text{closed} & \tilde{g}_{\alpha} - \text{closed} & g_{\text{sp}} - \text{closed} \\
\downarrow & \downarrow & \downarrow \\
\text{semi-preclosed} & sg - \text{closed} & g_{\text{pr}} - \text{closed}
\end{array}
\]

**References**


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