Stability of ice lenses in saline soils

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A model of the growth of an ice lens in a saline porous medium is developed. At high lens growth rates the pore fluid becomes supercooled relative to its equilibrium Clapeyron temperature. Instability occurs when the supercooling increases with distance away from the ice lens. Solute diffusion in the pore fluid significantly enhances the instability. An expression for the segregation potential of the soil is obtained from the condition for marginal stability of the ice lens. The model is applied to a clayey silt and a glass powder medium, indicating parameter regimes where the ice lens stability is controlled by viscous flow or by solute diffusion. A mushy layer, composed of vertical ice veins and horizontal ice lenses, forms in the soil in response to the instability. A marginal equilibrium condition is used to estimate the segregated ice fraction in the mushy layer as a function of the freezing rate and salinity.

Key words: frost heave, phase change, diffusion

1. Introduction

The freezing of soils and particulate suspensions has received increasing attention in recent years owing to its relevance to geophysical phenomena (Dash et al. 2006; Wettlaufer 2019), water purification (Gay & Azouni 2002; Gao et al. 2006) and materials science applications (Deville et al. 2006; Henderson et al. 2013; Deville 2017). In this process, ice is grown into a matrix of particles in water which may also contain dissolved salts and dispersants. For a range of freezing conditions and physical parameters, the ice segregates from the particle matrix, self-organizing into a variety of patterns (Taber 1929; Qian & Zhang 2011; Deville 2017). Materials scientists exploit this phenomenon to create novel microstructured and biomimetic materials (Deville 2013; Henderson et al. 2013). When soils freeze, the ice tends to segregate into a sequence of approximately planar horizontal layers called ice lenses (Taber 1929; Dash et al. 2006). While it is known that frost heave (the uplift of the soil surface in winter) is caused by the formation of ice lenses, the physical mechanisms governing their initiation and growth have yet to be fully explained (O’Neill 1983; Rempel 2010; Peppin & Style 2013; Wettlaufer 2019). In soils with a high clay content, ice lenses can become nonplanar, forming complex polygonal and reticulate vein structures (Taber 1929; Mackay 1974; Arenson et al. 2006; Peppin et al. 2006; Deville 2013; Wang et al. 2016; El Hasadi & Kodadadi 2015; Xu et al. 2016; You et al. 2018 b). In addition, most soils and colloidal suspensions contain dissolved solutes in the pore fluid, which significantly influence the freezing process (Hallet 1978; Chamberlain 1983; Arenson et al. 2006; Pekor 2014; Wang et al. 2016; Schollick et al. 2016; You et al. 2018a; Gïnot et al. 2019).

In the present work, a model of an ice lens growing in a saline porous medium is developed and a linear stability analysis is undertaken. A characteristic equation is obtained giving critical conditions for the onset of unstable lens growth, and a diagram is
constructed illustrating parameter regimes in which molecular diffusion or viscous flow controls the instability. The diagram indicates that dissolved impurities have significant effects on the growth and stability of ice lenses at the solute concentrations typically found in soils and freeze-casting applications. The results are applied to frost heave experiments on a silty clay soil (Konrad 1990a) and a glass powder medium (Watanabe et al. 2001; Watanabe 2002).

2. Ice lens growth in saline porous media

The system to be studied, shown in figure 1, consists of a solidification cell containing a column of soil freezing from the top down. The soil is made up of solid particles, and the pore fluid is composed of water and a dissolved solute. At the soil surface $z = 0$ is a planar ice lens that is growing without entering the soil pores (primary frost heave); an overburden pressure $P_{OB}$ acts on the top of the ice lens, so that the ice pressure is $p_i = P_{OB}$. Supplying the soil at position $z = H$ is a fluid reservoir at pressure $p_r$ and solute concentration $c_\infty$. A linear temperature gradient is imposed on the system, and a constant freezing speed $V$. This can be achieved by ramping the temperatures at the top and base (Penner 1986; Konrad 1989a) or by moving the soil at speed $V$ through a fixed temperature gradient (Watanabe & Mizoguchi 2000; Peppin et al. 2008; Anderson & Worster 2014; Schollick et al. 2016; You et al. 2018a; Saint-Michel et al. 2019). At steady state the ice lens growth speed $V_i$ is equal to $V$.

2.1. Governing equations

In the following the density differences between ice, water, particles and solute are neglected. The equations describing the system can then be written, in a frame of reference moving with the freezing speed $V$, as

$$T = T_0 + G_T z \quad (z > 0),$$

$$\frac{\partial p}{\partial t} - V \frac{\partial p}{\partial z} = C_v \frac{\partial^2 p}{\partial z^2} \quad (0 < z < H)$$

Figure 1. Schematic diagram of a directional solidification cell containing a saturated soil with an ice lens at $z = 0$ and a reservoir at $z = H$. The reservoir contains pore fluid at pressure $p_r$ and solute concentration $c_\infty$. The ice lens is growing at speed $V$ within a fixed temperature gradient $G_T$ under an overburden pressure $P_{OB}$. 
\[
\frac{\partial c}{\partial t} - V \frac{\partial c}{\partial z} = D \frac{\partial^2 c}{\partial z^2} \quad (0 < z < \infty),
\] (2.3)

where \( T_0 \) is the temperature at the ice lens surface \( z = 0 \), \( G_T \) is the temperature gradient, \( p \) is the Darcy pressure of the pore fluid, \( C_v \) is the consolidation coefficient of the soil, \( c \) is the solute concentration in the pore fluid and \( D \) is the solute diffusivity in the soil (Freeze & Cherry 1979; Wang 2000; Davis 2001; Bear 1972). Equation (2.2) is the pressure diffusion equation of consolidation theory written in a frame of reference moving with \( V \) (Appendix A). Typically \( C_v \gg D \) for silt or clay soils (Bear 1972; Wang 2000).

The boundary conditions on (2.2) and (2.3) are
\[
V = k \mu \frac{\partial p}{\partial z}, \quad (c_0 - c_s) V = -D \frac{\partial c}{\partial z} \quad (z = 0),
\] (2.4)
\[
p = p_r \quad (z = H) \quad \text{and} \quad c \to c_\infty \quad (z \to \infty),
\] (2.5)

where \( k \) is the soil permeability, \( \mu \) is the fluid viscosity, \( c_0 \) is the solute concentration at the ice lens surface, \( c_s = k_s c_0 \) is the solute concentration in the ice lens and \( k_s \) is the solute segregation coefficient. For \( k_s < 1 \) the ice lens tends to push ahead solute molecules, forming a concentrated boundary layer adjacent to the ice surface (Hallet 1978; Chamberlain 1983; Konrad & McCammon 1990b; Watanabe et al. 2001; Wang et al. 2016). The thickness of the solute boundary layer, \( D/V \sim 1 \text{mm} \), is assumed much smaller than the soil height \( H \sim 10 \text{ cm} \).

### 2.2. Steady-state solution

At steady state, \( c_s = c_\infty \) and equations (2.2)–(2.5) can be solved to give the pressure and solute profiles
\[
p = p_0 + \frac{\mu C_v}{k} (1 - e^{-zV/C_s}) \quad (0 \leq z \leq H),
\] (2.6)
\[
c = c_\infty + (c_0 - c_\infty) e^{-zV/D} \quad (z \geq 0),
\] (2.7)

where \( p_0 = p_r - (\mu C_v/k) (1 - e^{-H/V/C_s}) \) and \( c_0 = c_\infty/k_s \) are the pore pressure and concentration at the ice lens surface (Appendix B). In the case of a rigid porous matrix (Peppin 2009), \( C_v \to \infty \) and the pressure profile is linear
\[
p = p_r + \frac{\mu V}{k} (z - H) \quad (0 \leq z \leq H).
\] (2.8)

Equations (2.1), (2.6) and (2.7) are plotted on figure 2 for the case \( V = 1 \mu \text{m/s}, \ c_\infty = 0.1 \text{ g/L}, \ G_T = 20 \text{ K/m}, \ P_{OB} = 100 \text{ kPa}, \ H = 10 \text{ cm}, \) with physical and clay soil parameters \( \mu = 0.0015 \text{ Pa s}, \ C_v = 10^{-8} \text{ m}^2/\text{s}, \ k = 10^{-16} \text{ m}^2, \ D = 10^{-9} \text{ m}^2/\text{s} \) and \( k_s = 0.1 \).

### 2.3. Ice lens temperature

The equilibrium freezing temperature \( T_{CI} \) of the soil can be obtained from the generalized Clapeyron equation
\[
\frac{L_f(T_{CI} - T_m)}{T_m} = \nu_w (p - \pi) - \nu_i p_i,
\] (2.9)

where \( T_m \) is the melting temperature of pure ice at atmospheric pressure, \( \nu_w/\rho_i \) is the specific volume of water/ice, \( L_f \) is the latent heat per unit mass of ice, \( p_i \) is the ice pressure (relative to atmospheric pressure), \( p \) is the pore fluid pressure (suction) and \( \pi \) is the solute osmotic pressure (O’Neill 1983; Acker et al. 2001; Peppin & Style 2013).

Neglecting the density difference between water and ice (\( \nu_i = \nu_w = 1/\rho \)) and assuming
the solute is dilute, the Clapeyron equation (2.9) can be written as

\[ T_{Cl} = T_m - \frac{T_m}{\rho L_f} (p_i - p) - mc, \] (2.10)

where \( \rho \) is the density of water and \( mc = T_m \pi / (\rho L_f) \). For dilute NaCl in water, \( m = i K_f / M_w \) is the slope of the solute freezing temperature curve, where \( i = 2 \) is the Van’t Hoff coefficient, \( K_f = 1.86 \text{~KL/mol} \) is the molar cryoscopic constant (Schollick et al. 2016) and \( M_w = 58.4 \text{~g/mol} \) is the NaCl molar mass. The other parameters used in equation (2.10) are \( T_m = 273.15 \text{~K} \), \( \rho = 1000 \text{~kg/m}^3 \) and \( L_f = 3.3 \times 10^5 \text{~J/kg} \) (Table 1).

Assuming the ice lens pressure is equal to the overburden, \( p_i = P_{OB} \), the temperature \( T_0 \) at the ice lens–soil boundary \( z = 0 \) can be obtained from (2.10) as

\[ T_0 = T_m - \frac{T_m}{\rho L_f} (P_{OB} - p_0) - mc_0, \] (2.11)

where \( p_0 \) and \( c_0 \) are the pore pressure and solute concentration at \( z = 0 \).

### 2.3.1. Constitutional and geometric supercooling

In solidification of aqueous solutions, constitutional supercooling of the solution adjacent to the ice interface occurs when \( dT_L/dz > dT/dz \), where \( T_L = T_m - mc \) is the freezing temperature (liquidus) of the solution and \( dT/dz = G_T \) is the temperature gradient (Worster 1986; Davis 2001). Constitutional supercooling typically leads to instabilities because perturbations of the interface move into supercooled regions and experience accelerated growth (Tiller et al. 1953; Davis 2001).

Analogously, supercooling of the soil occurs when the gradient of the Clapeyron temperature is larger than the temperature gradient at the ice lens such that \( dT_{Cl}/dz > dT/dz \), or with (2.10),

\[ -m \frac{dc}{dz} + \frac{T_m}{\rho L_f} \frac{dp}{dz} > \frac{dT}{dz} \] (z = 0). (2.12)
Figure 3. (a) Illustration of a macroscopic perturbation to the shape of the ice lens. If the perturbation continues to grow in size the ice lens is unstable; if all such perturbations melt back to the planar state the ice lens is stable. (b) The perturbed region is magnified, showing the height $h(x)$ and the unit normal $\hat{n}$. (c) Further magnification shows the premelted films and the ice curvature on the pore scale.

The first term on the left-hand-side of (2.12) accounts for constitutional supercooling caused by solute diffusion, while the second term accounts for geometric (hydraulic) supercooling caused by viscous flow (Rempel 2008; Style et al. 2011; You et al. 2018b). In terms of dimensionless parameters equation (2.12) can be written as

$$M_c + M_p > 1,$$

where $M_c = -mG_c/G_T$ and $M_p = T_mG_p/(\rho L_f G_T)$ are the solutal and hydraulic morphological numbers, respectively; $G_c = d\delta/dz|_{z=0} = -(c_0 - c_\infty)V/D$ and $G_p = dp/dz|_{z=0} = \mu V/k$ are the solute and pressure gradients at the ice lens interface. Figure 2c plots $T_C(z)$ and $T(z)$ for the pressure and solute profiles obtained in Section 2.2, showing a supercooled region below the ice lens.

3. Linear stability analysis

In linear stability analysis, the shape of the ice interface is perturbed, as in figure 3a, and the governing equations are linearized about the planar steady-state solution. If the perturbations tend to diminish back to the planar state the ice lens is stable, while it is unstable if perturbations tend to grow. Assuming the linear temperature profile imposed by the boundary conditions is stable and not perturbed (frozen temperature approximation (Davis 2001)), the governing equations describing the evolution of the perturbed ice lens are

$$T = T_0 + G_T z,$$

$$\frac{\partial p}{\partial t} - V \frac{\partial p}{\partial z} = C_v \nabla^2 p,$$

$$\frac{\partial c}{\partial t} - V \frac{\partial c}{\partial z} = D \nabla^2 c,$$

$$V_n = \frac{k}{\eta} \nabla p \cdot \hat{n}, \quad c(1 - k_s)V_n = -D \nabla c \cdot \hat{n} \quad (z = h),$$

$$T_h = T_m - mc_h - \frac{T_m}{\rho L_f} (P_{OB} - p_h) \quad (z = h),$$

where $\eta$ is the viscosity of the mixture, $k$ is the thermal diffusivity of the mixture, $T_m$ is the melting temperature, $L_f$ is the latent heat of fusion, $\rho$ is the density of the mixture, $P_{OB}$ is the boundary pressure, and $P_h$ is the pressure at the ice lens interface.
where \( h = h(x,t) \) is the height of the deformed interface and

\[
\hat{n} = \frac{(-h_x,1)}{\sqrt{1+h_z^2}} \quad \text{and} \quad \hat{V}_n = V \frac{(1+h_t)}{\sqrt{1+h_z^2}}
\]  

(3.6)

are, respectively, the unit normal \( \hat{n} \) to the ice lens surface and the ice velocity \( \hat{V}_n \) along the normal direction (Mullins & Sekerka 1964; Davis 2001).

Latent heat and differing thermal conductivities have been neglected here, because many frost-heave experiments are designed to ensure that latent heat is quickly conducted away leading to linear temperature profiles (Watanabe & Mizoguchi 2000; Anderson & Worster 2014; Schollick 2015; Schollick et al. 2016; You et al. 2018a). The frozen temperature approximation also simplifies the analysis and allows the focus to be on the physical processes of diffusion and viscous flow that drive the ice lens instability. Accounting for latent heat effects leads to qualitatively similar results, with the thermal gradient \( G_T \) replaced by a weighted average \( \bar{G}_T \) of the gradients in the frozen and unfrozen regions (Mullins & Sekerka 1964; Sekerka 1968; Davis 2001).

Before linearizing the equations it is convenient to write them in dimensionless form. The solutions (2.6) and (2.7) suggest the scalings

\[
(x^*, z^*) = \left( \frac{x}{\delta_t}, \frac{z}{\delta_t} \right), \quad t^* = \frac{t}{\delta_t}, \quad p^* = \frac{(p-p_0)}{\delta_t G_p}, \quad c^* = \frac{(c-c_0)}{\delta_t G_c}, \quad T^* = \frac{(T-T_0)}{\delta_t G_T}
\]

where \( \delta_t = D/V \) and \( \delta_l = D/V^2 \) are length and time scales, \( G_p = \mu V/k \) is the steady-state Darcy pressure gradient at \( z = 0 \) and \( G_c = -c_0 V(1-k_s)/D \) is the steady-state solute gradient at \( z = 0 \). In dimensionless form the governing equations (3.1)–(3.3) become (upon dropping the stars)

\[
T = z, \quad p_0 - p = \epsilon_p \nabla^2 p, \quad c_t - c_z = \nabla^2 c, \quad (3.7)
\]

where \( \epsilon_p = C_v/D \) is the ratio of the soil consolidation coefficient to the solute diffusivity. The boundary conditions (3.4) and (3.5) become

\[
\hat{V}_n = \nabla p \cdot \hat{n}, \quad [c(1-k_s) + 1] \hat{V}_n = -\nabla c \cdot \hat{n} \quad (z = h),
\]  

(3.8)

\[
T_h = M_p \rho_h + M_c c_h \quad (z = h),
\]  

(3.9)

where \( \hat{V}_n = V_n/V \) and the dimensionless numbers

\[
M_p = \frac{T_m G_p}{\rho L_f \bar{G}_T} \quad \text{and} \quad M_c = -\frac{m G_c}{G_T}
\]

(3.10)

are, respectively, the hydraulic and diffusive morphological numbers introduced in Section 2.3.1. The steady base-state solutions (2.7) and (2.8) become, in terms of dimensionless quantities,

\[
\begin{bmatrix}
  c \\
  p \\
  h
\end{bmatrix}
= \begin{bmatrix}
  \hat{c} \\
  \hat{p} \\
  \hat{h}
\end{bmatrix}
= \begin{bmatrix}
  1 - e^{-z} \\
  \epsilon_p(1 - e^{-z}/\epsilon_p) \\
  0
\end{bmatrix}
\]  

(3.11)

Let the base state be perturbed such that

\[
p = \hat{p} + p', \quad c = \hat{c} + c', \quad h = \hat{h} + h',
\]  

(3.12)

where the perturbations \((p', c', h')\) take the form of normal modes

\[
p' = p_1(z)e^{iqx+\omega t}, \quad c' = c_1(z)e^{iqx+\omega t}, \quad h' = h_1(z)e^{iqx+\omega t}.
\]  

(3.13)

Here \( q \) is the wave number and \( \omega \) the growth rate of the perturbation (Mullins & Sekerka 1964; Davis 2001).
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1964; Davis 2001). If \( \omega < 0 \) for all wave numbers \( q \), the perturbations decay in time and the ice lens is stable, while if \( \omega > 0 \) for any wave number the perturbation grows and the ice lens is unstable. In general the growth rate can also be a complex number, with the imaginary component describing oscillatory instabilities (Davis 2001). An imaginary component to \( \omega \) can appear, for example, if the segregation coefficient \( k_s \) depends on the growth rate \( V \) (Coriell & Sekerka 1983; Davis 2001). In fact for lake ice \( k_s \) does depend on \( V \) (Weeks & Lofgren 1967), and experiments by Konrad & McCammon (1990b) suggest the same is true for ice lenses. Here, for simplicity, the segregation coefficient is treated as an averaged constant so that \( \omega \) is real (Sekerka 1968; Wollkind & Segel 1970) and the interface experiences only transverse wave instabilities.

Inserting (3.12) and (3.13) into (3.7), linearizing in the primed quantities, and solving the resulting system of ordinary differential equations leads to the following expression for the growth rate \( \omega \) (Appendix C)

\[
\mathcal{M}_p + \mathcal{M}_c = 1 + \mathcal{M}_p \left( \frac{\omega + \epsilon_p^{-1}}{r_p} \right) + \mathcal{M}_c \left( \frac{\omega + k_s}{k_s + r_c - 1} \right),
\]

where

\[
r_p = \frac{1}{2} \left[ 1 + \sqrt{1 + 4 \epsilon_p (\omega + \epsilon_p q^2)} \right] \quad \text{and} \quad r_c = \frac{1}{2} \left[ 1 + \sqrt{1 + 4 (\omega + q^2)} \right].
\]

Equation (3.14) is a nonlinear equation for \( \omega \) that must in general be solved numerically. However, a simpler result can be obtained in the large wave number limit \( q \gg 1 \) (Mullins & Sekerka 1964; Davis 2001). In this case \( r_p \approx r_c \approx q \) and, given that \( \epsilon_p^{-1} = D/C_v < 1 \) and \( k_s < 1 \), equation (3.14) becomes

\[
\omega \approx q \left( 1 - \left[ \mathcal{M}_p + \mathcal{M}_c \right]^{-1} \right).
\]

Equation (3.15) shows that \( \omega \) is positive when \( \mathcal{M}_p + \mathcal{M}_c > 1 \), and therefore the supercooling condition (2.13) leads to instability. The dimensional wavelength of the instability is \( \lambda = 2\pi D/qV \) so that, for a salt diffusivity \( D \sim 10^{-9} \text{m}^2/\text{s} \) and ice lens growth rate \( V \sim 1 \mu\text{m/s} \), the condition \( q \gg 1 \) implies \( \lambda \ll 1 \text{cm} \). Given the minimum observable size of segregated ice is of order the particle radius \( R \sim 1 \mu\text{m} \), this suggests bounds on the initial wavelength of the form \( 1 \mu\text{m} \ll \lambda \ll 1 \text{cm} \).

Setting \( \omega = 0 \) in (3.15) gives the marginal stability condition

\[
\mathcal{M}_p + \mathcal{M}_c = 1.
\]

In terms of dimensional parameters the marginal stability condition (3.16) can be written as

\[
\frac{T_m \mu V}{G_T \rho L_f k} + \frac{mc_v V (1 - k_s)}{D k_s G_T} = 1.
\]

(3.17)

The first term on the left-hand side of (3.17) accounts for geometric (hydraulic) supercooling, while the second term accounts for constitutional supercooling. This equation will be used in Section 5 to study the stability of ice lenses in a clayey silt and a soil composed of glass microspheres. First, some implications of the instability are discussed in Section 4.

4. Mushy layers in freezing porous media

4.1. Instability and mushy layer growth

Morphological instability during the solidification of fluids leads to the formation of a mushy layer composed of vertical ice dendrites and interstitial fluid (Worster 1986). The
Figure 4. Schematic diagram of a mushy layer in a freezing saline soil. The mushy layer is composed of vertical ice veins and horizontal ice lenses. The portion of the mushy layer between the warmest ice lens at temperature $T_0$ and the freezing front near $T_m = 0^\circ C$ is the frozen fringe. $T_{IE}$ is the temperature at which ice first enters the pore space. Here the system is experiencing primary frost heave ($T_{IE} < T_0$) and the soil surrounding the warmest ice lens is unfrozen. As in figure 1 there is a reservoir at $z = H$ containing pore fluid at pressure $p_r$ and solute concentration $c_\infty$. The system is moving at steady freezing speed $V$ within a fixed temperature gradient $G_T$ under an overburden pressure $P_{OB}$, yielding a sequence of equally spaced ice lenses.

dendrites are a nonlinear manifestation of the planar instability and tend to form side branches in order to more efficiently remove the supercooling (Kurz & Fisher 1998; Davis 2001). Figure 4 illustrates a freezing soil in which a similar mushy layer has formed. The vertical ice veins in figure 4 are a nonlinear manifestation of the normal mode instability in Section 3. In three dimensions ice veins tend to form a polygonal pattern in the soil cross section (Taber 1929; Mackay 1974; Chamberlain & Gow 1979; Arenson et al. 2006, 2008; Wang et al. 2018). Assuming the analogy with alloy solidification holds also for the vein spacing, the primary spacing $\lambda_1$ can be estimated as

$$\lambda_1 = K_1 C_v V^{-a} G_T^{-b},$$

where $a$, $b$ and $K_1$ are positive constants (Kurz & Fisher 1998). Equation (4.1) indicates that the vein spacing decreases with the freezing speed $V$, and increases with the consolidation coefficient $C_v$, in qualitative agreement with the observations of Mackay (1974, 1975) and Chamberlain & Gow (1979). The horizontal ice lenses in figure 4 are assumed to nucleate as side branches from the vertical veins at the position of maximum supercooling (Style et al. 2011; You et al. 2018a,b). This ice-lens nucleation mechanism is in contrast to secondary frost heave in which horizontal ice lenses form in the absence of ice veins via a regelation-induced fracture of the frozen fringe (Miller 1978; Rempel et al. 2004; Style et al. 2011).

4.2. Primary and secondary frost heave

In figure 4 $T_0$ is the temperature of the warmest ice lens. The soil in between the ice lenses is unfrozen until the ice-entry temperature $T_{IE}$ is reached and ice begins to enter
the pore space (Beskow 1935; Brown & Payne 1990). In soils with a uniform pore size distribution the ice-entry temperature is given by the Gibbs-Thompson equation, \( T_{IE} = T_m (1 - 2\gamma / (R_p \rho L_f)) \), where \( R_p \) is the pore size (Wettlaufer & Worster 2006; Peppin & Style 2013). Azmatch (2013) and Ginot et al. (2019) have shown that in soils with a broad size distribution or high salinity ice entry occurs over a finite range in temperature. In Miller’s classification scheme (Miller 1972, 1978) figure 4 is an example of (unstable) primary frost heave, because the warmest ice lens is forming in the absence of pore ice \((T_0 > T_{IE})\). This is in contrast to secondary frost heave that occurs after ice has entered the pores \((T_0 < T_{IE})\).

Current research is ongoing to distinguish between primary and secondary frost heave, with some works finding evidence for ice lens formation in soil containing pore ice (Loch & Miller 1975; Konrad & Morgenstern 1982; Walder & Hallet 1986; Anderson & Worster 2012, 2014; Schollick et al. 2016; You et al. 2018b) and others finding ice lenses forming in soil regions that do not contain pore ice (Beskow 1935; Mackay 1975; Brown & Payne 1990; Watanabe et al. 1997; Watanabe & Mizoguchi 2000; You et al. 2018a). It may be that both types of frost heave can occur in the same system depending on the freezing conditions (Schollick 2015; You et al. 2018b). Microscale experiments capable of observing the pore contents during freezing, such as Raman spectroscopy (Watanabe & Mizoguchi 2000), X-ray scattering (Spannuth et al. 2011) and confocal microscopy (Saint-Michel et al. 2019; Ginot et al. 2019) will play an important role in clarifying the situation.

### 4.3. The frozen fringe

In the context of secondary frost heave, Miller (1972, 1978) defined the frozen fringe as the region between the warmest ice lens and the warmest extent of pore ice. Figure 4 suggests that a frozen fringe-like region can also be defined in the context of unstable primary frost heave as the region between the warmest ice lens and the warmest extent of segregated ice (tips of the ice veins). In the absence of solutes the ice veins extend very close to the 0°C isotherm, in which case the primary frozen fringe has thickness \( d \approx (T_m - T_0) / G_T \). In general, the frozen fringe can be defined as the region of soil between the warmest ice lens and the freezing front (warmest location of ice) (Konrad 1989b; Arenson et al. 2008).

#### 4.3.1. Average permeability of the frozen fringe

In secondary frost heave the permeability \( k \) of the frozen fringe is reduced because of the presence of pore ice, which impedes the flow of water to the ice lens (Miller 1978; Konrad & Morgenstern 1981; Fowler & Krantz 1994; Rempel et al. 2004). In the unstable primary frost heave occurring in figure 4, the permeability of the frozen fringe region is also reduced because of consolidation of the soil between the ice veins, which reduces the pore size, and because of resistance to flow in premelted films at the ice lens–soil interface (Kuroda 1987; Worster & Wettlaufer 1999; Rempel 2008; Style & Peppin 2012; You et al. 2018b). Similarly to Konrad & Morgenstern (1980), the soil below the warmest ice lens can be modelled as a two-layer system with Darcy’s law written as

\[
V_{it} = \frac{p_r - p_0}{H \mu / k_u + d \mu / k},
\]

where \( V_{it} \) is the growth rate of the ice lens, \( k_u \) is the permeability of the unfrozen soil and \( k \) is the average permeability of the frozen fringe. In the case of no overburden pressure \((P_{OB} = p_r = 0)\) and zero salinity \((c_0 = 0)\), the Clapeyron equation (2.11) can
be combined with (4.2) to give
\[ V_l = \frac{(L_f/gT_m)(T_m - T_0)}{H/K_u + d/K}, \tag{4.3} \]
where \( K_u = \rho g k_u/\mu \) is the hydraulic conductivity of the unfrozen soil, \( K = \rho g k/\mu \) is the hydraulic conductivity of the frozen fringe and \( g \) is the acceleration of gravity. By measuring the ice lens growth rate \( V_l \), the ice lens temperature \( T_0 \), the temperature gradient \( G_T \), the unfrozen soil height \( H \) and hydraulic conductivity \( K_u \), the frozen fringe thickness can be calculated as \( d = (T_m - T_0)/G_T \), and equation (4.3) can be used to determine the frozen-fringe hydraulic conductivity \( K \) (Konrad & Morgenstern 1980).

5. Applications

5.1. Devon silt

Devon silt is a clayey silt soil with a broad particle size distribution that is highly frost-susceptible (Azmatch 2013). Konrad & Morgenstern (1980, 1981) measured the parameters in equation (4.3) for Devon silt at zero overburden and no salinity, and obtained a frozen fringe thickness \( d \approx 1 \text{ mm} \) and hydraulic conductivity \( K \approx 10^{-11} \text{ m/s} \). Konrad (1990a) also studied the effects of overburden pressure and salinity on ice lenses in Devon silt. The Clapeyron equation (2.11) shows that an overburden pressure \( P_{OB} \) moves the ice lens temperature \( T_0 \) to lower values. This has the effect of reducing the premelted film thicknesses and further reducing the permeability. Black (1990) reviewed various empirical equations that have been used to model the effect of an overburden pressure on the permeability of the frozen fringe. Here the Gardner equation is used in the form

\[ K = K_0 e^{-\sigma/\sigma_K}, \tag{5.1} \]

where \( \sigma = P_{OB} - p_0 \) is the effective pressure on the soil matrix at \( z = 0 \), \( \sigma_K \) is a constant quantifying the effects of the overburden pressure on consolidation and flow in the premelted films, and \( K_0 \) is the hydraulic conductivity of the frozen fringe at zero effective pressure (Gilpin 1982; Konrad & Morgenstern 1984; Black 1990). An overburden pressure will also reduce the pore space available for solute diffusion, suggesting the diffusion coefficient be written as

\[ D = D_0 e^{-\sigma/\sigma_D}, \tag{5.2} \]

where \( \sigma_D \) is a constant and \( D_0 \) is the diffusion coefficient of salt within the unstressed frozen fringe.

Azmatch (2013) measured the effective pressure \( P_{OB} - p_0 \) required for ice to enter the pores of Devon silt and found an ice-entry pressure of 175 kPa; the necessary pressure increased further when NaCl was added to the soil. The maximum overburden pressure in Konrad’s experiments was \( P_{OB} = 130 \text{ kPa} \) and the average pore pressure (suction) at the ice lens was \( p_0 = -10 \text{ kPa} \) (Konrad 1990a), suggesting that the effective pressure was not sufficient for ice to enter the pores below the ice lens, at least in the high salinity experiments. Konrad (1990a) noted that “layers of unfrozen soil” may have been present between the ice lenses in his experiments because of the high salt concentrations. These results suggest that pore ice was not present in the frozen fringe of Konrad’s experiments, and that the system was undergoing unstable primary frost heave.
5.1.1. The segregation potential

Konrad & Morgenstern (1980, 1981) introduced the segregation potential $SP$, defined as

$$SP = \frac{V_{il}}{G_T}, \quad (5.3)$$

where $V_{il}$ and $G_T$ are measurements of the ice lens growth rate and temperature gradient at the onset of the formation of the final ice lens during transient freezing. For sufficiently small freezing rates $SP$ is a property of the soil, independent of the freezing parameters, and larger values of $SP$ indicate a greater propensity of the soil to heave (Konrad & Morgenstern 1981; Konrad 1987). Konrad (1990a) measured the segregation potential of Devon silt over a range of overburden pressures and salinities, shown in figure 5 as a regime diagram for $SP$ versus $c_{\infty}$. Assuming the onset of the final ice lens corresponds to a state of marginal stability such that $V_{il} = V$, the segregation potential can be obtained from equation (3.17) as

$$SP = \left( \frac{T_m g}{L_f K} + \frac{mc_{\infty}(1 - k_s)}{Dk_s} \right)^{-1}. \quad (5.4)$$

5.1.2. Regime diagram

The curves in figure 5 show $SP$ versus $c_{\infty}$ at several overburden pressures calculated from (5.4), using best-fit parameters $k_s = 0.2$, $K_0 = 1.8 \times 10^{-11} \text{m/s}$, $\sigma_K = 80 \text{kPa}$, $D_0 = 10^{-9} \text{m}^2/\text{s}$ and $\sigma_D = 55 \text{kPa}$ (other parameters such as $m$ and $L_f$ are as given in Table 1). Below the curves the ice lens is predicted to grow stably (single ice lens), while above the curves it is unstable (mushy layer). There is a regime at low solute concentrations ($c_{\infty} < 0.1 \text{g/L}$) where the behaviour of the ice lens is independent of solute concentration, and the instability is caused by viscous flow through the soil layer and premelted films (geometric supercooling). In this regime the segregation potential (5.4)
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takes the limiting value \( SP = L_f K / T_m g \). At high solute concentrations the instability is owing mainly to solute diffusion (constitutional supercooling); the first term on the right-hand side of (5.4) becomes negligible in this regime and \( SP = D k_s / [m c_\infty (1 - k_s)] \).

As the background solute concentrations in soils are typically above 0.1 g/L (Hivon & Sego 1993), these results suggest that frost heave often occurs in the diffusion regime, supporting the conclusions of Hallet (1978) and Chamberlain (1983) that solutes have significant effects on frost heave in periglacial soils.

The best-fit value for the hydraulic conductivity \( K_0 \) of the frozen fringe is similar to that measured by Konrad & Morgenstern (1980) (10\(^{-11}\) m/s). The best-fit value for the NaCl segregation coefficient \( k_s \) is similar to the measured value \( k_s \approx 0.1 \) for lake ice (Weeks & Lofgren 1967; Leppäranta 2015) but much larger than the equilibrium value \( (k_s < 10^{-3}) \) for pure ice crystals (Gross et al. 1987). However Konrad & McCammon (1990b) also found that \( k_s \) for Devon silt is larger than 0.1. Similarly Butler (2001) found, when freezing aqueous sucrose and polymer solutions, that a large segregation coefficient \( (k_s > 0.1) \) was needed in the Mullins-Sekerka equation to accurately model the instability.

A potential explanation for the relatively large segregation coefficient is the presence of grain boundaries and defects in the ice. Weeks & Lofgren (1967) show that lake ice is highly polycrystalline, in which case solute can be concentrated in between the grain boundaries (Thomson et al. 2013). Penner (1961) showed that ice lenses are also polycrystalline, and Wang et al. (2018) found that ice lenses contain significant defects that could also entrap salt. Furthermore, in soils such as Devon silt with a broad size distribution, salt could be transported in premelted films adjacent to the largest particles as they are engulfed by the ice during particle sorting (Corte 1962; Körber 1988). The ice–soil interface during lens growth is therefore a complex non-equilibrium system, with multiple mechanisms for solute entrapment, in contrast to the equilibrium ice-solution interface with a single ice grain used to obtain the much smaller equilibrium segregation coefficient (Gross et al. 1987).

5.2. Glass powder

Watanabe et al. (2001) and Watanabe (2002) performed a series of frost-heave experiments on a soil composed of glass microparticles. Figure 4 is a schematic of their system, in which a Hele-Shaw cell containing the water-saturated powder is moved at speed \( V \) through a temperature gradient \( G_T \), yielding a sequence of equally-spaced ice lenses. Watanabe et al. (2001) varied the salinity \( c_\infty \) of the pore fluid and Watanabe (2002) varied the freezing speed \( V \), and the effects on the ice lens thicknesses \( t_h \) and spacings \( s_p \) were measured.

5.2.1. Steady-state mushy layer

The system in figure 4 can be modelled as a mushy layer containing segregated ice of volume fraction \( \Phi \) (Worster 1986; Peppin et al. 2007). Developing a full mushy layer model is beyond the scope of the present work; however, some results can be obtained by writing the boundary conditions at the position of the warmest ice lens \( z = 0 \) as

\[
V_{dl} = \frac{k}{\mu} \frac{dp}{dz} \quad \text{and} \quad (c_0 - c_s) V = -D \frac{dc}{dz} \quad (z = 0),
\]

(5.5)

where

\[
V_{dl} = \Phi V \quad \text{and} \quad c_0 = \Phi c_s / k_s + (1 - \Phi) c_s,
\]

(5.6)

are, respectively, the average growth speed \( V_{dl} \) of the warmest ice lens and the average solute concentration \( c_0 \) at the ice lens. The first term in the expression for \( c_0 \) accounts for
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The solute rejected by the warmest ice lens, while the second term accounts for the solute trapped in between the ice lenses. When \( \Phi = 1 \) (single stable ice lens), \( c_0 = c_\infty/k_s \) as in Section 2.2. At steady state \( c_s = c_\infty \) is the average solute concentration in the mushy layer.

Assuming that segregated ice forms to eliminate supercooling and return the system to local equilibrium (Worster 1986), the temperature everywhere in the mushy layer is given by the Clapeyron equation (2.10). In the experiments of Watanabe et al. (2001) the overburden and reservoir pressures were equal to atmospheric pressure so that \( P_{OB} = p_i = p_r = 0 \) and (2.10) becomes

\[
T_{Cl} = T_m + \frac{T_m}{\rho L_f} p - mc. \quad (5.7)
\]

Following Worster (1986), the segregated ice fraction \( \Phi \) in the mushy layer can be obtained by assuming marginal equilibrium in the mushy layer so that

\[
\frac{dT}{dz} = \frac{dT_{Cl}}{dz} \quad (z = 0), \quad (5.8)
\]
or, with (5.5)–(5.7),

\[
\Phi = \frac{G_T}{V} \left( \frac{T_m g}{L_f k} + \frac{mc_\infty (1 - k_s)}{k_s D} \right)^{-1}, \quad (5.9)
\]

where \( K = k\rho g/\mu \) is the hydraulic conductivity of the frozen fringe. To determine \( K \) from the Gardner equation (5.1) it is necessary to solve the full mushy layer model to calculate
the pore pressure $p$ and effective pressure $\sigma = P_{OB} - p$ profiles. However, since Watanabe (2002) did not use an overburden pressure, and used a small freezing cell ($H \sim 5$ cm), it can be assumed that $\sigma \ll \sigma_K$, and treat $K$ and $D$ as averaged constants.

Figure 6 shows $\Phi$ vs the freezing speed $V$ and salinity $c_\infty$ calculated from (5.9), along with experimental data from Watanabe et al. (2001) and Watanabe (2002), with best-fit parameters $k_s = 0.1, K = 0.7 \times 10^{-11}$ m/s and $D = 10^{-9}$ m$^2$/s. The best-fit hydraulic conductivity $K$ of the frozen fringe is similar to that for Devon silt (Konrad & Morgenstern 1980), while the NaCl segregation coefficient $k_s$ is similar to that for lake ice (Leppänta 2015). Also shown as the inset to figure 6 is the segregation potential estimated as

$$SP = \frac{V_{il}}{G_T} = \left( \frac{T_m g}{L_f K} + \frac{m c_\infty (1 - k_s)}{k_s D} \right)^{-1}. \quad (5.10)$$

Similarly to the case of Devon silt, the $SP$ diagram shows a viscous regime at low salinity and a diffusion regime at high salinity. The magnitude of the segregation potential is less than the zero-overburden $SP$ data in figure 5, indicating that the glass powder is somewhat less frost susceptible than Devon silt.

6. Conclusions

A model of the growth of an ice lens during primary frost heave in a saline soil has been developed, and a linear stability analysis undertaken, showing that a planar ice lens can become unstable because of supercooling. The characteristic equation for the marginal stability of the ice lens is related to the segregation potential of the soil. Segregation potential diagrams have been obtained for Devon silt and a glass powder medium, showing two distinct regimes: At low salinity the ice lens growth rate and stability is affected mainly by viscous flow in the soil and premelted films, while at high salinity it is controlled by salt diffusion. At unstable freezing rates a mushy layer, composed of a reticulate network of ice lenses and vertical ice veins, forms in order to remove the supercooling. A mushy layer model has been used to determine the segregated ice fraction in the mushy layer as a function of the salinity and freezing rate.

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Appendix A

Assuming the porous medium is composed of inert incompressible particles and an incompressible fluid, conservation of mass can be written as

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \phi \mathbf{v}_p = 0 \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0, \quad (A 1)$$

where $\phi$ is the volume fraction of solids in a representative volume element, $1 - \phi$ is the porosity, $\mathbf{v}_p$ is the average velocity of the porous matrix, $\mathbf{v} = \phi \mathbf{v}_p + (1 - \phi) \mathbf{v}_f$ is the
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volume average velocity of the mixture and \( \mathbf{v}_f \) is the average fluid velocity (deGroot & Mazur 1962; Bear 1972). Using the identity \( \nabla \cdot \phi \mathbf{v} = \phi \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \phi \) equations (A 1) can be combined to give

\[
\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \nabla \cdot \phi \mathbf{q},
\]

where \( \mathbf{q} = (\mathbf{v} - \mathbf{v}_p) = (1 - \phi)(\mathbf{v}_f - \mathbf{v}_p) \) is the volume velocity of the fluid relative to the porous matrix. Darcy’s law can be written in the form

\[
\mathbf{q} = -\frac{k}{\mu} \nabla p,
\]

where \( p \) is the pore pressure (Bear 1972; Bear & Corapcioglu 1981). Assuming the constitutive relation \( p = p(T, P, \phi) \), differentiating at constant temperature \( T \) and constant isotropic total pressure \( P \) gives

\[
\frac{d\phi}{dt} = -\frac{\Pi}{\mu} \phi,
\]

where \( \Pi \equiv P - p \) is the effective pressure on the porous matrix and \( \Pi_{\phi} \equiv \frac{\partial \Pi}{\partial \phi} \). Combining (A 4) and (A 3) with (A 2) and assuming \( \phi k/\mu \) is constant gives the pressure diffusion equation

\[
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = C_v \nabla \cdot \nabla p,
\]

where \( C_v = \frac{\phi k}{\mu} \Pi_{\phi} \) is the consolidation coefficient. Finally, considering a one-dimensional system with \( \mathbf{v} = -V \hat{z} \) gives equation (2.2).

Appendix B

At steady state \( \partial p/\partial t = \partial c/\partial t = 0 \) and equations (2.2) and (2.3) become

\[
C_v \frac{d^2 p}{dz^2} + V \frac{dp}{dz} = 0, \quad D \frac{d^2 c}{dz^2} + V \frac{dc}{dz} = 0,
\]

subject to the boundary conditions

\[
V = \frac{k}{\mu} \frac{dp}{dz}; \quad (c_0 - c_s)V = -D \frac{dc}{dz}; \quad (z = 0),
\]

and

\[
p = p_r; \quad (z = H) \quad \text{and} \quad c \to c_c; \quad (z \to \infty),
\]

where \( c_s = c_c \) and \( c_0 = c_c/k_\alpha \). The general solutions to (B 1) are of the form

\[
p = A_p + B_p e^{r_p z} \quad \text{and} \quad c = A_c + B_c e^{r_c z},
\]

where \( A_p, B_p, r_p, c \) are constants. Inserting (B 4) into (B 1) gives \( r_p = -V/C_v \) and \( r_c = -V/D \); inserting (B 4) into (B 2) gives \( B_p = -C_v \mu/k \) and \( B_c = c_0 - c_c \); equations (B 3) then require \( A_p = p_r + (C_v \mu/k) e^{-VH/C_v} \) and \( A_c = c_c \), giving (2.6) and (2.7).

Appendix C

Inserting (3.12) into (3.7) and linearizing in the primed quantities gives, bearing in mind that for the steady base state \( \dot{e}_z + \dot{e}_{zz} = 0 \) and \( \dot{p}_z + \dot{e}_p \dot{p}_{zz} = 0 \),

\[
p'_t - p'_z = \epsilon_p \nabla^2 p', \quad \text{(C 1)}
\]

\[
c'_t - c'_z = \nabla^2 c'. \quad \text{(C 2)}
\]
The boundary conditions at the perturbed ice interface \( z = h \) can be expanded via Taylor series about \( z = 0 \) (Davis 2001). To first order and neglecting terms nonlinear in primed quantities, bearing in mind that \( \bar{c}(0) = 0 \) and \( \bar{c}_z(0) = 1 \),

\[
c(h) \approx c(0) + hc_z(0) = [\bar{c}(0) + c'(0)] + h[\bar{c}_z(0) + c'_z(0)] \approx h' + c'(0).
\]

Similarly,

\[
c_z(h) \approx 1 - h + c'_z(0), \quad p(h) \approx h' + p'(0) \quad \text{and} \quad p_z(h) \approx 1 - h'e_p^{-1} + p'_z(0).
\]

Inserting these expansions into (3.8) gives, upon neglecting terms nonlinear in the primed quantities and using the linearized quantities \( \hat{n} \approx (-h'_x, 1) \) and \( V_n \approx V(1 + h'_t) \),

\[
p'_z = h'e_p^{-1} + h'_z \quad \text{and} \quad c'_z = c'(1 - k_s) + h'k_s + h'_t \quad (z = 0).
\]

Combining equations (3.7) and (3.9) at \( z = h' \) and expanding about \( z = 0 \) gives

\[
h' = M_p(h' + p') + M_c(h' + c') \quad (z = 0).
\]

Inserting \((p', c', h')\) from (3.13) into (C 1) and (C 2), and neglecting terms nonlinear in the perturbations, leads to the ordinary differential equations

\[
c_{pp} dz^2 + c_{pz} dz - (\omega + q^2)c_1 = 0 \quad (z > 0), \quad (C 5)
\]

\[
c_{pp} dz^2 + c_{pz} dz - (\omega + q^2)c_1 = 0 \quad (z > 0). \quad (C 6)
\]

With (3.13) the boundary conditions (C 3) become

\[
\frac{dp_1}{dz} = (\epsilon_p^{-1} + \omega)h_1, \quad \frac{dc_1}{dz} = c_1(k_s - 1) + (k_s + \omega)h_1 \quad (z = 0), \quad (C 7)
\]

and (C 4) gives

\[
h_1 = \frac{M_p p_1 + M_c c_1}{1 - (M_p + M_c)} \quad (C 8).
\]

Assuming the perturbations decay far from the ice lens, the far-field boundary conditions are

\[
p_1 \to 0, \quad c_1 \to 0 \quad (z \to \infty). \quad (C 9)
\]

The solutions to (C 5) and (C 6) subject to (C 9) are

\[
p_1 = Ae^{-rp_z}, \quad c_1 = Be^{-rc_z}, \quad (C 10)
\]

where \( A \) and \( B \) are constants,

\[
r_p = \frac{1}{2}r_p^{-1} \left[ 1 + \sqrt{1 + 4r_p(\omega + q^2)} \right] \quad \text{and} \quad r_c = \frac{1}{2} \left[ 1 + \sqrt{1 + 4(\omega + q^2)} \right].
\]

Eliminating \( h_1 \) between equations (C 7) and (C 8), and inserting the solutions (C 10) into the result, leads to the characteristic equation (3.14).

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